# Solution de la serie TD 2

## Exo1

* $f(t) = c\,x(t)$

$$c = \frac{1}{E_x}\int f(t)\,x(t)\,dt$$

$$E_x = \int_0^1 x^2(t)\,dt = \int_0^1 1\,dt = 1$$

$$c = \frac{1}{1}\int f(t)\,x(t)\,dt = \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}$$

$$c = \frac{1}{2} \Rightarrow f(t) = \frac{1}{2}x(t)$$

* $x(t) = c\,f(t)$

$$c = \frac{1}{E_f}\int_0^1 x(t)\,f(t)\,dt$$

$$E_f = \int_0^1 t^2\,dt = \frac{1}{3}t^3\Big|_0^1 = \frac{1}{3}$$

$$c = \frac{1}{E_f}\int_0^1 t\,dt = \frac{3}{2}t^2\Big|_0^1 = \frac{3}{2}$$

$$\Rightarrow x(t) = \frac{3}{2}f(t) = \frac{3}{2}t$$

## Exo2:

1/ $f(t) = t^2 \qquad -1 \leq t \leq 1$

$$f(t) = a_0 + \sum_{n=1}^{+\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{1}{T_0}\int_{-1}^{1} f(t)\,dt = \frac{1}{2}\int_{-1}^{+1} t^2\,dt = \frac{1}{3}$$

$$a_n = \frac{2}{T_0}\int_{-1}^{+1} f(t)\cos(n\omega_0 t)\,dt = \frac{2}{2}\int_{-1}^{1} t^2\cos(n\omega_0 t)\,dt$$

$$= \frac{4(-1)^n}{\pi^2 n^2} \quad \text{(voir la table)}$$

$$b_n = \frac{2}{T_0}\int_{-1}^{+1} f(t)\sin(n\omega_0 t) = 0 \quad (f \text{ pair})$$

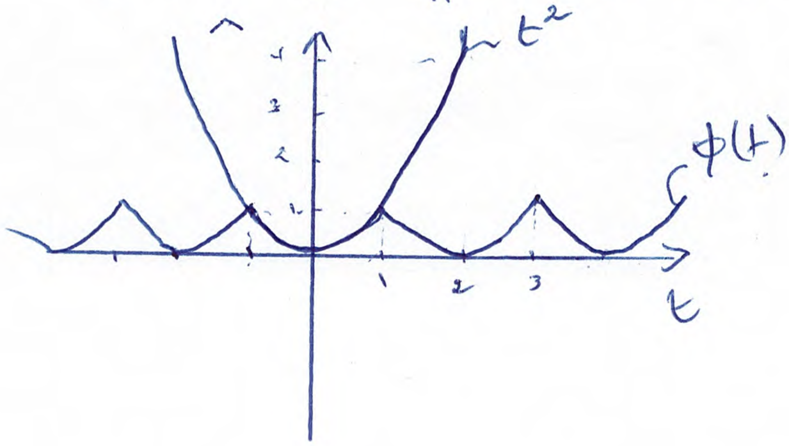
NB :

$$\int x^2 \cos(ax)\,dx = \frac{1}{a^3}\left(2ax\cos ax + 2\sin ax + a^2x^2\sin\right.$$

$$a = n\pi \implies a_n = \frac{4(-1)^n}{\pi^2 n^2}.$$

donc : $$\phi(t) = \frac{1}{3} + \sum_{n=}^{+\infty} \frac{4(-1)^n}{\pi^2 n^2}\cos n\pi t.$$

$$= \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=}^{+\infty}\frac{(-1)^n}{n^2}\cos(n\pi t)$$




Exo3

* la forme compacte.

1/ $f(t) = a_0 + \sum c_n \cos(n\omega_0 t + \theta_n)$

$c_n = \sqrt{a_n^2 + b_n^2}$ $\theta_n = \operatorname{artg}\left(\frac{-b_n}{a_n}\right)$, $a_0 = a_0$.

$T_0 = 4$, $\omega_0 = \frac{\pi}{2}$..

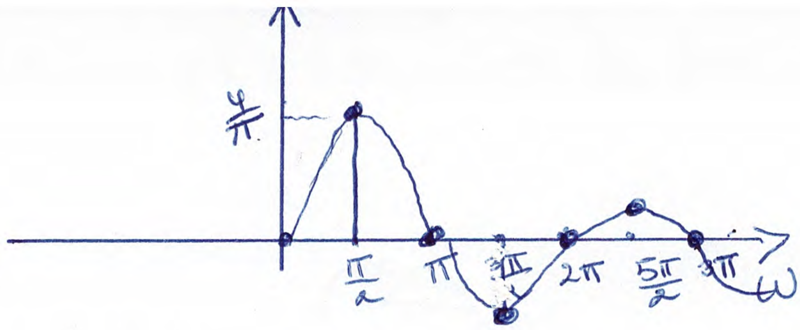
$f(t)$ est paire $\implies b_n = 0.$

$$a_0 = \frac{1}{T_0}\int_0^4 f(t)\,dt = 0.$$

$$a_n = \frac{2}{4}\left[\int_0^1 1\cos\left(n\frac{\pi}{2}t\right)dt + \int_1^3(-1)\cos\left(\frac{n\pi}{2}t\right)dt + \int_3^4 1.\cos\left(\frac{n\pi}{2}t\right)dt\right.$$

$$= \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

$$f(t) = \frac{4}{\pi}\left(\cos\frac{\pi}{2}t - \frac{1}{3}\cos\left(\frac{3}{2}\pi t\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2}t\right) - \frac{1}{7}\cos\left(\frac{7\pi}{2}t\right)\right.$$
$$\left. + \dots\right)$$




b/ $T_0 = 10\pi, \quad \omega_0 = \frac{1}{5}$.

$$f(t) = a_0 + \sum a_n \cos\left(\frac{n}{5}t\right) + b_n \sin\left(\frac{n}{5}t\right)$$

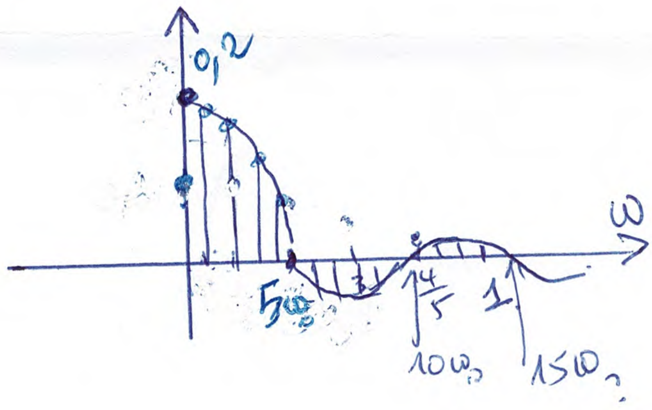
$$a_0 = \frac{1}{T_0}\int_{-5\pi}^{+5\pi} f(t)\,dt = \frac{1}{5}.$$

$$a_n = \frac{2}{10\pi}\int_{-5\pi}^{+5\pi} f(t)\cos\left(\frac{nt}{5}\right)dt = \frac{2}{\pi n}\sin\left(\frac{n\pi}{5}\right)$$

$$b_n = \frac{2}{10\pi}\int_{-\pi}^{\pi} \sin\left(\frac{nt}{5}\right)dt = 0$$

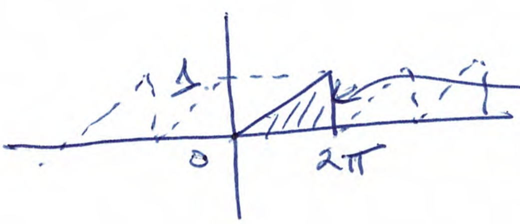
$$f(t) = \frac{1}{5} + \sum_{n=1}^{+\infty} \frac{2}{\pi n}\sin\left(\frac{n\pi}{5}\right)\cos\left(\frac{n}{5}t\right).$$

$$\begin{cases} a_0 = \frac{1}{5} \\ a_1 = \frac{2}{\pi}\sin\left(\frac{\pi}{5}\right) = 0{,}006981 \\ a_2 = \frac{1}{\pi}\sin\left(\frac{2\pi}{5}\right) = 0{,}006980 \\ a_3 = \frac{2}{3\pi}\sin\left(\frac{3\pi}{5}\right) = 0{,}06800 \end{cases}$$




c/ $T_0 = 2\pi, \quad \omega_0 = 1$

$$f(t) = a_0 + \sum_{n=1}^{+\infty} a_n \cos nt + b_n \sin nt,$$ avec $a_0 = 0{,}5$ (voir la figure.)

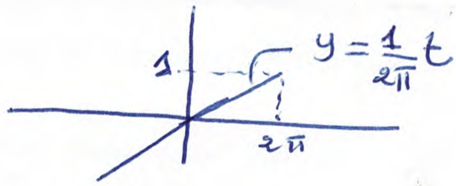



$$\longrightarrow a_0 = \frac{1}{2\pi}\boxed{\int_0^{2\pi} f(t)\,dt} \longrightarrow \text{مساحة المثلث}$$

$$= \frac{1}{2}(2\pi)\cdot 1$$
$$= \pi$$

$$a_0 = \frac{1}{2\pi}\cdot\pi = 0{,}5$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos(nt)\,dt = 0$$

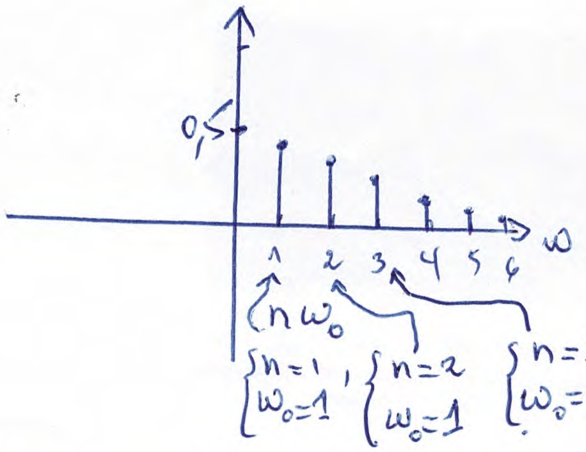


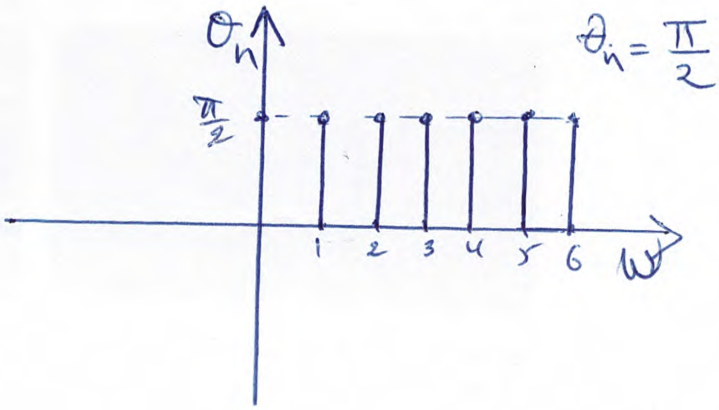

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin(nt) = \frac{-1}{\pi n}.$$

$$f(t) = 0{,}5 - \frac{1}{\pi}\left(\sin t + \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \frac{1}{4}\sin 4t \dots\right)$$

pour la forme compacte: $C_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = tg^{-1}\left(\frac{-b_n}{a_n}\right)$

$$f(t) = 0{,}5 + \frac{1}{\pi}\left(\cos\left(t + \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(2t + \frac{\pi}{2}\right) + \cos\left(3t + \frac{\pi}{2}\right) + \dots\right.$$




d) $T_0 = \pi$, $\omega_0 = 2$, $f(t) = \frac{4}{\pi}t$

$a_n = 0 \rightarrow f$ : impair

$a_0 = 0$

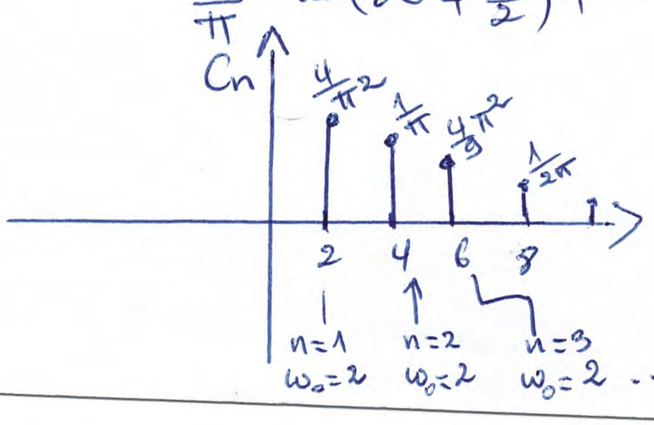
$$b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\sin(2nt)\,dt = \frac{2}{\pi}\times 2\int_0^{\frac{\pi}{2}} f(t)\sin(2nt)\,dt$$

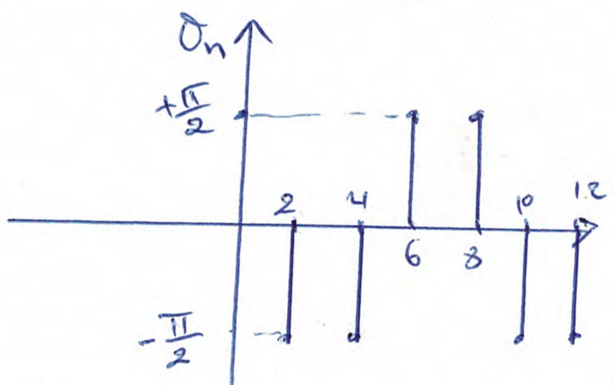
$$= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \frac{4}{\pi}t\sin(2nt)\,dt = \frac{2}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

$$f(t) = \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t + \frac{1}{2\pi}\sin 8t + \dots$$

$$= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right) +$$

$$\frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right) + \dots$$



* Pour les même signaux, on calcule les Series de Fourier exponentiel

a/ $T_0 = 4,\ \omega_0 = \frac{\pi}{2}$ et $D_0 = 0$

$$D_n = \frac{1}{2\pi}\int_{-1}^{+1} e^{-j\left(\frac{n\pi}{2}\right)t}\,dt - \int_{1}^{3} e^{-j\left(\frac{n\pi}{2}\right)t}\,dt = \frac{2}{\pi n}\sin\frac{n\pi}{2} \quad |n|\geq 1$$

$$f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\frac{\pi}{2}t}$$

b/ $T_0 = 10\pi,\ \omega_0 = \frac{1}{5}$, $f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{j\frac{n}{5}t}$

$$D_n = \frac{1}{10\pi}\int_{-\pi}^{+\pi} e^{-j\frac{n}{5}t}\,dt = \frac{1}{\pi n}\sin\left(\frac{n\pi}{5}\right), \quad D_0 = C_0 = \frac{1}{5}$$

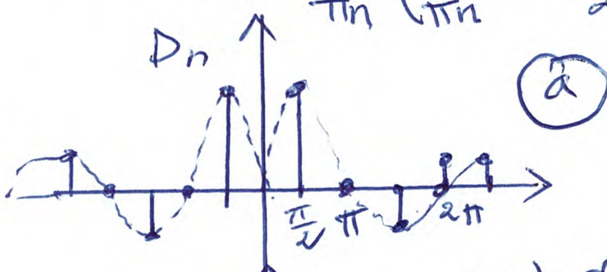
c/ $D_0 = 0{,}5$

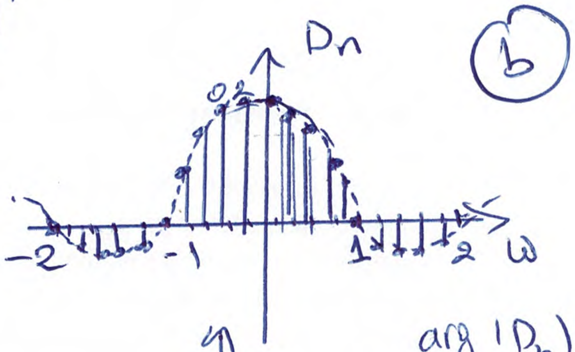
$$D_n = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{t}{2\pi} e^{-jnt}\,dt = \frac{j}{2\pi n}, \quad |D_n| = \frac{1}{2\pi n} \text{ et } \arg(D_n) = \begin{cases} \frac{\pi}{2} & n>0 \\ -\frac{\pi}{2} & n<0 \end{cases}$$

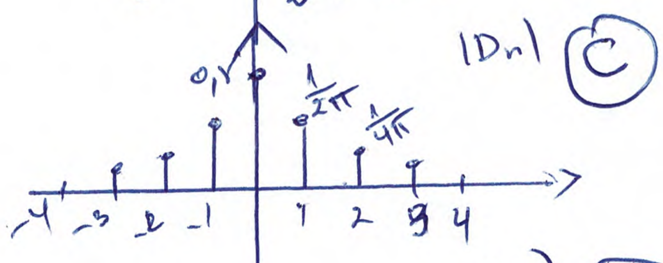
d/ $T_0 = \pi,\ \omega_0 = 2$ et $D_0 = 0$

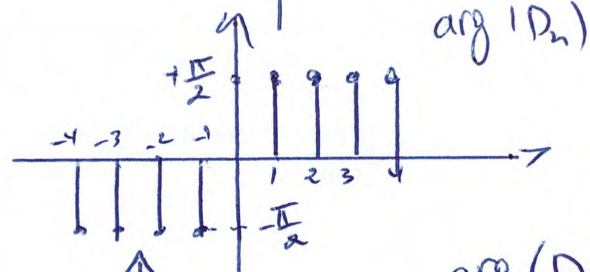
$$f(t) = \sum_{-\infty}^{+\infty} D_n e^{j2nt}, \quad D_n = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \frac{4t}{\pi} e^{-j2nt}\,dt$$

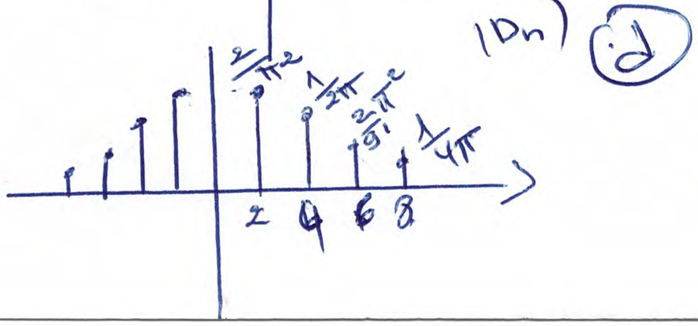
$$D_n = \frac{-j}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

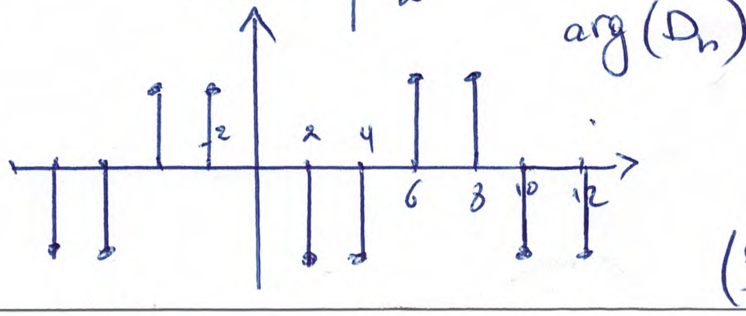


## Exo 4

$T_0 = \frac{\pi}{2}$ ; $\omega_0 = \frac{2\pi}{T_0} = 4$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 4nt + b_n \sin nt$$

$$a_0 = \frac{2}{T}\int_0^{\frac{\pi}{2}} e^{-t}\, dt = 0{,}504$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} e^{-t} \cos(4nt)\, dt = 0{,}504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} e^{-t} \sin(4nt)\, dt = 0{,}504\left(\frac{8n}{1+16n^2}\right)$$

$C_0 = a_0 = 0{,}504$ , $C_n = \sqrt{a_n^2 + b_n^2} = 0{,}504\left(\frac{2}{\sqrt{1+16n^2}}\right)$ , $\theta_n = -tg^{-1}(4n)$

la SF de cette fct est identique à celle de $f(t) = e^{-t/2}$ (voir le cour) avec $t$ remplacée par $2t$.

## Exo 5

$$f(t) = 3 + \sqrt{3}\cos 2t + \sin 2t + \sin 3t - \frac{1}{2}\cos\left(5t + \frac{\pi}{3}\right)$$

la forme compacte : toute les terme sont des cos et les amplitudes $> 0$. $f(t)$ peut être écrite :

$$f(t) = 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$$

* $\sqrt{3}\cos(2t) + \sin 2t = c\cos(2t + \theta)$

$a = \sqrt{(\sqrt{3})^2 + 1} = 2$ , $\theta = tg^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6}$.

* $\sin 3t = \cos\left(3t - \frac{\pi}{2}\right)$
* $-\frac{1}{2}\cos\left(5t + \frac{\pi}{3}\right) = +\frac{1}{2}\cos\left(5t + \frac{\pi}{3} - \pi\right) = \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$

$C_0 = 3 \rightarrow n = 0$

$C_1 = 0 \rightarrow n = 1$

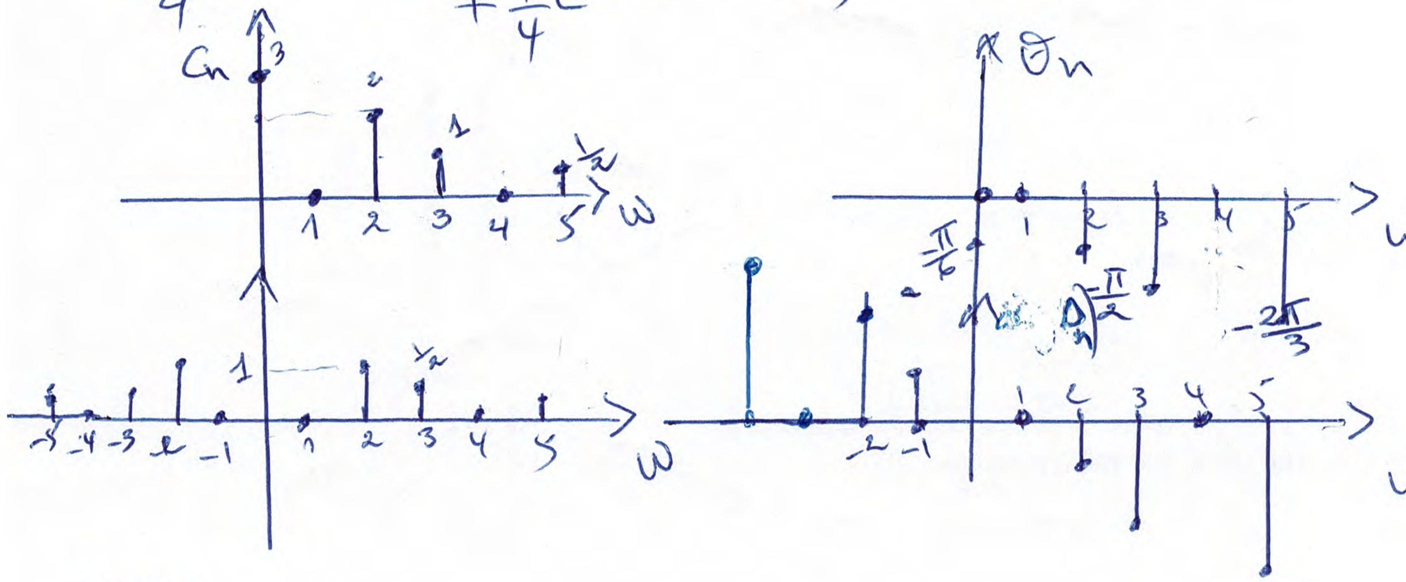
$\left\{\begin{array}{l} C_2 = 2 \\ \theta_2 = \frac{-\pi}{6} \end{array}\right\} n = 2$

$\left\{\begin{array}{l} C_3 = 1 \\ \theta_3 = \frac{-\pi}{2} \end{array}\right\} n = 3$

$C_4 = 0 \rightarrow n = 4$

$\left\{\begin{array}{l} C_5 = \frac{1}{2} \\ \theta_5 = -\frac{2\pi}{3} \end{array}\right\} n = 5$

$$f(t) = 3 + e^{j(2t-\frac{\pi}{6})} + e^{-j(2t-\frac{\pi}{6})} + \frac{1}{2}e^{j(3t-\frac{\pi}{2})} + \frac{1}{2}e^{-j(3t-\frac{\pi}{3})} + \frac{1}{4}e^{j(5t-\frac{2\pi}{3})} + \frac{1}{4}e^{-j(5t-\frac{2\pi}{3})}$$




<u>Exo 6 :</u>

1. La TF des signaux :

a) $$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt = \int_0^T e^{-at}e^{-j\omega t}\,dt$$

$$= \frac{1 - e^{-(j\omega + a)T}}{j\omega + a}.$$

b) $$F(\omega) = \int_0^T e^{at}e^{-j\omega t}\,dt = \frac{1 - e^{-(j\omega - a)T}}{j\omega - a}.$$

c) $$F(\omega) = \int_0^1 4e^{-j\omega t}\,dt + \int_1^2 2e^{-j\omega t}\,dt = \frac{4 - 2e^{-j\omega} - 2e^{-j2\omega}}{j\omega}$$

d) $$F(\omega) = 2\int_0^\tau \frac{t}{\tau}\cos(\omega t)\,dt$$

$f(t)$ est pair $\Longleftrightarrow$ $F(\omega) = 2\int_0^{+\infty} f(t)\cos(\omega t)\,dt.$

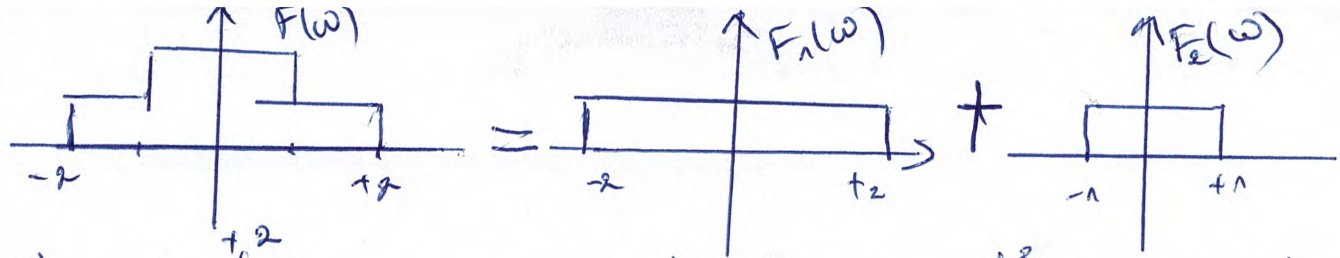
$$F(\omega) = \frac{2}{\tau\omega^2}\left[\cos\omega t + \omega\tau\sin\omega\tau - 1\right].$$

2/ La $TF^{-1}$ des signaux

$$f(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \omega^2 e^{j\omega t}\,d\omega = \frac{1}{2\pi}\frac{e^{j\omega t}}{(jt)^3}\left[-\omega^2t^2 - 2j\omega t + 2\right]_{-\omega_0}^{+\omega_0}$$

$$= \frac{(\omega_0^2t^2 - 2)\sin\omega_0 t + 2\omega_0 t\cos\omega_0 t}{\pi t^3}$$

b)




$$f(t) = \frac{1}{2\pi}\int_{-2}^{+2}\left[F_1(\omega)+F_2(\omega)\right]e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left\{\int_{-2}^{+2} e^{j\omega t}\,d\omega + \int_{-1}^{+1} e^{j\omega t}\,d\omega\right\}$$

$$= \frac{\sin 2t + \sin t}{\pi t}$$

c) $$f(t) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\omega\, e^{j\omega t}\,d\omega = \frac{1}{\pi(1-t^2)}\cos\left(\frac{\pi t}{2}\right).$$

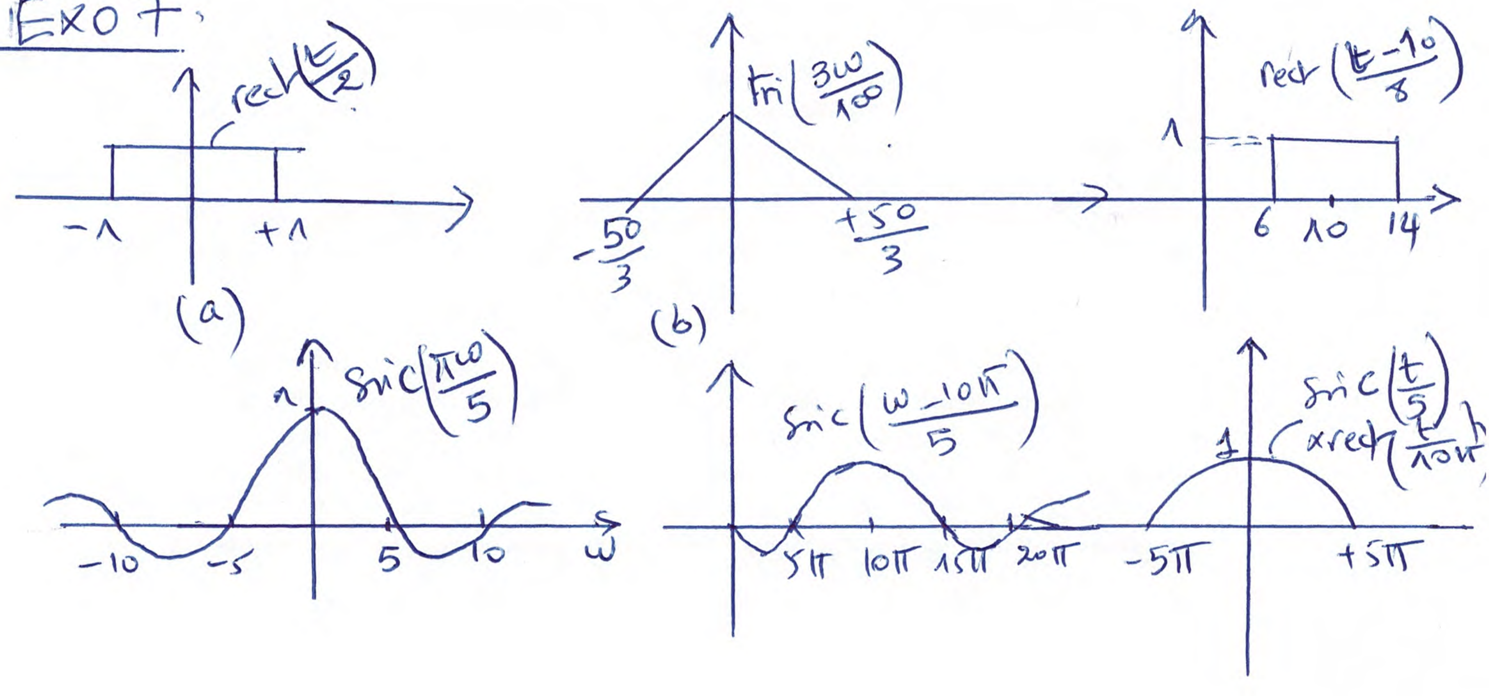
d) $$f(t) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} F(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} F(\omega)\cos\omega t\,d\omega + \underbrace{j\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} F(\omega)\sin(\omega t)\,d\omega}_{=0}\right]$$

$F(\omega)$ est paire le $2^{\text{ème}}$ terme $= 0$

$$f(t) = \frac{1}{\pi}\int_0^{\omega_0}\frac{\omega}{\omega_0}\cos t\omega\,d\omega = \frac{1}{\pi\omega_0}\left[\frac{\cos t\omega + t\omega\sin t\omega}{t^2}\right]_0^{\omega_0}$$

$$f(t) = \frac{1}{\pi\omega_0 t^2}\left[\cos\omega_0 t + \omega_0 t\sin\omega_0 t - 1\right].$$

## Exo 7.



EXO8 :

b) $f_1(t)=f(-t)$ et $F_1(\omega)=F(-\omega)=\frac{1}{\omega^2}\left[e^{-j\omega}+j\omega e^{-j\omega}-1\right]$

c) $f_2(t)=f(t-1)+f_1(t-1)$

$$F_3(\omega)=\left[F(\omega)+F_1(\omega)\right]e^{-j\omega}=\left[F(\omega)+F(-\omega)\right]e^{-j\omega}$$

$$=\frac{2e^{-j\omega}}{\omega^2}\left(\cos\omega+\omega\sin\omega-1\right)$$

d) $f_4(t)=f(t-1)+f(t+1)$

$$F_4(\omega)=F(\omega)e^{-j\omega}+F(-\omega)e^{j\omega}$$

$$=\frac{1}{\omega^2}\left[2-2\cos\omega\right]=\frac{4}{\omega^2}\sin^2\frac{\omega}{2}=\text{sinc}^2\left(\frac{\omega}{2}\right).$$

e) $f_5(t)=f(t-\frac{1}{2})+f_1(t+\frac{1}{2})$

$$F_5(\omega)=F(\omega)e^{-j\frac{\omega}{2}}+F_1(\omega)e^{j\frac{\omega}{2}}.$$

$$=\frac{e^{-j\frac{\omega}{2}}}{\omega^2}\left[e^{j\omega}-j\omega e^{j\omega}-1\right]+\frac{e^{j\frac{\omega}{2}}}{\omega^2}\left[e^{-j\omega}+j\omega e^{-j\omega}-1\right]$$

$$=\frac{1}{\omega^2}\left[2\omega\sin\frac{\omega}{2}\right]=\text{sinc}\,\frac{\omega}{2}.$$

هام

• في حالة $b_n=0$ يمكن السماح ل $C_n$ أخذ قيم سالبة
وتمثيل طيف السعات فقط دون $\theta_n$.

• في EXO1 (a) و (b) مثلنا فقط $C_n$ دون $\theta_n$ مع العلم
أن $C_n$ أخذت قيم سالبة مثلا:

a) $f(t)=\frac{4}{\pi}\left(\cos\frac{\pi}{2}t-\frac{1}{3}\cos\left(\frac{3}{2}\pi t\right)+\frac{1}{5}\cos\left(\frac{5\pi}{2}t\right)-\frac{1}{7}\cos\left(\frac{7\pi}{2}\right)+\dots\right.$

$$=\frac{4}{\pi}\left(\cos\frac{\pi}{2}t\right)+\frac{1}{3}\cos\left(\frac{3}{2}\pi t-\pi\right)+\frac{1}{5}\cos\left(\frac{5\pi t}{2}\right)+\frac{1}{7}\cos\left(\frac{7\pi}{2}-\pi\right)$$

$$\theta_n=\begin{cases}0 & n\neq 3,5,7,11\dots \\ -\pi & n=3,5,7,11\dots\end{cases}\qquad C_n=\begin{cases}0 & n \text{ pair} \\ \frac{4}{n\pi} & n=1,5,9,\dots \\ \frac{-4}{n\pi} & n=3,7,11,\dots\end{cases}$$

أي أن