Cours de Theorie des champs.

M1_ QFT
Physique théorique.

Théorème de Gauss.

$$\oint_{\Sigma} A_\mu \, dS^\mu = \int_{\Omega} \partial^\mu A_\mu \, d\Omega .$$

L'intégrale fermée, sur une hypersurface $\{dS^\mu\}$ enveloppants $d\Omega$, d'un vecteur A_μ est l'integrale divergence $(\partial^\mu A_\mu)$ étendu à l'élément $d\Omega$.

la forme covariante des éqs du champ de Maxwell

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad ; \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu .$$

Le Tenseur dual est défini par:

$$F^{*\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} . \quad , \quad F(\vec{E}, \vec{B}) \longleftrightarrow F^*(\vec{B}, -\vec{E})$$

- Les invariants du champ sont:

- <u>scalaire</u>: $F_{\mu\nu} F^{\mu\nu} = - F^*_{\mu\nu} F^{*\mu\nu} = -2(\vec{E}^2 - \vec{B}^2)$
- <u>pseudo scalaire</u>: $F_{\mu\nu} \tilde{F}^{\mu\nu} = -4 \vec{E} . \vec{B}$.

$$\partial_\mu F^{\mu\nu} = J^\nu \quad , \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 .$$

l'équation de continuité: $\partial_\mu J^\mu = 0$.

généralisation du principe d'action aux champs :

Soit l'action $S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$

$L(q, \dot{q}, t) = T - V$.

$T \equiv$ l'energie cinétique , $V \equiv$ l'energie potentielle .

Le principe variationnel implique : $\delta S = 0$.

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0 \quad \Rightarrow \int_{t_1}^{t_2} \delta L(q, \dot{q}, t)\, dt = 0.$$

$$\delta \mathcal{L}(q_i, \dot{q}_i, t) = \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta \dot{q}_i + \cancel{\frac{\partial L}{\partial t}\delta t}^{\,0}$$

Alors : $$\int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right] dt = 0.$$

donc $\delta \dot{q}_i = \delta\left(\frac{d}{dt} q_i\right) = \frac{d}{dt}\delta q_i$.

$$\int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q_i}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial}{\partial t}\delta q_i \right] dt = 0.$$

on a : $$\frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\delta q_i$$

$$\Rightarrow \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i}\delta q - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\delta q \right] dt + \cancel{\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i\right) dt}^{\,0} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \right] \delta q\, dt = 0.$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$ qui est l'equation d'Euler-Lagrange

$$L(t) = \int_V d^3x\, \mathcal{L}(\text{champ})$$

densité lagrangienne à $(t, \vec{x}) \equiv x^\mu$

Lagrangien du champ à l'instant t.

champ $\equiv \varphi(x^\mu)$, ou bien φ.

$$\mathcal{L} = \mathcal{L}(\varphi(x), \partial_\mu \varphi(x) ; x).$$

Dans le cas générale. $q \longrightarrow \varphi$.

nous utilisons La densité lagrangienne $\mathcal{L}$ au lieu du lagrangien L.

$$L(t) = \int_V d^3x\, \mathcal{L}(\varphi, \partial_\mu \varphi ; x).$$

nous avons : $S = \int_{t_1}^{t_2} L(t)\, dt.$

$$S = \int_{t_1}^{t_2} dt \int_V d^3x\, \mathcal{L}(\varphi, \partial_\mu \varphi ; x).$$

$$S = \int_\Omega d^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi ; x)$$

$d^4x = dt.dv.$ | Ω un domaine 4-dim.

- Les conditions aux bordes :

$$\delta\varphi(\vec{x}, t_1) = \delta\varphi(\vec{x}, t_2) = 0.$$

et que $\varphi(\vec{x}, t) \xrightarrow[\|\vec{x}\| \to \infty]{} 0$

$$\delta S = \int_{\Omega} d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right].$$

or $\delta(\partial_\mu \varphi) = \partial_\mu(\delta\varphi)$.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta\varphi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi .$$

L'intégrale devient :

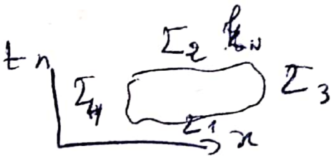
$$\delta S = \int d^4x . \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta\varphi + \cancel{\int_{\Omega} d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right)}$$ riante

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta\varphi = 0 .$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0 \rightarrow$$ les équations de la dynamique du champ $\varphi(x)$.

$$\int_{\Omega} d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) = \int_{\Sigma} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \, ds^\mu$$

↳ Théorème de Gauss.



$$= \int_{\Sigma_1 \cup \Sigma_2} d^3x \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} k_\mu \delta\varphi = 0 .$$

car $\varphi(\Sigma_3) = 0 = \varphi(\Sigma_4)$

$\delta\varphi(t_1) = 0 = \delta\varphi(t_2)$.

Principe d'action pour les équations de Maxwell:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu .$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\gamma} F_{\lambda\gamma} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\gamma}\left(\partial_\lambda A_\gamma - \partial_\gamma A_\lambda\right).$$

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu) - \partial_\mu(\partial^\nu A^\mu).$$

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) \quad \text{—(1)}.$$

nous avons écrit les éqs de Maxwell sous la forme covariante suivante: $\partial_\mu F^{\mu\nu} = j^\nu$ —(2) et $\partial_\mu \tilde{F}^{\mu\nu} = 0$.

d'après (1) et (2) $\Rightarrow$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu .$$

$$\square A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu . \quad \text{—}(*) .$$

avec : $\square = \partial_\mu \partial^\mu$.

cette éq est invariante par transformation de jauge:

$$\partial_\mu A^\mu \Rightarrow \partial_\mu A^\mu + \partial_\mu \partial^\mu \chi .$$

$$\partial^\nu(\partial_\mu A^\mu) \longrightarrow \partial^\nu(\partial_\mu A^\mu) + \partial^\nu \partial_\mu \partial^\mu \chi .$$

l'eq $\circledast$ devient :

$$\partial_\mu \partial^\mu A^\nu + \cancel{\partial_\mu \partial^\mu \partial^\nu \chi} - \partial^\nu(\partial_\mu A^\mu) - \cancel{\partial^\nu \partial_\mu \partial^\mu \chi} = j^\nu .$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu .$$

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu$$

$$j^\mu \longrightarrow j^\mu .$$

d'où l'invariance.

. On peut simplifier l'eq de mouvement en posant $\partial_\mu A^\mu = 0$ "jauge de Lorentz", i.e en fixant la jauge. $\Rightarrow \Box A^\nu = 0$.

. L'autre éq $\partial_\mu \tilde{F}^{\mu\nu} = 0$ donne l'identitée $0 = 0$.

* Cherchons maintenant un lagrangien adéquat à la théorie de Maxwell :

$$\mathcal{L}(A, \partial A, x) = \frac{1}{2}\left[a\, \partial_\mu A^\nu \partial^\mu A_\nu + b\, \partial_\mu A^\nu \partial_\nu A^\mu + c\,(\partial_\mu A^\mu)^2 + d\, A_\mu A^\mu + e\, A_\mu j^\mu \right].$$

c'est une forme quadratique quelconque de A et ∂A, puisque l'eq de mouvement est dérivative du seconde ordre.

$$\mathcal{L} = \frac{1}{2}\left[a\,\partial_\mu A^\nu \partial^\mu A_\nu + b\,\partial_\mu A^\nu \partial_\nu A^\mu + c(\partial_\mu A^\mu)^2 + d\,A_\mu A^\mu + e\,A_\mu J^\mu\right]$$

L'éq de Lagrange s'écrit comme suit :

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} - \partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta A_\alpha)} = 0 \quad —(*)$$

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = \frac{1}{2} d\left[2\,\delta^\alpha_\mu A^\mu\right] + \frac{1}{2} e\,\delta^\alpha_\mu J^\mu$$

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = d\,A^\alpha + \frac{e}{2} J^\alpha \quad —①$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A_\alpha)} = \frac{1}{2}\Big[2a\,\partial_\mu A^\nu \delta^\mu_\beta \delta^\alpha_\nu + b\,\partial^\mu A_\nu\, \delta^\nu_\beta\, \delta^\alpha_\mu + b\,\delta^\mu_\beta \delta^\alpha_\nu\, \partial^\nu A_\mu + 2c.(\partial_\mu A^\mu)\,\delta^\mu_\beta \delta^\alpha_\mu\Big]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A_\alpha)} = a\,\partial_\beta A^\alpha + \frac{b}{2}\partial^\alpha A_\beta + \frac{b}{2}\partial^\alpha A_\beta + c(\partial_\mu A^\mu)\,\delta^\alpha_\beta .$$

$$\partial^\beta\left(\frac{\partial \mathcal{L}}{\partial(\partial^\beta A_\alpha)}\right) = a\,\partial^\beta\partial_\beta A^\alpha + \frac{b}{2}\partial^\beta\partial^\alpha A_\beta + \frac{b}{2}\partial^\beta\partial^\alpha A_\beta + c\,\partial^\beta(\partial_\mu A^\mu)\,\delta^\alpha_\beta .$$

$$\partial^\beta\left(\frac{\partial \mathcal{L}}{\partial(\partial^\beta A_\alpha)}\right) = a\,\partial^\beta\partial_\beta A^\alpha + b\,\partial^\beta\partial^\alpha A_\beta + c\,\partial^\alpha(\partial_\mu A^\mu) \quad —②$$

En remplaçant ① et ② dans $(*)$ ⇒

$$d\,A^\alpha + \frac{e}{2}J^\alpha - a\,\partial^\beta\partial_\beta A^\alpha - b\,\partial^\beta\partial^\alpha A_\beta - c\,\partial^\alpha(\partial_\mu A^\mu) = 0.$$

$$d\,A^\alpha + \frac{e}{2}J^\alpha - a\,\Box A^\alpha - b\,\partial^\beta\partial^\alpha A_\beta - c\,\partial^\alpha(\partial_\mu A^\mu) = 0$$

$$d\,A^\alpha + \frac{e}{2}J^\alpha - a\,\Box A^\alpha - (b+c)\,\partial^\alpha(\partial^\beta A_\beta) = 0 \quad —(\alpha)$$

Nous avons précédement :

$$\Box A^\alpha - \partial^\alpha(\partial^\beta A_\beta) = J^\alpha \cdot \quad (\beta)$$

alors :

$$J^\alpha - \Box A^\alpha + \partial^\alpha(\partial^\beta A_\beta) = 0$$

si en comparant l'éq (α) avec l'éq (β) $\Rightarrow$.

$d = 0$.

$\frac{e}{2} = -1$. $\quad a = -1$. , $\quad (b+c) = +1$.

$e = -2$, $a = -1$, $\quad b + c = +1$.

$$\Rightarrow -J^\alpha + \Box A^\alpha - \partial^\alpha(\partial^\beta A_\beta) = 0.$$

$$\Rightarrow \Box A^\alpha - \partial^\alpha(\partial^\beta A_\beta) = J^\alpha$$

$$\mathcal{L} = \frac{1}{2}\left[-\partial_\mu A^\nu \partial^\mu A_\nu + \partial_\mu A^\nu \partial_\nu A^\mu\right. + C\left[-\partial_\mu A^\nu \partial_\nu A^\mu + (\partial_\mu A^\mu)^2\right]$$

$$-2 A_\mu J^\mu.$$

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - A_\mu J^\mu$$

$$+ \frac{c}{2}\left[(\partial_\mu A^\mu)^2 - \partial_\mu A^\nu \partial_\nu A^\mu\right]$$

Le dernier terme reste indeterminé et il est à une 4- divergence, i.e, il ne change rien dans les équations de mouvement, En effet:

$$(\partial_\mu A^\mu)^2 - \partial_\mu A^\nu \partial_\nu A^\mu = \partial_\mu \left(A_\nu \left(g^{\mu\nu} \partial_\rho A^\rho - \partial^\nu A^\mu \right) \right).$$

Alors on écrit le lagrangien simplement comme:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu .$$

L'action.

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right].$$

$$= \int d^4x \left[\frac{\vec{E}^2 - \vec{B}^2}{2} - \rho A^0 + \vec{J} \cdot \vec{A} \right].$$

Scanné avec CamScanner

Symétries et lois de conservations:

L'invariance de l'action par des transformations implique la conservation d'une quantité physique $\Leftrightarrow$ théorème de Noether

-> La quantité conservée est dite courant de Noether.

-> Le nom "courant" car la conservation est exhibée par une équation de continuité.

-> Il y a deux types de symétries
- -> géométriques (externes).
- -> internes.

① Symétries géométriques

a) Translation et conservation du tenseur Impulsion - Energie.

Considérons un lagrangien $\mathcal{L} = \mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$.

sous l'action d'une translation:

$$x \to x + a.$$

on a:
$$\delta\varphi = \varphi(x+a) - \varphi(x)$$
$$= \varphi(x) + \frac{\partial \varphi(x)}{\partial x^\mu} \delta a^\mu - \varphi(x)$$

$$\delta\varphi = \partial_\mu \varphi(x)\, \delta a^\mu$$

$$\delta(\partial_\mu \varphi) = [\partial_\mu \varphi(x+a) - \partial_\mu \varphi(x)] = \partial_\mu \delta\varphi.$$
$$= \partial_\mu [\partial_\nu \varphi(x)]\, \delta a^\nu + \partial_\nu \varphi(x) (\partial_\mu \delta a^\nu)$$

$\Rightarrow \delta\varphi = \partial_\mu \varphi(x)\, \delta a^\mu$

$\Rightarrow \delta(\partial_\mu \varphi) = \partial_\mu \partial_\nu \varphi(x)\, \delta a^\nu + \partial_\mu(\delta a^\nu)\, \partial_\nu \varphi(x)$

Voyons comment change l'action sous cette transformation.

$$\delta I = \int d^4x\, \delta\mathcal{L} = \int d^4x \left[\mathcal{L}(x+a) - \mathcal{L}(x)\right]$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right)$$

$$\delta I = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi(x)\, \delta a^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \left(\partial_\mu \partial_\nu \varphi(x)\right) \delta a^\nu \right.$$

$$\left. + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta a^\nu)\, \partial_\nu \varphi(x) \right).$$

On sait que.

$$\partial_\nu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \partial_\nu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \partial_\mu \varphi.$$

Alors.

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi = \partial_\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \partial_\nu \varphi.$$

la variation de l'action devient:

$$\delta I = \int d^4x \left(\partial_\mu \mathcal{L}\, \delta a^\mu - \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \partial_\nu \varphi\, \delta a^\mu} \right.$$

$$\left. + \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \left(\partial_\mu \partial_\nu \varphi(x)\right) \delta a^\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta a^\nu)\, \partial_\nu \varphi(x) \right).$$

$$\delta I = \int d^4x \left(\partial_\mu \mathcal{L}\, \delta a^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta a^\nu)\, \partial_\nu \varphi(x) \right).$$

on a : $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu(\delta a^\nu)\partial_\nu \varphi(x) = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta a^\nu \partial_\nu \varphi(x)\right]$$

$$- \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi(x)\right)\delta a^\nu .$$

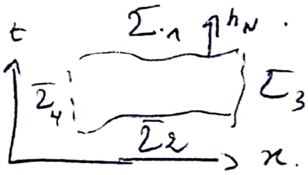
$$\delta I = \int d^4x \left(\partial_\nu \mathcal{L} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi(x)\right)\right) \delta a^\nu$$

$$+ \int d^4x \, \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi(x)\, \delta a^\nu\right].$$

Le dernier terme est une divergence et il s'écrit :

$$\int_\Omega d^4x \, \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi(x)\, \delta a^\nu\right] = \oint_\Sigma dS^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underbrace{\partial_\nu \varphi(x)\delta a^\nu}_{\delta\varphi}$$

$$= \int_{\Sigma_1 \cup \Sigma_2} d^3x \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} n_\mu \, \delta\varphi = 0$$



Car : $\varphi(\Sigma_3) = 0 = \varphi(\Sigma_4)$.
$\delta\varphi(t_1) = 0 = \delta\varphi(t_2)$.

donc ce dernier terme est nul, ce qui donne :

$$\delta I = \int d^4x \left(\partial_\nu \mathcal{L} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi(x)\right)\right) \delta a^\nu .$$

Si, on impose l'invariance de I, i.e. $\delta I = 0$

on obtient l'éq :

$$\partial_\nu \mathcal{L} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi(x)\right) = 0 .$$

qui se réécrit comme : $\partial_\mu T^{\mu\nu} = 0$

$$\partial_\nu \mathcal{L} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi \right) = 0$$

$$\Leftrightarrow \partial_\mu \left(g^{\mu\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi \right) = 0$$

$$\Rightarrow T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}$$ $\to$ non-symétrique !!

↳ c'est le "tenseur impulsion - énergie".

. "L'impulsion - énergie du champ" est le 4-vecteur défini par :

$$p^\nu = \int d^3x \, T^{0\nu}$$

et il est facile de voir que :

$$\frac{dp^\nu}{dt} = \int_V d^3x \, \partial_0 T^{0\nu} , \quad \partial_\mu T^{\mu\nu} = 0 .$$

$$\partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0$$

$$\Rightarrow \partial_0 T^{0\nu} = - \partial_i T^{i\nu} .$$

$$\frac{dp^\nu}{dt} = \int_V d^3x . \partial_0 T^{0\nu} = - \int_V d^3x \, \partial_i T^{i\nu} .$$

et suivant le théorème de Gauss nous pouvons écrire :

frontières $T^{\mu\nu} = 0$.

$$= - \oint_S dS_i \, T^{i\nu} = 0 .$$

↳ . car sur les frontières de V le tenseur $T^{\mu\nu}$ s'annulle.

$\Rightarrow$ D'où la conservation de l'impulsion - énergie du champ.

Remarque : $T^{\mu\nu}$ est par construction non-symétrique.

Ceci est connu sous le nom du théorème de Noether impliquant par l'invariance de l'action, l'eq locale de continuité d'un courant "j^μ" l'intégrale $\int d^3x\, j^0$, $\partial_\mu j^\mu = 0$ génère une conservation de la charge :

$$Q = \int d^3x\, j^0 \quad \text{avec} \quad \dot{Q} = 0.$$

$$Q = \int d^3x\, p^0, \quad \frac{dQ}{dt} = \int d^3x \frac{\partial p^0}{\partial t}.$$

nous avons : $\frac{\partial p^0}{dt} = \int d^3x\, \partial_0 T^{00}$

$$= -\int d^3x\, \partial_i T^{i0}$$

$$= -\oint ds_i\, T^{i0} = 0.$$

$$\frac{dQ}{dt} = 0 \quad \Leftarrow \quad \_\_\_ = 0.$$

Application champ de Maxwell.

Nous avons le Lagrangien de Maxwell :

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu .$$

Le tenseur impulsion - énergie est :

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}.$$

$$\mathcal{L} = -\frac{1}{4} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - J_\mu A^\mu .$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} = -\frac{1}{4} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\rho)} F^{\alpha\beta} - \frac{1}{4} F_{\alpha\beta} \frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\rho)}$$

$$= -\frac{1}{4} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\rho)} F^{\alpha\beta} - \frac{1}{4} g^{\alpha\delta} g^{\alpha\gamma} F_{\alpha\beta} \cdot \frac{\partial F_{\delta\gamma}}{\partial(\partial_\mu A_\rho)} .$$

$$= -\frac{1}{\cancel{4}} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\rho)} F^{\alpha\beta} - \frac{1}{4} F^{\delta\gamma} \frac{\partial F_{\delta\gamma}}{\partial(\partial_\mu A_\rho)} .$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} = -\frac{1}{2} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\rho)} F^{\alpha\beta}$$

$$= -\frac{1}{2} \left[\left(\delta^\mu_\alpha \delta^\rho_\beta - \delta^\mu_\beta \delta^\rho_\alpha \right) \right] F^{\alpha\beta} .$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} = -\frac{1}{2} \left[F^{\mu\rho} - F^{\rho\mu} \right] = F^{\rho\mu} = -F^{\mu\rho} .$$

Puisque : $F^{\mu\rho} = -F^{\rho\mu}$.

$$\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} = F^{\rho\mu}}$$

$$T^{\mu\nu} = -F^{\mu\rho} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L} .$$

$$\boxed{T^{\mu\nu} = -F^{\mu\rho} \partial^\nu A_\rho - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + g^{\mu\nu} J_\alpha A^\alpha}$$

$$T^{\mu\nu'} = -F'^{\mu\lambda}\partial^\nu A'_\lambda + \frac{1}{4} g^{\mu\nu} F'_{\alpha\beta} F'^{\alpha\beta} + g^{\mu\nu} J_\alpha A'^\alpha \quad \ldots (I).$$

avec : $A'_\lambda \longrightarrow A_\lambda + \partial_\lambda \phi$ —(*)

il est facile de montrer que $F'_{\alpha\beta} F'^{\alpha\beta} = F_{\alpha\beta} F^{\alpha\beta}$

(est invariant sous la transformation (*))

$$F'^{\mu\lambda} = (\partial^\mu A'^\lambda - \partial^\lambda A'^\mu)$$

$$= (\partial^\mu A^\lambda - \partial^\lambda A^\mu + \cancel{\partial^\mu \partial^\lambda \phi} - \cancel{\partial^\lambda \partial^\mu \phi})$$

$$F'^{\mu\lambda} = (\partial^\mu A^\lambda - \partial^\lambda A^\mu) \quad ① $$

$$\partial^\nu A'_\lambda = \partial^\nu A_\lambda + \partial^\nu \partial_\lambda \phi \quad ②$$

$$A'^\alpha = A^\alpha + \partial^\alpha \phi \quad \ldots ③$$

$$T'^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda - F^{\mu\lambda}\partial^\nu\partial_\lambda\phi + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} + g^{\mu\nu} J_\alpha A^\alpha + g^{\mu\nu} J_\alpha \partial^\alpha \phi$$

$$T'^{\mu\nu} = T^{\mu\nu} - F^{\mu\lambda}\partial^\nu\partial_\lambda\phi + g^{\mu\nu} J_\alpha \partial^\alpha \phi.$$

$$T'^{\mu\nu} = T^{\mu\nu} + \partial_\lambda (g^{\mu\nu} J^\lambda \phi - F^{\mu\lambda}\partial^\nu \phi) - J^\mu \partial^\nu \phi.$$

en absence de courant extérieur $J^\mu = 0$, la différence est une divergence et ne contribue pas ~~ds~~ dans la valeur de l'impulsion-energie du champ.

le tenseur $T^{\mu\nu}$ est donc dépendant de jauge, En plus il n'est pas symétrique

$$T^{\mu\nu} - T^{\nu\mu} = - F^{\mu\lambda}\partial^{\nu}A_{\lambda} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} + g^{\mu\nu} \partial_{\alpha} A^{\alpha}$$
$$+ F^{\nu\lambda}\partial^{\mu}A_{\lambda} - \frac{1}{4} g^{\nu\mu} F_{\alpha\beta}F^{\alpha\beta} - g^{\mu\nu} \partial_{\alpha} A^{\alpha}$$

$$T^{\mu\nu} - T^{\nu\mu} = + F^{\nu\lambda}\partial^{\mu}A_{\lambda} - F^{\mu\lambda}\partial^{\nu}A_{\lambda}$$

(لازم يكون $T^{\mu\nu}$ symétrique باش نقدرو نطلعو ونحسبو P^{μ} ..)

Alors il serait préférable de rajouter à ce tenseur $T^{\mu\nu}$ une quantité qui soit une divergence $T^{\mu\nu} \longrightarrow T^{\mu\nu} + \Delta T^{\mu\nu}$.

$$\Delta T^{\mu\nu} \overset{\text{déf}}{=} \partial_{\lambda} T^{\mu\lambda,\nu}$$

$$\tilde{T}^{\mu\nu} \longrightarrow T^{\mu\nu} + \Delta T^{\mu\nu}.$$

$$\partial_{\mu}\tilde{T}^{\mu\nu} = 0 = \partial_{\mu}T^{\mu\nu} + \partial_{\mu}\Delta T^{\mu\nu}$$

$$\Rightarrow \partial_{\mu}(\Delta T^{\mu\nu}) = 0$$

$T^{\mu\lambda,\nu}$ est un tenseur antisymétrique en (μ,λ) de manière à avoir :

$$\partial_{\mu}\Delta T^{\mu\nu} = \partial_{\mu}\partial_{\lambda}T^{\mu\lambda,\nu} = 0$$

et en plus $\Delta P^{\nu} = \int d^3x \; \Delta T^{0\nu}$.

$$\Delta T^{0\nu} = \partial_{\lambda}T^{0\lambda,\nu} = \partial_{0}T^{00,\nu} + \partial_{i}T^{0i,\nu}$$

puisque $T^{\mu\lambda,\nu}$ est antisymétrique en (μ,λ).

$$\Rightarrow T^{00,\nu} = 0$$
$$\Rightarrow \partial_{0}T^{00,\nu} = 0.$$

alors.

$$\Delta T^{0\nu} = \partial_{i}T^{0i,\nu}.$$

$$\Delta P^{\nu} = \int_{V} d^3x \; \partial_{i}T^{0i,\nu} = \oint ds_{i}\, T^{0i,\nu} = 0.$$

Scanné avec CamScanner

et on voudrait bien d'avoir en relation avec la jauge, En effet, on a vu que :

$$\Delta_{\text{jauge}} T^{\mu\nu} = -\partial_\lambda (F^{\mu\lambda} \partial^\nu \phi) \quad \hookrightarrow (j^\mu = 0)$$

Ce qui nous pousse à écrire :

$$\Delta T^{\mu\nu} = \partial_\lambda (F^{\mu\lambda} A^\nu)$$

Alors, on définit le tenseur impulsion - energie, par :

$$T^{\mu\nu} = T^{\mu\nu} + \Delta T^{\mu\nu}$$

$$T^{\mu\nu} = g^{\mu\nu} \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + J_\alpha A^\alpha \right) - F^{\mu\lambda} \partial^\nu A_\lambda + \partial_\lambda (F^{\mu\lambda} A^\nu).$$

En utilisant l'éq de mouvement :

$\partial_\lambda F^{\mu\lambda} = -J^\mu$, il se réduit à

$$T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\lambda} \partial^\nu A_\lambda + (\partial_\lambda F^{\mu\lambda}) A^\nu + g^{\mu\nu} J_\alpha A^\alpha + F^{\mu\lambda} \partial_\lambda A^\nu$$

$$T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} (\partial_\lambda A^\nu - \partial^\nu A_\lambda) + g^{\mu\nu} J_\alpha A^\alpha - J^\mu A^\nu \quad \text{--- } (*)$$

En absence de courant : $T^{\mu\nu}$ est
- ① invariant de jauge
- ② conservé
- ③ symétrique
- ④ et de trace nulle

Confirmation

① pour montrer que $T^{\mu\nu}$ est invariant de jauge, en posant en premier lieu $J^\mu = 0$, et on remplace $A'_\mu \rightarrow A_\mu + \partial_\mu \phi$ dans l'eq (*), bien que $F'_{\alpha\beta} F'^{\alpha\beta} = F_{\alpha\beta} F^{\alpha\beta}$

et $F'^{\mu\lambda} F'_\lambda{}^\nu = F^{\mu\lambda} F_\lambda{}^\nu$

Alors :

$$T'^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F'_{\alpha\beta} F'^{\alpha\beta} + F'^{\mu\lambda} F'_\lambda{}^\nu .$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} F_\lambda{}^\nu = T^{\mu\nu}$$

et donc $T^{\mu\nu}$ est invariant de jauge ✓

② pour montrer que $T^{\mu\nu}$ est conservé, il faut montrer que $\partial_\mu T^{\mu\nu} = 0$.

$$\partial_\mu T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} \partial_\mu (F_{\alpha\beta} F^{\alpha\beta}) + \partial_\mu (F^{\mu\lambda} F_\lambda{}^\nu).$$

$$\partial_\mu T^{\mu\nu} = -\frac{1}{2} g^{\mu\nu} \underbrace{\partial_\mu (\vec{E}^2 - \vec{B}^2)}_{=0} + \overbrace{(\partial_\mu F^{\mu\lambda})}^{=0} F_\lambda{}^\nu + F^{\mu\lambda} \partial_\mu F_\lambda{}^\nu .$$

⋮