Solution du TD n° 3

Exo1 : $I = \int_C z^2\,dz = \int_C (x^2 - y^2 + i\,2xy)\,dz$

$$= \int_C (x^2-y^2)\,dx - 2xy\,dy + i\int_C (x^2-y^2)\,dy + 2xy\,dx.$$

l'equation de la droite qui passe par o et $2+i$ est

$y = \frac{1}{2}x$.

$$I = \int_0^2 (x^2-y^2)\,dx - \int_0^1 2xy\,dy + i\int_0^1 (x^2-y^2)\,dy + i\int_0^2 2xy\,dx.$$

$$= \int_0^2 \left(x^2 - \frac{1}{4}x^2\right)dx - 2\int_0^1 2y^2\,dy + i\int_0^1 (2y^2 - y^3)\,dy + i\int_0^2 x^2\,dx$$

$$= \frac{3}{4}\int_0^2 x^2\,dx - 4\int_0^1 y^2\,dy + \frac{i}{3}y^3\Big|_0^1 + \frac{i}{3}x^3\Big|_0^2$$

$$= \frac{1}{4}x^3\Big|_0^2 - \frac{4}{3}y^3\Big|_0^1 + \frac{i}{3} + \frac{i8}{3}$$

$$= 2 - \frac{4}{3} + i\frac{9}{3} = \frac{6-4}{3} + i3 = \boxed{\frac{2}{3} + i3.}$$

Exo2 :

1- $\int_{|z|=1} z\cdot\bar{z}\,dz = \int_{-\pi}^{\pi} e^{it}\cdot e^{-it} \times i e^{it}\,dt$ (sur $|z|=1$: $z(t) = e^{it}$, $-\pi \leq t \leq \pi$).

$$= e^{it}\Big|_{-\pi}^{\pi} = 0.$$

2- $\int_{1+i}^{-1-i} (2z+1) = \frac{1}{4}(2z+1)^2\Big|_{1+i}^{-1-i} = \frac{1}{4}\left\{(-2-2i+1)^2 - (2+2i+1)^2\right\}$

$$= \frac{1}{4}(-8-8i) = -2(1+i).$$

3- $\int_0^{1+i} z^3\,dz = \frac{1}{4}z^4\Big|_0^{1+i} = -1$ / (4) $\int_C \frac{dz}{\sqrt{z}}$, $\sqrt{z} = r^{1/2}\left[\cos\frac{\theta + 2k\pi}{2} + i\sin\frac{\theta+2k\pi}{2}\right]$

$$\sqrt{1} = \cos\frac{2k\pi}{2} + i\sin\frac{2k\pi}{2} \ , \quad k=\overline{0,1}$$

$$= \begin{cases} 1 & k=0 \\ -1 & k=1 \end{cases}$$

on choisit donc la branche $k=0$.

$$\int_C \frac{dz}{\sqrt{z}} = \int_0^{\pi} \frac{ie^{it}\,dt}{e^{it/2}} = \int_0^{\pi} ie^{it/2}\,dt = 2e^{it/2}\Big|_0^{\pi}$$

$$= 2\left(\sqrt{e^{i\pi}} - \sqrt{e^{i0}}\right)$$

$$= 2\left(e^{i\pi/2} - 1\right) = 2(i-1).$$

b) $|z|=1$, $\operatorname{Re} z \geq 0$ $\qquad z = e^{it} \quad -\frac{\pi}{2} \leq t \leq \pi/2$

$$\int_C \frac{dz}{\sqrt{z}} = \int_{-\pi/2}^{\pi/2} ie^{it/2}\,dt = 2\sqrt{e^{it}}\Big|_{-\pi/2}^{\pi/2} = 2\left(\sqrt{i} - \sqrt{-i}\right)$$

$$\sqrt{-i} = \frac{\sqrt{2}}{2}(1-i).$$

$$\sqrt{-i} = \cos\left(\frac{-\pi/2+2k\pi}{2}\right) + i\sin\frac{-\pi/2+2k\pi}{2} \ , \quad k=0,1$$

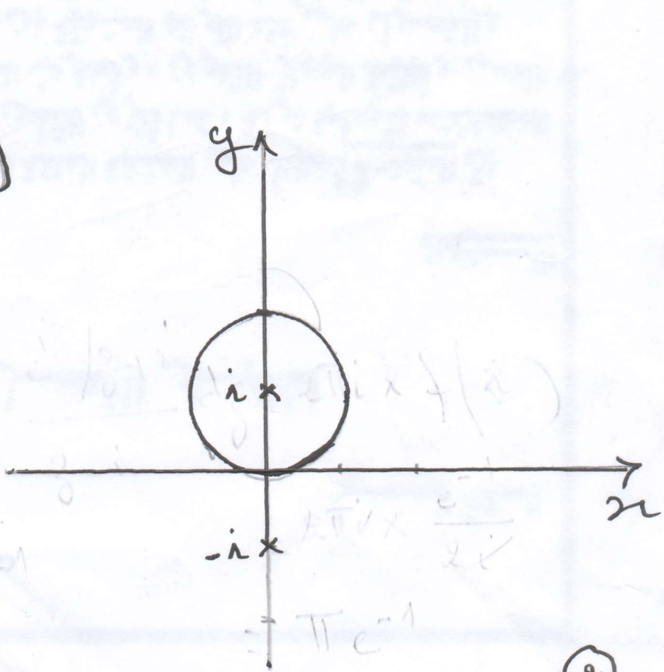
Pour que $\sqrt{-i} = \frac{\sqrt{2}}{2}(1-i)$, on choisit la branche pour $k=0$

Donc

$$\int_C \frac{dz}{\sqrt{z}} = 2\left\{\frac{\sqrt{2}}{2}(1+i) - \frac{\sqrt{2}}{2}(1-i)\right\}^2$$

$$= 2\sqrt{2}.$$

Exo 3:



1) $$\int_{|z-i|=1} \frac{e^{iz}}{z^2+1}\,dz = \int_{|z-i|=1} \frac{e^{iz}}{(z-i)(z+i)}\,dz$$

$$= \int_{|z-i|=1} \frac{e^{iz}/z+i}{z-i}$$

$$= 2\pi i\, f(i) \;/\; f(z) = e^{iz}/z+i$$

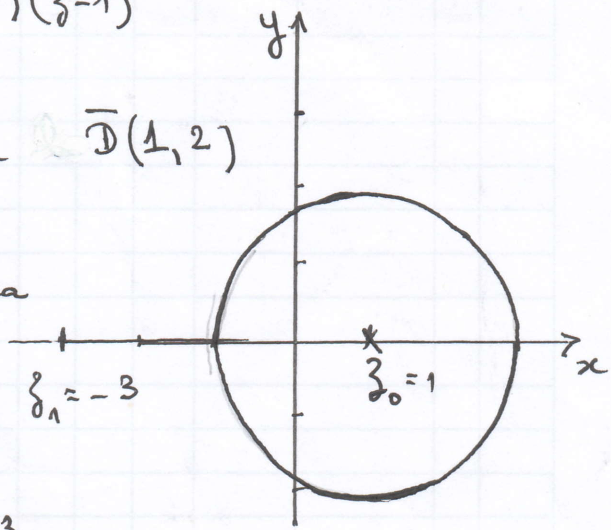
$$= 2\pi i \times \frac{e^{-1}}{2i}$$

$$= \pi e^{-1}.$$

$$\int_{|z-i|=1} \frac{e^{iz}}{z^2+1}\,dz = \pi e^{-1}$$

2) $$\int_{|z-1|=2} \frac{\sin\frac{\pi z}{2}}{z^2+2z-3}\,dz = \int_{|z-1|=2} \frac{\sin\frac{\pi z}{2}}{(z+3)(z-1)}\,dz$$

$z \mapsto \frac{\sin \pi z/2}{(z+3)(z-1)}$ est holomorphe sur $\overline{D}(1,2)$ sauf au point $z_0 = 1$, D'après la formule intégrale de Cauchy



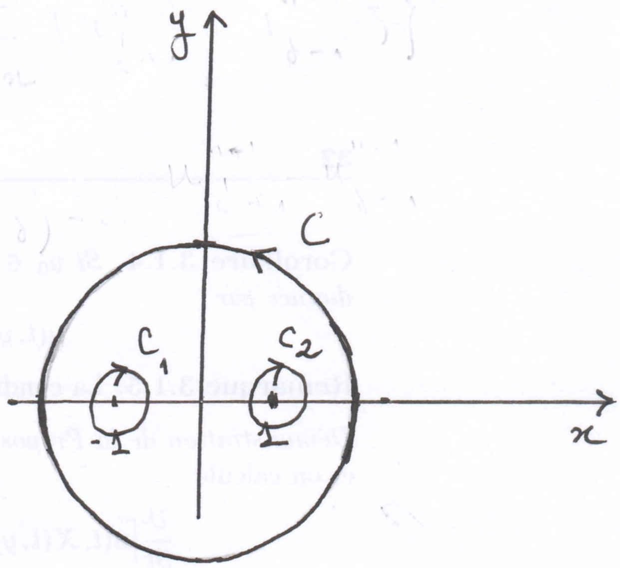
$$\int_{|z-1|=2} \frac{\sin \pi z/2}{z^2+2z-3}\,dz = \int_{|z-1|=2} \frac{\sin\frac{\pi z}{2} / z+3}{z-1}\,dz$$

$$= 2\pi i \times f(1) \;|\; f(z) = \frac{\sin\frac{\pi z}{2}}{z+3}$$

$$= 2\pi i \times \frac{\sin\frac{\pi}{2}}{4} = i\,\frac{\pi}{2}.$$

$$\int_{|z-1|=2} \frac{\sin \pi z/2}{z^2+2z-3}\,dz = i\,\pi/2.$$

3) $\displaystyle\int_{|z|=2} \frac{\cosh z}{(z+1)^2(z-1)}\, dz$



On pose $\Gamma = C \cup (-C_1) \cup (-C_2)$.

$z \mapsto \dfrac{\cosh z}{(z+1)^2(z-1)}$ est holomorphe sur Γ et $\mathring{\Gamma}$ et donc d'après cauchy

$$\int_\Gamma \frac{\cosh z}{(z+1)^2(z-1)}\, dz = 0.$$

d'où

$$\int_C \frac{\cosh z}{(z+1)^2(z-1)}\, dz = \int_{C_1} \frac{\cosh z}{(z+1)^2(z-1)}\, dz + \int_{C_2} \frac{\cosh z}{(z+1)^2(z-1)}\, dz$$

$$= \frac{2\pi i}{1!} \times f_1'(-1) + 2\pi i\, f_2(1)$$

où $f_1(z) = \dfrac{\cosh z}{z-1}$, $f_2(z) = \dfrac{\cosh z}{(z+1)^2}$

$$f_1'(z) = \frac{\operatorname{sh} z \times (z-1) - \cosh z}{(z-1)^2} \Rightarrow f_1'(-1) = \frac{2\operatorname{sh} 1 - \cosh 1}{4}$$

$$f_2(1) = \frac{\cosh 1}{4}$$

$$\int_{|z|=2} \frac{\cosh z}{(z+1)^2(z-1)}\, dz = 2\pi i \left\{ \frac{2\operatorname{sh} 1 - \cosh 1}{4} + \frac{\cosh 1}{4} \right\}$$

$$= \pi i \operatorname{sh} 1$$

$$= \int_{|z|=1} \frac{dz / i\cancel{z}}{\dfrac{z^2+3iz-1}{i\cancel{z}}} = \int_{|z|=1} \frac{dz}{\left(z+i\frac{3+\sqrt5}{2}\right)\left(z-i\frac{\sqrt5-3}{2}\right)}$$

La fonction $z \longmapsto \dfrac{1}{\left(z+i\frac{3+\sqrt5}{2}\right)\left(z-i\frac{\sqrt5-3}{2}\right)}$ est holomorphe sur $C(0,1)$ et $\mathring{C}(0,1)$ sauf au point $z_1 = i\frac{\sqrt5-3}{2} \in \mathring{C}(0,1)$ $\left(z_0 = -i\frac{3+\sqrt5}{2} \notin C(0,1)\right)$ et donc d'après la formule intégrale de Cauchy

$$\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = \int_{|z|=1} \frac{dz / z+i\frac{3+\sqrt5}{2}}{\left(z-i\frac{\sqrt5-3}{2}\right)}$$

$$= 2\pi i \, f\left(i\frac{\sqrt5-3}{2}\right) \Big| \, f(z) = \frac{1}{z+i\frac{3+\sqrt5}{2}}$$

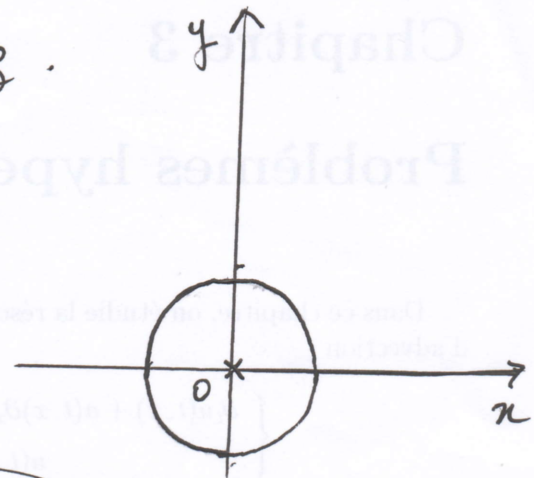
où f est une fct holomorphe au point $z_1 = i\frac{\sqrt5-3}{2}$.

$$f\left(i\frac{\sqrt5-3}{2}\right) = \frac{1}{i\frac{\sqrt5-3}{2}+i\frac{\sqrt5-3}{2}} = \frac{1}{i\sqrt5} = \frac{\sqrt5}{i5}$$

d'où $$\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = 2\pi\cancel{i}\,\frac{\sqrt5}{\cancel{i}5} = 2\pi\frac{\sqrt5}{5}.$$

$$\int_{|z|=2} \frac{\cosh z}{(z+1)^2(z-1)}\,dz = \pi i \,\mathrm{Sh}\, 1$$

4- $\displaystyle\int_{|z|=1} \frac{\cos z}{z^3}\,dz = \frac{2\pi i}{2!} f''(0) \;\Big|\; f(z) = \cos z.$



$f'(z) = -\sin z.$

$f''(z) = -\cos z.$

$$\int_{|z|=1} \frac{\cos z}{z^3}\,dz = \pi i \times (-\cos 0) = -\pi i$$

<u>Exo 4</u> : $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = 2\pi \frac{\sqrt{5}}{5}$

posons $z = e^{i\theta} \Rightarrow dz = i e^{i\theta}\, d\theta.$ et $|z| = 1$

$\Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}.$

on a $\sin\theta = \dfrac{z - \bar{z}}{2i} = \dfrac{1}{2i}\left(z - \tfrac{1}{z}\right) = \dfrac{1}{2iz}(z^2-1).$

Alors, $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = \int_{|z|=1} \frac{dz/iz}{3 + \frac{2}{2iz}(z^2-1)}$