Chapter 1 :

Real numbers

Motivation.

The Babylonians show that if A is a square of side unity and B a square of side equal to the diagonal d of A, then the area of B is double that of A, in other words: $d^2 = 2$. Afterwards, the Pythagoreans showed that d (which is equal to $\sqrt{2}$) is not a rational number. That is to say, we cannot write $\sqrt{2}$ in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. So, we will need another set containing $\mathbb{Q}$, as well as the solutions of the algebraic equations.

“ *Real numbers are used to represent any physical measurement such as: the price of a product, the time between two events, the altitude of a geographic site, the mass of an atom or the distance of the nearest galaxy. These measurements depend on the choice of a unit of measurement, and the result is expressed as the product of a real number by a unit.* ". **(Wikipedia Encyclopedia)**

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1.1. Definitions and properties

Reminder.

- The set of natural numbers : $\mathbb{N} = \{0, 1, 2, 3, \dots \dots\}$.
- The set of integers : $\mathbb{Z} = \{\dots \dots, -2, -1, 0, 1, 2, \dots \dots\}$.
- The set of rational numbers : $\mathbb{Q} = \left\{\frac{p}{q} \setminus p, q \in \mathbb{Z}, \ q \neq 0\right\}$.

Definition 1.

The set of real numbers, denoted $\mathbb{R}$, is the complement of the set of rational numbers $\mathbb{Q}$. It is an ordered commutative field equipped with the operations +, ×.

Properties.

The order relation $\leq$ is compatible with the operations $+$ et $\times$, i.e. for $a, b, c, d \in \mathbb{R}$ we have:

1) $a \leq b$ and $c \leq d \Longrightarrow a + c \leq b + d$.

2) $a \leq b$ and $c \geq 0 \Longrightarrow a \times c \leq b \times c$.

3) $a + c \leq b + c \ \Longrightarrow a \leq b$.

4) $a \leq b$ and $b \leq c \Longrightarrow a \leq c$.

5) $a \leq b \Longrightarrow b - a \in \mathbb{R}_+$, $a \leq b$ and $b \leq a \Longrightarrow a = b$.

6) $0 < a \leq b \iff 0 < \frac{1}{b} \leq \frac{1}{a}$ and $0 < a \leq b \iff 0 < a^n \leq b^n$, $\forall n \in \mathbb{N}^*$.

Theorem 1. (Property of Archimedes)

The field $\mathbb{R}$ is Archimedean, i.e.

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}: \quad n > x$$

1.2. Decimal representation and density

Definition 2. (Decimal representation - Ecriture décimale)

Decimal representation is the expansion in base 10 of a positive real number, given by:

$$x = c_n 10^n + \cdots + c_1 10 + c_0 + d_1 \frac{1}{10} + \cdots + d_m \frac{1}{10^m} + \cdots$$

$$= \sum_{k=0}^{n} c_k 10^k + \sum_{i=1}^{+\infty} \frac{d_i}{10^i}$$

with $c_n, \dots, c_1, c_0, d_1, \dots, d_n \in \{0,1,2,3,4,5,6,7,8,9\}$, they are called "**digits**".

We write $\quad x = c_n \dots c_1 c_0 , d_1 \dots d_m \dots$

Remark. We can consider another base of development other than 10, for example: base 2 (binary), base 8 (octal), base 16 (hexadecimal), base p (p-adic).

Examples.

1) For $x = 125{,}3269$, we have

$$125{,}3269 = 1.10^2 + 2.10 + 5 + 3.\frac{1}{10} + 2.\frac{1}{10^2} + 6.\frac{1}{10^3} + 9.\frac{1}{10^4}$$

2) For $x = 80764$, we have

$$80764 = 8.10^4 + 0.10^3 + 7.10^2 + 6.10 + 4 + 0.\frac{1}{10} + 0.\frac{1}{10^2} + \cdots$$

Proposition 1.

A number is rational if and only if its decimal expansion is periodic or finite.

Examples.

1) For $x = \frac{1253269}{10000}$, we have

$$\frac{1253269}{10000} = 125{,}3369 = 1.10^2 + 2.10 + 5 + 3.\frac{1}{10} + 2.\frac{1}{10^2} + 6.\frac{1}{10^3} + 9.\frac{1}{10^4}$$

2) For $x = \frac{78}{7}$, we have

$$\frac{78}{7} = 11{,}142857142857142857142857 = 11,\overline{142857}$$

$$= 1.10 + 1 + 1.\frac{1}{10} + 4.\frac{1}{10^2} + 2.\frac{1}{10^3} + 8.\frac{1}{10^4} + 5.\frac{1}{10^5} + 7.\frac{1}{10^6} + \ \dots\dots$$

3) For $x = \frac{1}{3}$, we have

$$\frac{1}{3} = 0{,}33333333 = 0 + 3.\frac{1}{10} + 3.\frac{1}{10^2} + 3.\frac{1}{10^3} + \ \dots\dots$$

Theorem 2. **(Density of $\mathbb{Q}$ - Densité de $\mathbb{Q}$)**

The set $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e.

$$\forall x, y \in \mathbb{R} \ \text{ with } \ x < y\,, \exists q \in \mathbb{Q} : \ \ x < q < y$$

Definition 3. (Irrational numbers)

Irrational numbers are numbers that are not rational. We denote the set of irrationals by $\mathbb{R} \setminus \mathbb{Q}$ or $\mathbb{Q}^c$.

Example. The numbers $\sqrt{2}$, π , e , ...are irrational numbers.

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1.3. integer part et absolute value

Definitions 4. (Interger part and fractional part)

- The integer part of a real number x is the largest integer $r \in \mathbb{Z}$ such as : $r \leq x$.
- We denote the integer part by $E(x)$ or $[x]$, so we have : $E(x) \leq x < E(x) + 1$.
- In the decimal representation of x the integer part is equal to : $[x] = c_n \dots c_1 c_0$.
- The fractional part, denoted $\{x\}$, is given by :

$$\{x\} = 0, d_1 d_2 \dots d_m \dots$$

- We have : $x = [x] + \{x\}$ and $\{x\} = x - [x]$.

Examples.

1) For $x = \frac{1253269}{10000} = 125{,}3269$, we have : $[x] = 125$, $\{x\} = 0{,}3269$.
2) For $x = 2023$, we have : $[x] = 2023 = x$, $\{x\} = 0$.
3) For $x = \frac{1}{3} = 0.3333333$, we have : $[x] = 0$, $\{x\} = 0{,}33333333$.
4) For $x = -5{,}86$, we have : $[x] = -6$, $\{x\} = 0{,}86$.
5) If $E(x) = 2$ then $2 \leq x < 3$.

Remark. There is another integer part called "**superior**" , denoted $\lceil x \rceil$, defined by:

$$\lceil x \rceil - 1 < x \leq \lceil x \rceil$$

In this case, the integer part $[x]$ is called the "**lower** integer part", denoted $\lfloor x \rfloor$.

Proposition 2.

Let $x, y \in \mathbb{R}$. We have the following properties :

1) For $\in \mathbb{Z}$: $E(x+n) = E(x) + n$.
2) $E(x) + E(-x) = \begin{cases} 0 & , \text{ if } x \in \mathbb{Z} \\ -1 & , \text{ Otherwise} \end{cases}$.
3) $E(x) + E(y) \leq E(x+y) \leq E(x) + E(y) + 1$.
4) For $\in \mathbb{N}^*$: $nE(x) \leq E(nx) \leq nE(x) + n - 1$.

Proof.

We will prove the last property. In fact, we have:

$$E(x) \leq x < E(x) + 1 \Longrightarrow nE(x) \leq nx < nE(x) + n$$

So : $nE(x) \leq E(nx) \leq nx \leq E(nx) + 1 < nE(x) + n$ ∎

Definition 5. (Valeur absolue – absolute value)

The absolute value, denoted $|x|$, is defined by :

$$|x| = \begin{cases} x & , \text{ si } x \geq 0 \\ -x & , \text{ si } x < 0 \end{cases}$$

Proposition 3.

Let $x, y \in \mathbb{R}$, We have the following properties :

1) $|x| = 0 \Leftrightarrow x = 0$, $|x| = |-x|$, $|x| = \sqrt{x^2}$.
2) For $\in \mathbb{R}^*$: $|x| \leq a \Leftrightarrow -a \leq x \leq a$.
3) The triangular inequality : $|x+y| \leq |x| + |y|$.
4) Generalization : $|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|$
5) $|xy| = |x||y|$.
6) $||x| - |y|| \leq |x+y|$, $||x| - |y|| \leq |x-y|$

Proof.

We will prove the triangular inequality. In fact we have : $-|x| \leq x \leq |x|$ and $-|y| \leq x \leq |y|$

We add up the inequalities, we find : $-(|x| + |y|) \leq x + y \leq |x| + |y|$

So $|x + y| \leq |x| + |y|$. ■

1.4. Intervals and Extended real line

Definitions 6. (Intervalles - Intervals)

Intervals are subsets of $\mathbb{R}$ defined for , $b \in \mathbb{R}$, by :

$$[a,b] = \{x \in \mathbb{R} \setminus a \leq x \leq b\} \quad ; \quad [a,b[= \{x \in \mathbb{R} \setminus a \leq x < b\}$$

$$]a,b] = \{x \in \mathbb{R} \setminus a < x \leq b\} \quad ; \quad]a,b[= \{x \in \mathbb{R} \setminus a < x < b\}$$

$$[a,+\infty[= \{x \in \mathbb{R} \setminus a \leq x\} \quad ; \quad]a,+\infty[= \{x \in \mathbb{R} \setminus a < x\}$$

$$]-\infty,b] = \{x \in \mathbb{R} \setminus x \leq b\} \quad ; \quad]-\infty,b[= \{x \in \mathbb{R} \setminus x < b\}$$

$$]-\infty,+\infty[= \mathbb{R} \quad ; \quad]0,+\infty[= \mathbb{R}_+^* \quad ; \quad [0,+\infty[= \mathbb{R}_+$$

$$]-\infty,0[= \mathbb{R}_-^* \quad ; \quad]-\infty,0] = \mathbb{R}_+$$

Remarks.

1) $[a,a] = \{a\}$, $]a,a[= \emptyset$.
2) The length of the interval $[a,b]$ equal to $b - a$. The center of this interval is the point $\frac{a+b}{2}$.
3) The intersection of intervals $\mathbb{R}$ is always an interval. The union of intervals $\mathbb{R}$ is not always an interval .

Definition 7. **(Droite réelle achevée - Extended real number line)**

The Extended real number line denoted $\overline{\mathbb{R}}$ is defined by : $]-\infty,+\infty[= \overline{\mathbb{R}}$

Operations on $\overline{\mathbb{R}}$. For every $x \in \mathbb{R}$, we have :

- ✓ $x + (+\infty) = +\infty$, $x + (-\infty) = -\infty$.
- ✓ $+\infty + (+\infty) = +\infty$, $-\infty + (-\infty) = -\infty$.
- ✓ $x \times (+\infty) = +\infty$, $x > 0$ and $x \times (+\infty) = -\infty$, $x < 0$.
- ✓ $x \times (-\infty) = -\infty$, $x > 0$ and $x \times (-\infty) = +\infty$, $x > 0$.
- ✓ $+\infty \times (+\infty) = +\infty$, $-\infty \times (-\infty) = -\infty$, $+\infty \times (-\infty) = -\infty$.
- ✓ Indeterminate forms : $0 \times (\pm\infty)$, $+\infty + (-\infty)$.

1.5. Upper bound and lower bound

Definitions 8. (Majorant, minorant – Upper and lower bounds)

Let $A \subset \mathbb{R}$ and $M, m \in \mathbb{R}$.

- We say that M is a ***majorant*** of A if : $\forall a \in A \ , a \leq M$.
  In this case we say that A is bounded from above or majorized.
- We say that m is a ***minorant*** of A if : $\forall a \in A \ , a \geq m$.
  In this case we say that A is bounded from below or minorized.
- We say that A is ***bounded*** if it is majorized and minorized.

Examples.

1) Let $A = [1,3[$, the numbers: $3, \frac{9}{2}, \sqrt{21}$ are upper bounds of A. The numbers: $1, 0, -1000$ are lower bounds of A. In this case, the interval $[1,3[$ is bounded.
2) Let $B =]-\infty, 2]$, the numbers: $2, \pi, 2023$ are upper bounds of B. There are no lower bounds of B(is not minorized, therefore is not bounded).
3) Let $C = \{x \in \mathbb{R} \setminus \sqrt{x} \leq 2\}$, the numbers: $4, \sqrt{5}, 2050$ are upper bounds of C. The numbers: $-1, 0, -\sqrt{1000}$ are lower bounds of C. In this case, the set C is bounded.
4) **Attention !** For $A = [0,1[\cup \{5\}$, the numbers: $3, 4, \frac{9}{2}$ are not upper bounds of A.

Definitions 9. (Maximum, minimum)

Let $A \subset \mathbb{R}$ and $M, m \in \mathbb{R}$.

- We say that M is the *largest element* of A if: $\forall a \in A \ , a \leq M$ and $M \in A$.
  It is called the *maximum* and is noted **max A**.
- We say that m is the *smallest element* of A if : $\forall a \in A \ , a \geq m$ and $m \in A$.
  It is called the *minimum* and is noted **min A**.

Remark. The maximum and the minimum do not always exist. If they exist, they are unique.

Examples.

1) For $A_1 = [1,3]$, we have 3 is the largest element of A_1, then $\max A_1 = 3$. The number 1 is the smallest element of A_1, then $\min A_1 = 1$.
2) For $A_2 = [1,3[$, we have $3 \notin A_2$, then $\max A_2$ does not exist. The number 1 remains the $\min A_2$.
3) For $A_3 =]-\infty, 2]$, we have $\max A_3 = 2$, the $\min A_3$ does not exist (A_3 is not bounded).
4) That is $A_4 = \{x \in \mathbb{R} \setminus \sqrt{x} \leq 2\}$, we have $\max A_4 = 4$ and $\min A_4 = 0$.

Definitions 10. (Supremum, infimum)

- If A is a majorized part of $\mathbb{R}$, we call the smallest of the upper bounds of A a *« supremum »* of A. We note it **sup A**.
- If A is a minorized part of $\mathbb{R}$, we call the largest of the lower bounds of A a *« infimum »* of A. We note it **inf A**.

Remarks.

1) If the upper bound and the lower bound exist, then they are unique.
2) If the upper bound does not exist, we write : $\sup A = +\infty$.
If the lower bound does not exist, we write : $\inf A = -\infty$.
3) It is not obligatory that **sup A** and **inf A** belong to A.
If this is the case, we have: $\sup A = \max A$ and $\inf A = \min A$.

Examples.

4) Let $A = [1,3[$ we have: $\sup A = 3$ ($\max A$ does not exist) and $\inf A = \min A = 1$.
5) Let we $B =]-\infty, 2]$ we have: $\sup B = \max B = 2$ and $\inf B = -\infty$. (since B it is not minorized).
6) Let we $C = \{x \in \mathbb{R} \setminus x^2 \leq 4\}$ we have: $\sup C = \max C = 2$ and $\inf C = \min C = -2$.
7) Let we $D = [0,1[\cup \{5\}$ we have: $\sup D = \max D = 5$ and $\inf D = \min D = 0$.

***Theorem* 3.** (Bolzano)

- Every majorized part of $\mathbb{R}$ has an upper bound in $\mathbb{R}$.
- Every minorized part of $\mathbb{R}$ has a lower bound in $\mathbb{R}$.

Remark. This property is not valid in $\mathbb{Q}$. For example, the set $A = \{x \in \mathbb{Q} \setminus x^2 < 2\}$ is bounded by $\sqrt{2}$, but the upper bound does not belong to $\mathbb{Q}$.

Proposition 4. (Characterization of the supremum)

The supremum of A is the unique element such that :

$$\sup A = M \Leftrightarrow \begin{cases} \mathbf{1)}\ \forall a \in A \quad , \quad a \leq M \\ \mathbf{2)}\ \forall \varepsilon > 0, \exists a_0 \in A\ : M - \varepsilon < a_0 \end{cases}$$

Proposition 5. (Characterization of the infimum)

The infimum of A is the unique element such that :

$$\inf A = m \Leftrightarrow \begin{cases} \mathbf{1)}\ \forall a \in A \quad , \quad a \geq m \\ \mathbf{2)}\ \forall \varepsilon > 0, \exists a_1 \in A\ : m + \varepsilon > a_1 \end{cases}$$

Example.

Let the set $A = \left\{x_n = 1 - \frac{1}{n} \ \backslash \ n \in \mathbb{N}^*\right\}$. We have: $\min A = 0$ and $\max A$ does not exist . So :

- $\inf A = \min A = 0$.
- We will show that $\sup A = 1$. In fact, we have $x_n < 1 \ , \ \forall n \in \mathbb{N}^*$.
  On the other hand, we must show that:
  $$\forall \varepsilon > 0, \exists\, N \in \mathbb{N}^* \ (a_0 = x_N \in A) \ : \quad 1 - \varepsilon < 1 - \frac{1}{N}$$
  That is to say $\frac{1}{\varepsilon} < N$. So, just choose $N = \left[\frac{1}{\varepsilon}\right] + 1$.

Example.

Let the set $B = \left\{b_n = \frac{n+1}{2n+1} \ \backslash \ n \in \mathbb{N}\right\}$. For every $n \in \mathbb{N}$, we have: $\frac{1}{2} \le \frac{n+1}{2n+1} \le 1$.

- Then, $\frac{1}{2}$ is a lower bound and 1 is an upper bound of B. So, the part B is bounded, hence $\inf B$ and $\sup B$ exist (according to Bolzano's theorem).
- We observe that $b_0 = 1$. So, $\sup B = \max B = 1$.
- We will show that $\inf B = \frac{1}{2}$. We have $\frac{1}{2} \le b_n \ , \ \forall n \in \mathbb{N}^*$.
  On the other hand, we must show that:
  $$\forall \varepsilon > 0, \exists\, N \in \mathbb{N}^* \ (a_0 = b_N \in A) \ : \quad \frac{n+1}{2n+1} < \frac{1}{2} + \varepsilon$$
  That is to say : $\frac{1-\varepsilon}{2\varepsilon} < N$. So, just choose $N = \left[\frac{1-\varepsilon}{2\varepsilon}\right] + 1$.

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