Chapter 4:

Functions of a real variable - 1st Part

Motivation.

A **function** allows to define a result (most often numerical) for each value of a set called *domain* . This result can be obtained by a series of arithmetic calculations or by a list of values, particularly in the case of taking physical measurements, or by other processes such as solving equations or crossing the limit . The actual calculation of the result or its approximation possibly relies on the development of a **computer** function . "Wikipedia"

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4.1. Definitions and properties

Definitions 1.

- A function is a relation f from a set E to a set F , such that every element x of E admits at most one image in F, we write:

$$f: \quad E \longrightarrow F$$
$$x \longmapsto f(x)$$

- If $F \subseteq \mathbb{R}$ we say that f is a real function. If further $E \subseteq \mathbb{R}$, we say that f is a real function of a real variable.
- The definition set of the function f noted D_f is the subset of E of the values taken by x for which $f(x)$ is computable, i.e. the elements of E which have an image by f.
- We call a graph , or representative curve , of a function f the set:

$$\Gamma = \{(x, f(x)) \in \mathbb{R}^2 \ / \ x \in D_f\}$$

It is the set of points M on the coordinate plane (x, y) $where$: $y = f(x)$.

Examples.

1) The "inverse of a number" relation is a function defined by : $f(x) = \frac{1}{x}$.
2) The polynomial $f(x) = x^2 + x - 3$ is a function defined on $\mathbb{R}$.
3) **The affine** function is in the form: $f(x) = ax + b$.
4) The **integer power function** is given by : $f(x) = x^n$, $n \in \mathbb{N}$.
5) The **n-th root function** is given by : $f(x) = x^{\frac{1}{n}}$, $n \in \mathbb{N}^*$.
6) The **homographic function** is a fraction : $f(x) = \frac{ax+b}{cx+d}$.
7) The **integer part function** is given by : $f(x) = E(x)$.
8) **The absolute value** function is given by : $f(x) = |x|$.

Definitions 2.

Let $f: D \longrightarrow \mathbb{R}$ a real function. We say that :

- f is **increasing** on D if : $\forall x_1, x_2 \in D \;,\; x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2)$.
- f is **strictly increasing** on D if : $\forall x_1, x_2 \in D \;,\; x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$.
- f is **decreasing** on D if : $\forall x_1, x_2 \in D \;,\; x_1 \leq x_2 \Longrightarrow f(x_1) \geq f(x_2)$.
- f is **strictly decreasing** on D if : $\forall x_1, x_2 \in D \;,\; x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$.
- f is **monotonic** if it is increasing or decreasing.
- f is **strictly monotonic** if it is strictly increasing or strictly decreasing.

Examples.

1) The function $f(x) = x + 2$ is strictly increasing on $\mathbb{R}$.
2) The function $f(x) = |x|$ is strictly increasing on $\mathbb{R}_+$. But, it is neither increasing nor decreasing on $\mathbb{R}$.
3) The function $f(x) = \ln x$ is strictly increasing on $\mathbb{R}$. The function $f(x) = -\ln x$ is strictly decreasing on $\mathbb{R}$.

Definitions 3.

- The function f is **constant** if : $\exists \alpha \in \mathbb{R} \;, \forall x \in D \;:\; f(x) = \alpha$.
- The function f is **periodic** if : $\exists T \in \mathbb{R}, \forall x \in D \;:\; f(x + T) = f(x)$.

Definitions 4.

Consider $f: D \longrightarrow \mathbb{R}$ a function, with D is symmetrical about 0. We say that:

- The function f is **even** if : $\forall x \in D \;,\; f(-x) = f(x)$.
- The function f is **odd** if : $\forall x \in D \;,\; f(-x) = -f(x)$.

Remarks : (Geometric interpretation)

- A function is even if its graph is symmetrical about the y-axis.
- A function is odd if its graph is symmetrical about the origin.

Examples.

1) The function $f(x) = \sqrt{x^2 - 1}$ is even on the domain of definition $D =]-\infty, -1[\cup]1, +\infty[$.
2) The domain of definition of the function $f(x) = \sqrt{x}$ is not symmetrical about 0.
3) The functions $f(x) = x^2 \;,\; f(x) = x^4 , \dots , f(x) = x^{2n}$ are even on $\mathbb{R}$.
4) The functions $f(x) = x \;,\; f(x) = x^3 , \dots , f(x) = x^{2n+1}$ are odd on $\mathbb{R}$.

Definitions 5. Let $f: D \longrightarrow \mathbb{R}$ a real function. We say that :

- f is **increased** on D if : $\exists M \in \mathbb{R}, \; \forall x \in D \;:\; f(x) \leq M$.
- f is **reduced** on D if : $\exists m \in \mathbb{R}, \; \forall x \in D \;:\; f(x) \geq m$.

- We say that the function f is **bounded** if : $\exists M \in \mathbb{R}, \forall x \in D \ : \ |f(x)| \leq M$.

  That is to say that f is **increased** and **reduced** .

Examples.

1) The function $f(x) = e^x$ is minimized on $D = \mathbb{R}$ by $m = 0$, it is not increased.
2) The function $f(x) = -e^x$ is increased on $D = \mathbb{R}$ by $M = 0$, it is not reduced.
3) The function $f(x) = \cos x$ is bounded on $D = \mathbb{R}$ by 1 et -1 . We have : $\forall x \in \mathbb{R}$, $|\cos x| \leq 1$.

Definitions 6. Let $f: D_f \longrightarrow F_f$ and be $g: D_g \longrightarrow F_g$ two functions such that $D_f \subset D_g$. We define the function composed gof by:

$$gof : \begin{array}{l} D_f \longrightarrow F_g \\ x \longrightarrow g(f(x)) \end{array}$$

Notes :

1) The condition $D_f \subset D_g$ is essential for the image by the function g to $f(x)$ have meaning.
2) It is necessary to pay attention to the order of functions, in general gof and fog are not equal.

4.2. Limits

Definition 7. Let be $f: D \longrightarrow \mathbb{R}$ a real function, $\ell \in \mathbb{R}$ and $x_0 \in D$ where an endpoint of D.

We say that the function f admits a limit ℓ in x_0, if :

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D \ : \ |x - x_0| < \delta \Longrightarrow \ |f(x) - \ell| < \varepsilon$$

We write : $\lim_{x \to x_0} f(x) = \ell$

Explanation: If x is in the neighborhood of x_0 , i.e. $x \in]x_0 - \delta , x_0 + \delta[$, then $f(x)$ is in the neighborhood of ℓ, ie $f(x) \in]\ell - \varepsilon , \ell + \varepsilon[$.

Example. Let the function $f(x) = 7x + 1$ be defined on $D = \mathbb{R}$.

For $x_0 = 1$, we have $\lim_{x \to 1} f(x) = 8$. In fact, we have:

$$|f(x) - 8| < \varepsilon \Leftrightarrow 7|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \frac{\varepsilon}{7}$$

So just take $\delta = \frac{\varepsilon}{3}$, so that the definition of the limit is verified:

$$\forall \varepsilon > 0, \exists \delta = \frac{\varepsilon}{7} > 0, \forall x \in D \ : \ |x - 1| < \delta \Longrightarrow \ |f(x) - 8| < \varepsilon$$

Proposition 1. If f admit a limit at a point, then this limit is unique.

Definitions 8. Let be $f: D \longrightarrow \mathbb{R}$ a real function, $x_0 \in D$ where an endpoint of D.

- We say that the function f tends to $+\infty$ in x_0, if :

$$\forall A > 0, \exists \delta > 0, \forall x \in D : \quad |x - x_0| < \delta \Longrightarrow \quad f(x) > A$$

We write : $\lim_{x \to x_0} f(x) = +\infty$

- We say that the function f tends to $-\infty$ in x_0, if :

$$\forall A > 0, \exists \delta > 0, \forall x \in D \quad : \quad |x - x_0| < \delta \Longrightarrow \quad f(x) < -A$$

We write : $\lim_{x \to x_0} f(x) = -\infty$

Definitions 9. Let be $f : D \longrightarrow \mathbb{R}$ a real function and $\ell \in \mathbb{R}$.

- We say that the function f tends to ℓ in $+\infty$(we write $\lim_{x \to +\infty} f(x) = \ell$), if :

$$\forall \varepsilon > 0, \exists B > 0, \forall x \in D \quad : \quad x > B \Longrightarrow \quad |f(x) - \ell| < \varepsilon$$

- We say that the function f tends to ℓ in $-\infty$ (we write $\lim_{x \to -\infty} f(x) = \ell$), if :

$$\forall \varepsilon > 0, \exists B > 0, \forall x \in D \quad : \quad x < -B \Longrightarrow \quad |f(x) - \ell| < \varepsilon$$

- We say that the function f tends to $+\infty$ in $+\infty$ (we write $\lim_{x \to +\infty} f(x) = +\infty$), if :

$$\forall A > 0, \exists B > 0, \forall x \in D \quad : \quad x > B \Longrightarrow \quad f(x) > A$$

- We say that the function f tends to $-\infty$ in $+\infty$ (we write $\lim_{x \to +\infty} f(x) = -\infty$), if :

$$\forall A > 0, \exists B > 0, \forall x \in D \quad : \quad x > B \Longrightarrow \quad f(x) > -A$$

- We say that the function$f$ tends to$+\infty$ in $-\infty$ (we write $\lim_{x \to -\infty} f(x) = +\infty$), if :

$$\forall A > 0, \exists B > 0, \forall x \in D \quad : \quad x < -B \Longrightarrow \quad f(x) < A$$

- We say that the function$f$ tends towards $-\infty$en $-\infty$ (we write $\lim_{x \to -\infty} f(x) = -\infty$), if :

$$\forall A > 0, \exists B > 0, \forall x \in D \quad : \quad x < -B \Longrightarrow \quad f(x) > -A$$

Examples.

1) For $n \in \mathbb{N}$, we have:

$$\lim_{x \to +\infty} x^n = +\infty \qquad et \qquad \lim_{x \to -\infty} x^n = \begin{cases} +\infty & \text{si} \quad n \text{ est pair} \\ -\infty & \text{si} \quad n \text{ est impair} \end{cases}$$

$$\lim_{x \to +\infty} \frac{1}{x^n} = \lim_{x \to -\infty} \frac{1}{x^n} = 0$$

2) To $n, m \in \mathbb{N}$be $f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$, with $a_n , b_m \in \mathbb{R}_+^*$. We have :

$$\lim_{x \to +\infty} f(x) = \begin{cases} +\infty & \text{si} \quad n > m \\ \dfrac{a_n}{b_m} & \text{si} \quad n = m \\ -\infty & \text{si} \quad n < m \end{cases}$$

Definitions 10.

- We say that the function $f$admits a right limit in x_0, if :

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D \quad : \quad 0 < x - x_0 < \delta \Longrightarrow \quad |f(x) - \ell| < \varepsilon$$

We write: $\lim\limits_{\substack{x \to x_0 \\ >}} f(x) = \ell$ or $\lim\limits_{x \to x_0^+} f(x) = \ell$

- We say that the function f admits a left limit in x_0, if :

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D \ : \ \delta < x - x_0 < 0 \Longrightarrow \ |f(x) - \ell| < \varepsilon$$

We write: $\lim\limits_{\substack{x \to x_0 \\ <}} f(x) = \ell$ or $\lim\limits_{x \to x_0^-} f(x) = \ell$

Proposition 2. We have the following equivalence:

$$\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \lim_{\substack{x \to x_0 \\ >}} f(x) = \lim_{\substack{x \to x_0 \\ <}} f(x) = \ell$$

Noticed :

- To demonstrate that $\lim\limits_{x \to x_0} f(x) \neq \ell$ it is enough to demonstrate that one of the two limits (right or left) is different from ℓ.
- To demonstrate that f there is no limit at the point x_0, it is enough to demonstrate that the two limits (on the right and on the left) are different.

Example : Let the function:

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{si} \quad x > 0 \\ -1 & \text{si} \quad x < 0 \end{cases}$$

We have $\lim\limits_{\substack{x \to 0 \\ <}} f(x) = -1 \neq \lim\limits_{\substack{x \to 0 \\ >}} f(x) = 1$. So f does not admit a limit to the point $x_0 = 0$.

Proposition 3.

Let be $f: D \longrightarrow \mathbb{R}$ a real function, $x_0 \in D$ where one end of D. Then $\lim\limits_{x \to x_0} f(x) = \ell$ iff for every sequence $(x_n)_{n \in \mathbb{N}}$ converges to x_0, we have $(f(x_n))_{n \in \mathbb{N}}$ converged to ℓ. ie .

$$\lim_{x \to x_0} f(x) = \ell \quad \Leftrightarrow \quad \forall (x_n)_{n \in \mathbb{N}} \subset D \ \text{ tel que } \ \lim_{n \to +\infty} x_n = x_0 \ \text{ on a } \ \lim_{n \to +\infty} f(x_n) = \ell$$

Example . The function defined by $f(x) = \cos\frac{1}{x}$ does not admit a limit or point $x_0 = 0$.

Indeed, the following two sequences:

$$x_n = \frac{1}{2n\pi} \quad , \quad t_n = \frac{1}{(2n+1)\pi}$$

Converge towards $x_0 = 0$. On the other hand, we have:

$$\lim_{n \to +\infty} f(x_n) = \lim_{n \to +\infty} \cos(2n\pi) = 1 \neq -1 = \lim_{n \to +\infty} \cos\big((2n+1)\pi\big) = \lim_{n \to +\infty} f(t_n)$$

Proposition 4.

Let f, g two functions be such that $\lim\limits_{x \to x_0} f(x) = A$ and $\lim\limits_{x \to x_0} g(x) = B$, then:

1) $\lim\limits_{x \to x_0} (f(x) + g(x)) = \lim\limits_{x \to x_0} f(x) + \lim\limits_{x \to x_0} g(x) = A + B.$

2) $\lim\limits_{x \to x_0} (f(x) - g(x)) = \lim\limits_{x \to x_0} f(x) - \lim\limits_{x \to x_0} g(x) = A - B.$

3) $\lim_{x\to x_0}(f(x)\times g(x)) = \lim_{x\to x_0} f(x)\times \lim_{x\to x_0} g(x) = A\times B.$

4) $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{A}{B}$, $B\neq 0.$

Theorem 1. (Comparison rule)

- Let f, g two functions be such that : $\forall x\in D, f(x)\leq g(x)$. SO :

$$\lim_{x\to x_0} f(x)\leq \lim_{x\to x_0} g(x).$$

- In addition we have:

$$\lim_{x\to x_0} f(x) = +\infty \implies \lim_{x\to x_0} g(x) = +\infty .$$

Theorem 2. (Squeeze' theorem)

Consider f, g and h three functions , such as:

$$\forall x\in D,\ \ g\leq f\leq h \qquad \text{And} \qquad \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = \ell.$$

So we have : $\lim_{x\to x_0} f(x) = \ell.$

Proposition 5 . (Limit of product)

Let f and g two functions, such as g is bounded and $\lim_{x\to x_0} f(x) = 0$. So :

$$\mathbf{\lim_{x\to x_0} f(x)g(x) = 0}$$

Proposition 6 . (Limit of composition)

Let f and g two functions, such as $\lim_{x\to x_0} f(x) = \ell$ and $\lim_{t\to \ell} g(t) = \text{L}$. So :

$$\mathbf{\lim_{x\to x_0} gof(x) = L}$$

Special limitations:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad , \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$$\lim_{x\to \infty}(1+\frac{1}{x})^x = \text{e} \qquad , \qquad \lim_{x\to 0+}(1+x)^{1/x} = \text{e}$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1 \qquad , \qquad \lim_{x\to 0}\frac{\ln(x+1)}{x} = 1$$