c) $N\, a(q)\, |n(k)\rangle = \int d^3k\; N(k)\, a(q)\, |n(k)\rangle$

$= \int d^3k \left\{ [N(k), a(q)] + a(q)\, N(k)\, |n(k)\rangle \right.$

$= \int d^3k \left\{ - a(q)\, \delta(\vec{k}-\vec{q}) + a(q)\, n(k)\, |n(k)\rangle \right.$

$= \left[- a(q) + a(q) \int d^3k\, n(k) \right] |n(k)\rangle$

$= (n-1)\, a(q)\, |n(k)\rangle$

$N\, a(q)\, |n(k)\rangle = (n-1)\, a(q)\, |n(k)\rangle$

d) $N\, a^+(q)\, |n(k)\rangle = \int d^3k\; N(k)\, a^+(q)\, |n(k)\rangle$

$= \int d^3k \left\{ [N(k), a^+(q)]\, |n(k)\rangle + a^+(q)\, N(k)\, |n(k)\rangle \right\}$

$= \int d^3k \left\{ a^+(q)\, \delta(\vec{k}-\vec{q}) + n(k)\, a^+(q) \right\} |n(k)\rangle$

$= \left[\int d^3k\, n(k)\, a^+(q) + a^+(q) \right] |n(k)\rangle$

$N\, a^+(q)\, |n(k)\rangle = (n+1)\, a^+(q)\, |n(k)\rangle$

Scanné avec CamScanner

. Ces relations apparaissent alors que :

- $a(k)$ de une particule d'impulsion $\vec{k}$ et de masse m.
- $a^+(k)$ crée une particule " " " .

ces particules satisfont à l'éq de Klein - Gordon.

$|0\rangle$ est l'état qui ne contient aucune particule.

. Comme $[a^+(q), a^+(k)] = 0$ l'ordre de l'action $a^+(q)$ et $a^+(k)$ est sans importante.

. Un état est complètement déterminé par la donnée des impulsions $\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_m$.

$$|k_1, k_2, \ldots, k_m\rangle = a^+(k_1)\, a^+(k_2) \ldots a^+(k_m)\, |0\rangle .$$

et on a : $|k_1, k_2, \ldots k_m\rangle = |k_2, k_1, \ldots k_m\rangle$.
= une permutation quelconque $|k_1 \ldots k_m\rangle$.

on dit que l'état est complètement symétrique

$$|k_1, k_2, \ldots k_m\rangle = \frac{1}{m!} \sum_{\text{permutation}} a^+(k_1)\, a^+(k_2) \ldots a^+(k_m)\, |0\rangle .$$

c'est un état à m particules.

L'espace de Fock admet alors la base de tous les états à m particules, avec $m = 0, 1, 2, 3, \ldots, \infty$.

Ces m particules sont indiscernables au sens de la m.q et suivant la statistique de Bose - Einstein (B.E).

Ceci découle divergent des relations de commutations de φ et π.

et aussi on a : $a^+(k)\,a(k) = (2\pi)^3\, 2\omega_k\, N(k)$

$$|a(k)|n(k)\rangle|^2 = \langle n(k)|\,a^+(k)\,a(k)\,|n(k)\rangle$$
$$= (2\pi)^3\, 2\omega_k\, \langle n(k)|\,N(k)\,|n(k)\rangle.$$
$$= (2\pi)^3\, 2\omega_k\, n(k)\, \langle n(k)|n(k)\rangle.$$
$$= (2\pi)^3\, 2\omega_k\, n(k)\, |n(k)\rangle^+ |n(k)\rangle$$
$$= (2\pi)^3\, 2\omega_k\, n(k)\, \||n(k)\rangle\|^2.$$

$$\Rightarrow N(k) \geq 0,\; n(k) \geq 0.$$

$$a(k)|0\rangle = 0 \Rightarrow N(k)|0\rangle = 0.$$

c'est à dire, que le vide ne contient aucune particule.

* Montrons que $N(k)$ est un opérateur densité de nombre de particules:

$$H = \frac{1}{2}\int d^3x\, [\pi^2 + (\vec\nabla\varphi)^2 + m^2\varphi^2].$$
$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \frac{\omega_k}{2} \{a(k)\,a^+(k) + a^+(k)\,a(k)\}.$$
$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \frac{\omega_k}{2} \{[a(k), a^+(k)] + 2a^+(k)\,a(k)\}$$
$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\, \omega_k \left\{\frac{(2\pi)^3\, 2\omega_k\, \delta^3(0)}{2} + (2\pi)^3\, 2\omega_k\, N(k)\right\}.$$

$$H = \int d^3k\, \omega_k \left[N(k) + \frac{1}{2}\delta(0)\right].$$
$$H = \int d^3k\, \omega_k N(k) + \frac{\omega_k}{2}\int d^3k\, \delta(0) \to \infty$$

$$H = \int d^3k\, \omega_k\, N(k)$$ pour éliminer ∞.

$$:H: = H = \int d^3x\, \omega_k\, N(k)$$

$$[:H:, N(k)] = \int d^3x\, \omega q\, [N(q), N(k)] = 0.$$

Définition de la forme normale :

Définissons d'abord l'énergie de l'état $|n(k)\rangle$.

$$H = \int d^3K \left(\omega_k N(k) + \frac{1}{2}\omega_k \delta(0) \right).$$

Comme le terme infini $\int d^3k \frac{1}{2}\omega_k \delta(\vec{0})$ est indépendant de l'état, on définit l'énergie en retranchant ce terme (on mesure les différences d'énergie) d'où :

$$H = \int d^3k \, \omega_k N(k) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k a^+(k) a(k).$$

ce qui est ~~le~~ même si on définit H par $:H:$ forme normale où a^+ à gauche et a à droite.

Puisque $[a^+, a^+] = [a, a] = 0$, cette définition est sans ambiguïté.

Exemple : $:\varphi(x): = \varphi(x)$.

$$:\varphi(x)\varphi(y): = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left[a(k) e^{-ikx} + a^+(k) e^{ikx} \right]$$
$$\times \left[a(q) e^{-iqy} + a^+(q) e^{iqy} \right].$$

[(

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \Big[a(k) a(q) e^{-i(kx+qy)}$$
$$+ a(k) a^+(q) e^{-i(kx-qy)} + a^+(k) a(q) e^{+i(kx-qy)}$$
$$+ a^+(k) a^+(q) e^{i(kx+qy)} \Big].$$

Commutateur des champs :

$$[\varphi(x), \varphi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \Big[a(k) e^{-ikx} + a^+(k) e^{ikx} , \; a(q) e^{-iqy} + a^+(q) e^{iqy} \Big].$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \Big\{ \underbrace{[a(k), a(q)]}_{=0} e^{-i(kx+qy)} + [a^+(k), a(q)] e^{i(kx-qy)} + [a(k), a^+(q)] e^{-i(kx-qy)} + \underbrace{[a^+(k), a^+(q)]}_{=0} e^{i(kx+qy)} \Big\}.$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \Big\{ -(2\pi)^3 2\omega_k \, \delta(\vec{k}-\vec{q}) e^{i(kx-qy)} + (2\pi)^3 2\omega_k \, \delta(\vec{k}-\vec{q}) e^{-i(kx-qy)} \Big\}$$

$$[\varphi(x), \varphi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik(x-y)} - e^{ik(x-y)} \right).$$

$$= \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \frac{k_0}{|k_0|} e^{-ik(x-y)}.$$

C'est un invariant de Lorentz qui s'annule pour $(x-y)^2 < 0$.

causalité de la théorie (pas d'interférence entre deux points de genre espace).

On note : commutateur à temps quelconque.

$$[\varphi(x), \varphi^+(y)] =: \Delta(x-y)$$

Démonstration de H en fonction de a^+ et a.

$$H = \frac{1}{2} \int d^3x \left[\pi^2 + (\vec{\nabla}\varphi)^2 + m^2 \varphi^2 \right] \quad \text{-- } (\Delta).$$

avec : $\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)\, e^{-ikx} + a^+(k)\, e^{ikx} \right)$ .....

$$\frac{\partial \varphi(x)}{\partial t} = \pi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k) \frac{\partial}{\partial t} e^{-i(\omega_k t - \vec{k}\vec{x})} + a^+(k) \frac{\partial}{\partial t} e^{i(\omega_k t - \vec{k}\vec{x})} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left((-i\omega_k)\, a(k)\, e^{-ikx} + (i\omega_k)\, a^+(k) e^{ikx} \right).$$

$$\pi(x) = \dot{\varphi}(x) = (-i) \int \frac{d^3k}{(2\pi)^3\, 2} \left(a(k)\, e^{-ikx} - a^+(k)\, e^{ikx} \right).$$

* Calculons $\pi^2(x)$:

$$\pi^2(x) = -\int \frac{d^3k}{(2\pi)^3 2} \int \frac{d^3q}{(2\pi)^3 2} \left\{ \left(a(k)\, e^{-ikx} - a^+(k)\, e^{ikx} \right) \times \left(a(q)\, e^{-iqx} - a^+(q)\, e^{iqx} \right) \right\}.$$

$$\pi^2 = -\int \frac{d^3k}{2(2\pi)^3} \int \frac{d^3q}{2(2\pi)^3} \left[a(k) a(q)\, e^{-i(k+q)x} - a(k) a^+(q)\, e^{-i(k-q)x} - a^+(k) a(q)\, e^{i(k-q)x} + a^+(k) a^+(q)\, e^{i(k+q)x} \right]$$

---- (1).

* Calculons $(\vec{\nabla}\varphi)^2$.

$$\vec{\nabla}\varphi = \frac{\partial}{\partial x^i}\varphi = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left(a(k)\frac{\partial}{\partial x^i} e^{-i(\omega_k t - k^i x^i)} + a^+(k)\frac{\partial}{\partial x^i} e^{i(\omega_k t - k^i x^i)}\right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left((ik^i)\, a(k)\, e^{-ikx} - (ik^i)\, a^+(k)\, e^{ikx}\right)$$

$$\vec{\nabla}\varphi = i\int \frac{d^3k\; k^i}{(2\pi)^3 2\omega_k}\left(a(k)\, e^{-ikx} - a^+(k)\, e^{ikx}\right)$$

$$(\vec{\nabla}\varphi)^2 = (i)^2 \int \frac{d^3k}{(2\pi)^3 2\omega_k} k^i \int \frac{d^3q}{(2\pi)^3 2\omega_q} q^i \left[a(k)\, e^{-ikx} - a^+(k)\, e^{ikx}\right]\left[a(q)\, e^{-iqx} - a^+(q)\, e^{iqx}\right].$$

$$(\vec{\nabla}\varphi)^2 = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\int \frac{d^3q}{(2\pi)^3 2\omega_q}\, \vec{k}.\vec{q}\, \Big\{ - a(k)a(q)\, e^{-i(k+q)x} + a(k)a^+(q)\, e^{-i(k-q)x} + a^+(k)a(q)\, e^{i(k-q)x} - a^+(k)\, a^+(q)\, e^{i(k+q)x}\Big\} \quad -(2).$$

Calculons $m^2\varphi^2$,

$$m^2\varphi^2 = \int \frac{d^3k \quad m}{(2\pi)^3 2\omega k} \int \frac{d^3q \quad m}{(2\pi)^3 2\omega q} \left(a(k) e^{-ikx} + a^+(k) e^{ikx}\right)$$

$$\times \left(a(q) e^{-iqx} + a^+(q) e^{iqx}\right).$$

$$m^2\varphi^2 = m^2 \int \frac{d^3k}{(2\pi)^3 2\omega k} \int \frac{d^3q}{(2\pi)^3 2\omega q} \Big[a(k) a(q) e^{-i(k+q)x}$$

$$+ a(k) a^+(q) e^{-i(k-q)x} + a^+(k) a(q) e^{i(k-q)x}$$

$$+ a^+(k) a^+(q) e^{i(k+q)x} \Big] \quad - (3).$$

on remplace (1), (2) et (3) dans l'eq (1),

$$H = \frac{1}{2} \int d^3x \cdot \int \frac{d^3k}{(2\pi)^3 2\omega k} \int \frac{d^3q}{(2\pi)^3 2\omega q}$$

$$\Big\{ m^2 a(k) a(q) \underline{e^{-i(k+q)x}} + m^2 a(k) a^+(q) \cancel{e^{-i(k-q)x}}$$

$$+ m^2 a^+(k) a(q) e^{i(k-q)x} + m^2 a^+(k) a^+(q) e^{i(k+q)x}$$

$$- \omega k \omega q \; a(k) a(q) \underline{e^{-i(k+q)x}} + \omega k \omega q \; a(k) a^+(q) \cancel{e^{-i(k-q)x}}$$

$$+ \omega k \omega q \; a^+(k) a(q) e^{i(k-q)x} - \omega k \omega q \; a^+(k) a^+(q) e^{i(k+q)x}$$

$$- \vec{k}\vec{q} \; a(k) a(q) \underline{e^{-i(k+q)x}} + \vec{k}\vec{q} \; a(k) a^+(q) \cancel{e^{-i(k-q)x}}$$

$$+ \vec{k}.\vec{q} \; a^+(k) a(q) e^{i(k-q)x} - \vec{k}\vec{q} \; a^+(k) a^+(q) e^{i(k+q)x} \Big\}$$

$$H = \frac{1}{2}\int d^3x \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q}$$

$$\Big\{ [m^2 - \omega_k\omega_q - \vec{k}.\vec{q}]\, a(k)a(q)\, e^{-i(k+q)x}$$

$$+ [m^2 + \omega_k\omega_q + \vec{k}\vec{q}]\, a(k)a^+(q)\, e^{-i(k-q)x}$$

$$+ [m^2 + \omega_k\omega_q + \vec{k}.\vec{q}]\, a^+(k)a(q)\, e^{i(k-q)x}$$

$$+ [m^2 - \omega_k\omega_q - \vec{k}\vec{q}]\, a^+(k)a^+(q)\, e^{i(k+q)x} \Big\}.$$

Nous avons : $\displaystyle\int \frac{d^3x}{(2\pi)^3}\, e^{\pm i(k\pm q)x} = \delta(\vec{k}\pm\vec{q})\, e^{-i(\omega_k \pm \omega_q)x^0}$

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{2\omega_q} \Big\{ [m^2 - \omega_k\omega_q - \vec{k}.\vec{q}]\, a(k)a(q) \int \frac{d^3x}{(2\pi)^3} e^{-i(k+q)x}$$

$$+ [m^2 + \omega_k\omega_q + \vec{k}.\vec{q}]\, a(k)a^+(q) \int \frac{d^3x}{(2\pi)^3} e^{-i(k-q)x}$$

$$+ [m^2 + \omega_k\omega_q + \vec{k}.\vec{q}]\, a^+(k)a(q) \int \frac{d^3x}{(2\pi)^3} e^{+i(k-q)x}$$

$$+ [m^2 - \omega_k\omega_q - \vec{k}\vec{q}]\, a^+(k)a^+(q) \int \frac{d^3x}{(2\pi)^3} e^{i(k+q)x} \Big\}.$$

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{2\omega_q} \Big\{ [m^2 - \omega_k\omega_q - \vec{k}.\vec{q}]\, a(k)a(q)\, \delta(\vec{k}+\vec{q})\, e^{-i(\omega_k+\omega_q)\dot{x}^0}$$

$$+ [m^2 + \omega_k\omega_q + \vec{k}\vec{q}]\, a(k)a^+(q)\, \delta(\vec{k}-\vec{q})\, e^{-i(\omega_k-\omega_q)x^0}$$

$$+ [m^2 + \omega_k\omega_q + \vec{k}\vec{q}]\, a^+(k)a(q)\, \delta(\vec{k}-\vec{q})\, e^{i(\omega_k-\omega_q)x^0}$$

$$+ [m^2 - \omega_k\omega_q - \vec{k}\vec{q}]\, a^+(k)a^+(q)\, \delta(\vec{k}+\vec{q})\, e^{i(\omega_k+\omega_q)x^0} \Big\}.$$

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3(2\omega_k)^2}\Big\{ [m^2 - \omega_k^2 + \vec{k}^2]\, a(k)a(k)$$

$$+ [m^2+\omega_k^2 + k^2]\, a(k)a^+(k) + [m^2+\omega_k^2 + k^2]\, a^+(k)a(k)$$

$$+ [m^2 - \omega_k^2 + k^2]\, a^+(k)a^+(k)\Big\}$$

Remarque : $\vec{k}^2 + m^2 - \omega_k^2 = 0$ Pour $\vec{q} = -\vec{k}$.

$\omega_k^2 + \vec{k}^2 + m^2 = 2\omega_k^2$ Pour $\vec{k} = \vec{q}$

donc :

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3(2\omega_k)^2}\left\{2\times 2\omega_k^2\, a^+(k)\, a(k)\right\}$$

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\, \omega_k\, a^+(k)\, a(k).$$

Quantification du champ scalaire complexe :

La densité lagrangienne du champ complexe libre est obtenue à partir de celle du champ réel :

$$\mathcal{L}(\varphi, \varphi^*, \partial\varphi, \partial\varphi^*) = \mathcal{L}_1 + \mathcal{L}_2 .$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{m^2}{2}\varphi_1^2 + \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{m^2}{2}\varphi_2^2 .$$

$$\mathcal{L} = (\partial_\mu \varphi)^* (\partial^\mu \varphi) - m^2 \varphi^* \varphi .$$

$$\begin{cases} \varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) . \\ \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2) \end{cases}$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \partial^\nu \varphi^* - g^{\mu\nu}\mathcal{L} .$$

$$= (\partial^\mu \varphi)^* \partial^\nu \varphi + (\partial^\mu \varphi)\partial^\nu \varphi^* - g^{\mu\nu}\left((\partial_\alpha \varphi)^* (\partial^\alpha \varphi) - m^2 \varphi^* \varphi\right) .$$

$$= (\partial^\mu \varphi)^* \partial^\nu \varphi + (\partial^\mu \varphi)(\partial^\nu \varphi^*) - g^{\mu\nu}(\partial_\alpha \varphi)^* \partial^\alpha \varphi + g^{\mu\nu} m^2 \varphi^* \varphi .$$

$$T^{00} = (\partial^0 \varphi)^* \partial^0 \varphi + (\partial^0 \varphi)(\partial^0 \varphi^*) - \underbrace{g^{00}}_{1} (\partial_\alpha \varphi)^* \partial^\alpha \varphi + \underbrace{g^{00}}_{1} m^2 \varphi^* \varphi .$$

$$= \dot\varphi^* \dot\varphi + \dot\varphi \dot\varphi^* - (\partial_0 \varphi)^* (\partial^0 \varphi) - (\partial_i \varphi)^* \partial^i \varphi + m^2 \varphi^* \varphi .$$

$$= \dot\varphi^* \dot\varphi + \dot\varphi \dot\varphi^* - \dot\varphi^* \dot\varphi - \underbrace{g^{ii}}_{-1} (\partial_i \varphi)^* (\partial_i \varphi) + m^2 \varphi^* \varphi .$$

$$T^{00} = \dot\varphi^* \dot\varphi + (\vec\nabla \varphi)^* (\vec\nabla \varphi) + m^2 \varphi^* \varphi .$$

$$H = \int T^{00} d^3x = \int d^3x \left[\dot\varphi^* \dot\varphi + (\vec\nabla \varphi)^* (\vec\nabla \varphi) + m^2 \varphi^* \varphi \right] .$$

$$T^{0i} = (\partial^0 \varphi)^* \partial^i \varphi + (\partial^0 \varphi)(\partial^i \varphi^*) - \underbrace{g^{0i}}_{0} (\partial_\alpha \varphi)^* (\partial^\alpha \varphi) + \underbrace{g^{i0}}_{0} m^2 \varphi^* \varphi$$

$$P^i = \int d^3x \, T^{0i} = \int d^3x \left[\partial^0 \varphi^* \partial^i \varphi + \partial^i \varphi^* \partial^0 \varphi \right] .$$

Ce champ conserve la charge de Noether.

$Q = \int d^3x\, j^0$, avec : $j^\mu = i\varphi^* \overleftrightarrow{\partial}^\mu \varphi$.

En utilisant l'expansion de φ_1 et φ_2.

$$\varphi_1(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a_1(k)\, e^{-ikx} + a_1^+(k)\, e^{ikx} \right].$$

$$\varphi_2(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a_2(k)\, e^{-ikx} + a_2^+(k)\, e^{ikx} \right].$$

$$\varphi(x) = \frac{1}{\sqrt{2}} \left(\varphi_1(x) + i\varphi_2(x) \right).$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[\frac{a_1(k) + i a_2(k)}{\sqrt{2}}\, e^{-ikx} + \frac{a_1^+(k) + i a_2^+(k)}{\sqrt{2}}\, e^{ikx} \right].$$

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k)\, e^{-ikx} + b^+(k)\, e^{ikx} \right].$$

$$\varphi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a^+(k)\, e^{ikx} + b(k)\, e^{-ikx} \right].$$

avec : $$\begin{cases} a(k) = \frac{1}{\sqrt{2}} \left(a_1(k) + i a_2(k) \right) \\ b^+(k) = \frac{1}{\sqrt{2}} \left(a_1^+(k) + i a_2^+(k) \right) \end{cases} \quad \sim (*)$$

et : $$\begin{cases} a^+(k) = \frac{1}{\sqrt{2}} \left(a_1^+(k) - i a_2^+(k) \right). \\ b(k) = \frac{1}{\sqrt{2}} \left(a_1(k) - i a_2(k) \right) \end{cases} \quad \sim (**).$$

Pour quantifier φ, il suffit de reprendre les résultats de quantification du champ réel, c-à-d les relations de commutations précédentes et de les appliquer aux opérateurs de création et d'annihilation de deux champs φ_1 et φ_2, on trouve :

$$[a(k), a(q)] = [a^+(k), a^+(q)] = 0 .$$

$$[b(k), b(q)] = [b^+(k), b^+(q)] = 0 .$$

$$[a(k), b(q)] = [a^+(k), b^+(q)] = [a^+(k), b(q)] = [a(k), b^+(q)] = 0 .$$

$$[a(k), a^+(q)] = (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{q}) .$$

$$[b(k), b^+(q)] = (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{q})$$

Démonstration de ces relations :

Notons que : $[a_i(k), a_i^+(q)] = (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{q})$

$$[a_i(k), a_i(q)] = 0 = [a^+_i(k), a^+_i(q)] .$$

$$[a(k), a(q)] = \frac{1}{2}\left[a_1(k) + i a_2(k),\ a_1(q) + i a_2(q)\right]$$

$$= \frac{1}{2}\Big\{ \underbrace{[a_1(k), a_1(q)]}_{=0} + i\,\underbrace{[a_1(k), a_2(q)]}_{=0} + i\,\underbrace{[a_2(k), a_1(q)]}_{=0} - \underbrace{[a_2(k), a_2(q)]}_{=0} \Big\} .$$

$$[a(k), a(q)] = 0 .$$

$$\ast\,[a^+(k), a^+(q)] = \frac{1}{2}\,[a_1(k) - i a_2(k)\,,\, a_1(q) - i a_2(q)].$$

$$= \frac{1}{2}\Big\{ \overbrace{[a_1(k), a_1(q)]}^{=0} - i\,\overbrace{[a_1(k), a_2(q)]}^{=0}$$

$$- i\,\underbrace{[a_2(k), a_1(q)]}_{=0} - \underbrace{[a_2(k), a_2(q)]}_{=0}\Big\}.$$

$$[a^+(k), a^+(q)] = 0$$

$$\ast\,[b(k), b(q)] = \frac{1}{2}\,[a_1(k) - i a_2(k)\,,\, a_1(q) - i a_2(q)]$$

$$= \frac{1}{2}\Big\{ \overbrace{[a_1(k), a_1(q)]}^{=0} - i\,\overbrace{[a_1(k), a_2(q)]}^{=0}$$

$$- i\,\underbrace{[a_2(k), a_1(q)]}_{=0} - \underbrace{[a_2(k), a_2(q)]}_{=0}\Big\}.$$

$$[b(k), b(q)] = 0$$

$$\ast\,[b^+(k)\,,\, b^+(q)] = \frac{1}{2}\Big[a_1^+(k) + i a_2^+(k)\,,\, a_1^+(q) + i a_2^+(q)\Big].$$

$$= \frac{1}{2}\Big\{ \overbrace{[a_1^+(k), a_1^+(q)]}^{=0} + i\,\overbrace{[a_2^+(k), a_1^+(q)]}^{=0}$$

$$+ i\,\underbrace{[a_1^+(k), a_2^+(q)]}_{=0} - \underbrace{[a_2^+(k), a_2^+(q)]}_{=0}\Big\}.$$

$$[b^+(k), b^+(q)] = 0$$

$$\ast\,[a(k), b(q)] = \frac{1}{2}\Big[a_1(k) + i a_2(k)\,,\, a_1(k) - i a_2(k)\Big].$$

$$= \frac{1}{2}\Big\{ \underbrace{[a_1(k), a_1(k)]}_{=0} + i\,\underbrace{[a_2(k), a_1(k)]}_{=0}$$

$$- i\,\underbrace{[a_1(k), a_2(k)]}_{=0} + \underbrace{[a_2(k), a_2(k)]}_{=0}\Big\}.$$

$$[a(k), b(q)] = 0$$

* $[a^+(k), b^+(q)] = \frac{1}{2}\left[a_1^+(k) - i a_2^+(k),\ a_1^+(q) + i a_2^+(q)\right].$

$$= \frac{1}{2}\Big\{ [a_1^+(k), a_1^+(q)] + i\,\underbrace{[a_1^+(k), a_2^+(q)]}_{=0}$$

$$- i\,\underbrace{[a_2^+(k), a_1^+(q)]}_{=0} + \underbrace{[a_2^+(k), a_2^+(q)]}_{=0}\Big\}.$$

$$\boxed{[a^+(k), b^+(q)] = 0}$$

* $[a^+(k), b(q)] = \frac{1}{2}\left[a_1^+(k) - i a_2^+(k),\ a_1(q) - i a_2(q)\right].$

$$= \frac{1}{2}\Big\{ [a_1^+(k), a_1(q)] - i\,\underbrace{[a_1^+(k), a_2(q)]}_{=0}$$

$$- i\,\underbrace{[a_2^+(k), a_1(q)]}_{=0} - [a_2^+(k), a_2(q)]\Big\}.$$

$$= \frac{1}{2}\Big\{ [a_2(q), a_2^+(k)] - [a_1(q), a_1^+(k)]\Big\}.$$

$$= \frac{1}{2}\Big\{ (2\pi)^3 2\omega_k\, (\delta(\vec{k}-\vec{q}) - (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{q})\Big]$$

$$\boxed{[a^+(k), b(q)] = 0}$$

* $[a(k), b^+(q)] = \frac{1}{2}\left[a_1(k) + i a_2(k),\ a_1^+(q) + i a_2^+(q)\right].$

$$= \frac{1}{2}\Big\{ [a_1(k), a_1^+(q)] + i\,\underbrace{[a_1(k), a_2^+(q)]}_{=0}$$

$$+ i\,\underbrace{[a_2(k), a_1^+(q)]}_{=0} - [a_2(k), a_2^+(q)]\Big\}.$$

$$= \frac{1}{2}\Big\{ (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{q}) - (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{q})\Big\}$$

$$\boxed{[a(k), b^+(q)] = 0}$$

$$[a(k), a^+(q)] = \frac{1}{2}\left[a_1(k) + i a_2(k) , a_1^+(q) - i a_2^+(q)\right]$$

$$= \frac{1}{2}\Big\{ [a_1(k), a_1^+(q)] - i [a_1(k), a_2^+(q)] + i [a_2(k), a_1^+(q)] + [a_2(k), a_2^+(q)] \Big\}$$

$$= \frac{1}{2}\left\{ (2\pi)^3 2\omega_k \, \delta(\vec{k}-\vec{q}) + (2\pi)^3 2\omega_k \, \delta(\vec{k}-\vec{q}) \right\}.$$

$$[a(k), a^+(q)] = (2\pi)^3 2\omega_k \, \delta(\vec{k}-\vec{q})$$

$$[b(k), b^+(q)] = \frac{1}{2}\left[a_1(k) - i a_2(k) , a_1^+(q) + i a_2^+(q)\right]$$

$$= \frac{1}{2}\Big\{ [a_1(k), a_1^+(q)] + i [a_1(k), a_2^+(q)] - i [a_2(k), a_1^+(q)] + [a_2(k), a_2^+(q)] \Big\}.$$

$$= \frac{1}{2}\left\{ (2\pi)^3 2\omega_k \, \delta(\vec{k}-\vec{q}) + (2\pi)^3 2\omega_k \, \delta(\vec{k}-\vec{q}) \right\}.$$

$$[b(k), b^+(q)] = (2\pi)^3 2\omega_k \, \delta(\vec{k}-\vec{q})$$

Ces résultats montrent que les opérateurs $a^+(k)$ et $b^+(k)$ jouent le rôle d'opérateurs de création, alors que leurs conjugués $a(k)$ et $b(k)$ sont des opérateurs d'annihilation.

Les règles de commutation canonique à des temps quelconques peuvent être directement déduites de la décomposition du champ complexe et de :

$$\begin{cases} \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k) e^{-ikx} + b^+(k) e^{ikx} \right] . \\ \varphi(y) = \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left[a(q) e^{-iqy} + b^+(q) e^{+iqy} \right] \end{cases}$$

$$\begin{cases} \varphi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a^+(k) e^{ikx} + b(k) e^{-ikx} \right] . \\ \varphi^+(y) = \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left[a^+(q) e^{iqy} + b(q) e^{-iqy} \right] . \end{cases}$$

* $[\varphi(x), \varphi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q}$

$$\times \left\{ \left[a(k) e^{-ikx} + b^+(k) e^{ikx} , a(q) e^{-iqy} + b^+(q) e^{iqy} \right] \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left\{ \underbrace{[a(k), a(q)]}_{=0} e^{-i(kx+qy)} + \underbrace{[a(k), b^+(q)]}_{=0} e^{-i(kx-qy)} + \underbrace{[b^+(k), a(q)]}_{=0} e^{i(kx-qy)} + \underbrace{[b^+(k), b^+(q)]}_{=0} e^{i(kx+qy)} \right\} .$$

$$= \int \frac{d^3q}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \{0\} .$$

$$[\varphi(x), \varphi(y)] = 0$$

$$[\varphi^+(x), \varphi^+(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q}$$

$$\times \left\{ \left[a^+(k)\, e^{ikx} + b(k)\, e^{-ikx}, a^+(q)\, e^{iqy} + b(q)\, e^{-iqy} \right] \right\}.$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left\{ \underbrace{[a^+(k), a^+(q)]}_{=0} e^{i(kx+qy)} \right.$$

$$+ \underbrace{[a^+(k), b(q)]}_{=0} e^{i(kx-qy)} + \underbrace{[b(k), a^+(q)]}_{=0} e^{-i(kx-qy)}$$

$$\left. + \underbrace{[b(k), b(q)]}_{=0} e^{-i(kx+qy)} \right\}.$$

$$[\varphi^+(x), \varphi^+(y)] = 0$$

$$[\varphi(x), \varphi^+(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q}$$

$$\times \left[a(k)\, e^{-ikx} + b^+(k)\, e^{ikx}, a^+(q)\, e^{iqy} + b(q)\, e^{-iqy} \right]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q}$$

$$\times \left\{ [a(k), a^+(q)]\, e^{-i(kx-qy)} + \underbrace{[a(k), b(q)]}_{=0} e^{-i(kx+qy)} \right.$$

$$\left. + \underbrace{[b^+(k), a^+(q)]}_{=0} e^{i(kx+qy)} + [b^+(k), b(q)]\, e^{i(kx-qy)} \right\}.$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left\{ [a(k), a^+(q)]\, e^{-i(kx-qy)} \right.$$

$$\left. - [b(q), b^+(k)]\, e^{i(kx-qy)} \right\}.$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left\{ (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{q})\, e^{-i(kx-qy)} \right.$$

$$\left. - (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{q})\, e^{i(kx-qy)} \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ik(x-y)} - e^{ik(x-y)} \right] = i\Delta(x-y).$$

$$[\varphi(x), \varphi^+(y)] = i\Delta(x-y)$$

L'espace de Fock se construit de la même manière en agissant $a(k)$ et $b(k)$ sur le vide $|0\rangle$ $\forall\ k(\omega_k, \vec{k})$

Pour pouvoir interpréter, calculons d'abord la charge:

$$Q = \int d^3x\, j^0 \quad , \text{avec: } j^0 = :ie\varphi^+ \overleftrightarrow{\partial}^0 \varphi: \quad \text{-- (k)}$$

posons $e=1$.

$$Q = \int d^3x\, j^0 = i\int d^3x \left(\varphi^+ \overleftrightarrow{\partial}^0 \varphi\right).$$

$$Q = i\int d^3x \left(\varphi^+ \partial^0\varphi - (\partial_0 \varphi^+).\varphi\right) \quad \text{-- } (\square)^0$$

- $$\varphi = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(k)\, e^{-i(k_0t - \vec{k}\vec{x})} + b^+(k)\, e^{i(k_0t - \vec{k}.\vec{x})}\right].$$

$$\partial^0\varphi = \frac{\partial\varphi}{\partial t} = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(k)\frac{\partial}{\partial t} e^{-i(k_0t - \vec{k}\vec{x})} + b^+(k)\frac{\partial}{\partial t} e^{i(k_0t - \vec{k}\vec{x})}\right]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[-i\omega_k\, a(k)\, e^{-ikx} + i\omega_k\, b^+(k)\, e^{ikx}\right]$$

$$k_0 = \omega_k.$$

- $$\partial^0\varphi = \left(\frac{-i}{2}\right)\int \frac{d^3k}{(2\pi)^3}\left[a(k)\, e^{-ikx} - b^+(k)\, e^{ikx}\right].$$

On a aussi:

- $$\varphi^+ = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a^+(k)\, e^{ikx} + b(k)\, e^{-ikx}\right].$$

$$\partial^0\varphi^+ = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a^+(k)\frac{\partial}{\partial t} e^{i(\omega_k t - \vec{k}\vec{x})} + b(k)\frac{\partial}{\partial t} e^{-i(\omega_k t - \vec{k}\vec{x})}\right]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[i\omega_k\, a^+(k)\, e^{ikx} - i\omega_k\, b(k)\, e^{-ikx}\right]$$

- $$\partial^0\varphi^+ = \left(\frac{i}{2}\right)\int \frac{d^3k}{(2\pi)^3}\left[a^+(k)\, e^{ikx} - b(k)\, e^{-ikx}\right]$$

$$\psi^+ \gamma^0 \psi = \left(\frac{-i}{2}\right) \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3} \Big[\left\{ a^+(k) e^{ikx} + b(k) e^{-ikx} \right\} \times \left\{ a(q) e^{-iqx} - b^+(q) e^{iqx} \right\} \Big]$$

$$\psi^+ \gamma^0 \psi = \left(\frac{-i}{2}\right) \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3} \Big\{ a^+(k) a(q) e^{i(k-q)x} - a^+(k) b^+(q) e^{i(k+q)x} + b(k) a(q) e^{-i(k+q)x} - b(k) b^+(q) e^{-i(k-q)x} \Big\} \quad \ldots (1)^\circ$$

$$(\gamma^0 \psi^+) \psi = \left(\frac{i}{2}\right) \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \Big[\left\{ a^+(k) e^{ikx} - b(k) e^{-ikx} \right\} \times \left\{ a(q) e^{-iqx} + b^+(q) e^{iqx} \right\} \Big]$$

$$(\gamma^0 \psi^+) \psi = \left(\frac{i}{2}\right) \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \Big\{ a^+(k) a(q) e^{i(k-q)x} + a^+(k) b^+(q) e^{i(k+q)x} - b(k) a(q) e^{-i(k+q)x} - b(k) b^+(q) e^{-i(k-q)x} \Big\} \quad -(2)^\circ$$

on remplace (1)° et (2)° dans (α), on obtient :

$$j^0 = i \left(\psi^+ \gamma^0 \psi - \gamma^0 \psi^+ \psi \right)$$

$$j^0 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3} \Big\{ a^+(k) a(q) e^{i(k-q)x} - \cancel{a^+(k) b^+(q) e^{i(k+q)x}} + b(k) a(q) e^{-i(k+q)x} - b(k) b^+(q) e^{-i(k-q)x} + a^+(k) a(q) e^{i(k-q)x} + \cancel{a^+(k) b^+(q) e^{i(k+q)x}} - b(k) a(q) e^{-i(k+q)x} - b(k) b^+(q) e^{-i(k-q)x} \Big\}.$$

$$j^0 = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3} \Big\{ a^+(k) a(q) e^{i(k-q)x} - b(k) b^+(q) e^{-i(k-q)x} \Big\}$$