3) Pour montrer que $T^{\mu\nu}$ est symétrique, on change $\mu \longleftrightarrow \nu$,

$$T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} F_{\lambda}{}^{\nu}.$$

$$T^{\nu\mu} = \frac{1}{4} g^{\nu\mu} F_{\alpha\beta} F^{\alpha\beta} + F^{\nu\lambda} F_{\lambda}{}^{\mu} \qquad \Big/ \quad g^{\mu\nu} = g^{\nu\mu}$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu}{}_{\lambda} F^{\lambda\nu}. \qquad F^{\nu\lambda} = -F^{\lambda\nu}.$$

$$F_{\lambda}{}^{\mu} = -F^{\mu}{}_{\lambda}.$$

et donc :

$$T^{\nu\mu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} F_{\lambda}{}^{\nu} = T^{\mu\nu}$$

et donc symétrique.

4) Pour montrer que $T^{\mu\nu}$ est de trace nulle, il faut montrer que $T^{\mu}{}_{\mu} = 0$.

$$T^{\mu}{}_{\nu} = \frac{1}{4} g^{\mu}{}_{\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} F_{\lambda\nu}.$$

$$tr(T^{\mu}{}_{\nu}) = T^{\mu}{}_{\mu} = \frac{1}{4} \underbrace{g^{\mu}{}_{\mu}}_{4} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} F_{\lambda\mu}.$$

$$= F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} \underbrace{F_{\lambda\mu}}_{= -F_{\mu\lambda}}.$$

$$= F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\lambda} F_{\mu\lambda}.$$

$$T^{\mu}{}_{\mu} = 0$$

et donc $T^{\mu\nu}$ est de trace nulle.

Scanné avec CamScanner

Si j_μ n'est pas nul, on a:

$$\partial_\mu T_0{}^{\mu\nu} = A_\lambda \partial^\nu j^\lambda .$$

qui peut aussi s'écrire:

$$\partial_\mu T_0{}^{\mu\nu} = \partial_\mu \left(\frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} F_\lambda{}^\nu \right) + \partial_\mu \left[g^{\mu\nu} (j_\lambda A^\lambda) - j^\mu A^\nu \right] = A_\lambda \partial^\nu j^\lambda .$$

$$\partial_\mu \left(\frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} F_\lambda{}^\nu \right) = A_\lambda \partial^\nu j^\lambda + \partial_\mu (j^\mu A^\nu) - g^{\mu\nu} \partial_\mu (j_\lambda A^\lambda).$$

$$\partial_\mu \left(\frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\lambda} F_\lambda{}^\nu \right) = A_\lambda \partial^\nu j^\lambda + (\partial_\mu j^\mu) A^\nu + j^\mu \partial_\mu A^\nu - g^{\mu\nu} (\partial_\mu j_\lambda) A^\lambda - g^{\mu\nu} j_\lambda (\partial_\mu A^\lambda)$$

$$\partial_\mu \left(\frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\lambda} F_\lambda{}^\nu \right) = \cancel{A_\lambda \partial^\nu j^\lambda} + (\partial_\mu j^\mu) A^\nu + j^\mu \partial_\mu A^\nu - \cancel{(\partial^\nu j_\lambda) A^\lambda} - j_\lambda (\partial^\nu A^\lambda).$$

$$\begin{aligned}
\partial_\mu \left(\frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\lambda} F_\lambda{}^\nu \right) &= (\partial_\mu j^\mu) A^\nu - j_\lambda (\partial^\nu A^\lambda) + j^\mu \partial_\mu A^\nu \\
&= \underbrace{(\partial_\lambda j^\lambda)}_{=0} A^\nu - j_\lambda (\partial^\nu A^\lambda) + j^\lambda \partial_\lambda A^\nu . \\
&= j^\lambda (\partial_\lambda A^\nu - \partial^\nu A^\lambda)
\end{aligned}$$

$$\boxed{\partial_\mu \left(\frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\lambda} F_\lambda{}^\nu \right) = j^\lambda F_\lambda{}^\nu .}$$

on identifie alors, le tenseur du champ pur à : ($J_\lambda = 0$).

$$T_{(0)}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} F_\lambda{}^\nu .$$

$$T_{(0)}^{00} = \frac{1}{4} g^{00} F_{\alpha\beta} F^{\alpha\beta} + F^{0\lambda} F_\lambda{}^0 .$$

$$= \frac{1}{4} \times -2(\vec{E}^2 - \vec{B}^2) + F^{01} F_1{}^0 + F^{02} F_2{}^0 + F^{03} F_3{}^0 .$$

$$= -\frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \vec{E}^2 .$$

$$T_{(0)}^{00} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$ densité d'énergie du champ.

et

$$T^{i0}{}_{(0)} = \frac{1}{4} \underbrace{g^{i0}}_{=0} F_{\alpha\beta} F^{\alpha\beta} + F^{i\lambda} F_\lambda{}^0 .$$

$$= \underbrace{F^{i0} F_0{}^0}_{=0} + F^{i1} F_1{}^0 + F^{i2} F_2{}^0 + F^{i3} F_3{}^0 .$$

$$= F^{01} F_1{}^0 + F^{21} F_1{}^0 + F^{31} F_1{}^0$$
$$+ F^{02} F_2{}^0 + F^{12} F_2{}^0 + F^{32} F_2{}^0$$
$$+ F^{03} F_3{}^0 + F^{13} F_3{}^0 + F^{23} F_3{}^0 .$$

$$T^{i0}{}_{(0)} = (\vec{E} \wedge \vec{B})^i$$ densité de l'impulsion du champ vecteur de Poynting.

Scanné avec CamScanner

et on a : $-\frac{d}{dt}\int d^3x\, T^{00}_{(0)} = ?$

$$\partial_\mu T_{(0)}^{\mu\nu} = A_\lambda \partial^\nu j^\lambda.$$

$$\partial_0 T_{(0)}^{0\nu} = A_\lambda \partial^\nu j^\lambda - \partial_i T_{(0)}^{i\nu}.$$

$$\partial_0 T_{(0)}^{00} = A_\lambda \partial^0 j^\lambda - \partial_i T_{(0)}^{i0}.$$

alors :

$$-\frac{d}{dt}\int d^3x\, T^{00}{}_{(0)} = \int d^3x\, \partial_i T_{(0)}^{i0} - \int d^3x\, A_\lambda \partial^0 j^\lambda.$$

$$= \int d^3x\, \partial_i T_{(0)}^{i0} - \int d^3x\, \partial^0 (\underbrace{A_\lambda j^\lambda}_{=0}) + \int d^3x\, j^\lambda \partial^0 A_\lambda.$$

$$= \int d^3x\, \partial_i T_{(0)}^{i0} + \int d^3x\, j^\lambda \partial^0 A_\lambda.$$

$$= \oint_S dS_i\, T_{(0)}^{i0} + \int d^3x\, \vec{j}\cdot\vec{E}$$

D'après le théorème de Gauss.

et donc :

$$\boxed{-\frac{d}{dt}\int d^3x\, T^{00}{}_{(0)} = \oint_S dS_i\, (\vec{E}\wedge\vec{B})^i + \int d^3x\, \vec{E}\cdot\vec{j}.}$$

Rotation de Lorentz et conservation du moment angulaire généralisé:

Dans ce cas on a:

$$x \longrightarrow \bar{x} = \Lambda x . \qquad , \Lambda = e^{\delta\omega_{\mu\nu}}$$

sous sa forme infinitesimale:

$$\bar{x} = x + \delta x_\mu .$$

où $\delta x_\mu = \delta\omega_{\mu\nu}\, x^\nu$ .. $\quad | \ \delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu}$.

$$\delta x^\mu = \delta\omega^{\mu\nu}\, x_\nu .$$

$$\delta\varphi = \varphi(x+\delta x) - \varphi(x) = \partial_\mu\varphi\, \delta x^\mu .$$

$$\delta(\partial_\nu\varphi) = \partial_\nu \delta\varphi = \partial_\nu(\partial_\mu\varphi\, \delta x^\mu) = (\partial_\nu\partial_\mu\varphi)\, \delta x^\mu + \partial_\mu\varphi\, (\partial_\nu \delta x^\mu) .$$

$$\delta I = \int d^4x\, \delta I = \int d^4x \left[\frac{\partial L}{\partial\varphi}\, \delta\varphi + \frac{\partial L}{\partial(\partial_\mu\varphi)}\, \delta(\partial_\mu\varphi) \right] .$$

$$= \int d^4x \left[\frac{\partial L}{\partial\varphi}\, \partial_\nu\varphi\, \delta x^\nu + \frac{\partial L}{\partial(\partial_\mu\varphi)} (\partial_\mu\partial_\nu\varphi)\, \delta x^\nu + \frac{\partial L}{\partial(\partial_\mu\varphi)}\, \partial_\nu\varphi\, (\partial_\mu \delta x^\nu) \right] .$$

$$\partial_\nu \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\, \partial_\nu\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\, \partial_\nu(\partial_\mu\varphi) .$$

$$\frac{\partial \mathcal{L}}{\partial \varphi}\partial_\nu \varphi = \partial_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu(\partial_\mu \varphi).$$

Alors :

$$\delta I = \int d^4x \Big[\partial_\nu \mathcal{L}\,\delta x^\nu \cancel{- \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu(\partial_\mu \varphi)\,\delta x^\nu}$$

$$\cancel{+ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}(\partial_\mu \partial_\nu \varphi)\,\delta x^\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\,(\partial_\mu \delta x^\nu)\Big]$$

$$\delta I = \int d^4x \left[\partial_\nu \mathcal{L}\,\delta x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\,\partial_\mu(\delta x^\nu)\right].$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\,\partial_\mu(\delta x^\nu) = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\,\delta x^\nu\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\right).\delta x^\nu.$$

$$\delta I = \int d^4x \left[\partial_\mu \mathcal{L}\,\delta x^\mu - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\right)\right]\delta x^\nu$$

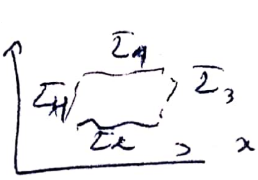
$$+ \underbrace{\int d^4x\,\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\,\delta x^\nu\right)}_{I_0}.$$

$$I_0 = \int_\Omega d^4x\,\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\,\delta x^\nu\right)$$

$$= \int_\Sigma dS^\mu\,\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi\,\delta x^\nu = \int_\Sigma dS^\mu\,\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta \varphi$$

$$= \int_{\Sigma_1 \cup \Sigma_2} d^3x\,\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}n_\mu\,\delta \varphi = 0$$

ce terme est nul.



Car : $\varphi(\Sigma_3) = \varphi(\Sigma_4) = 0$
$\varphi(\Sigma_1) = \varphi(\Sigma_2) = 0.$

et donc :

$$\delta I = \int d^4x \left[\partial_\nu \mathcal{L} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi\right)\right]\delta x^\nu .$$

$$\delta I = \int d^4x \left[g^\mu{}_\nu\, \partial_\mu \mathcal{L} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi\right)\right]\delta x^\nu .$$

$$= \int d^4x\, \partial_\mu \left[g^\mu{}_\nu \mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi\right]\delta x^\nu .$$

$$\delta I = -\int d^4x \left(\partial_\mu T^{\mu\nu}\right) x^\rho\, \delta\omega_{\nu\rho} .$$

$$\Rightarrow T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi - g^{\mu\nu}\mathcal{L} .$$

Pour $T^{\mu\nu}$ symétrique on aura :

$$\delta I = -\frac{1}{2}\int_\Omega d^4x\, \partial_\mu J^{\mu,\nu\rho}\, \delta\omega_{\nu\rho} .$$

avec : $J^{\mu,\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$.

Confirmation

$$\partial_\mu J^{\mu,\nu\rho} = \partial_\mu T^{\mu\nu}x^\rho + T^{\mu\nu}\delta^\rho_\mu - \partial_\mu T^{\mu\rho}x^\nu - T^{\mu\rho}\delta^\nu_\mu .$$

$$= \partial_\mu T^{\mu\nu}x^\rho + \cancel{T^{\rho\nu}} - \cancel{T^{\nu\rho}} - \partial_\mu T^{\mu\rho}x^\nu .$$

$$\partial_\mu J^{\mu,\nu\rho} = \partial_\mu T^{\mu\nu}x^\rho - \partial_\mu T^{\mu\rho}x^\nu .$$

$$\delta I = -\frac{1}{2}\int d^4x \left[\partial_\mu T^{\mu\nu}x^\rho\,\delta\omega_{\nu\rho} - \partial_\mu T^{\mu\rho}x^\nu \underbrace{\delta\omega_{\nu\rho}}_{-\delta\omega_{\rho\nu}}\right] .$$

$$= -\frac{1}{2}\int d^4x \left[2\partial_\mu T^{\mu\nu}x^\rho\,\delta\omega_{\nu\rho}\right]$$

$$\delta I = \int d^4x\, \partial_\mu T^{\mu\nu}x^\rho\,\delta\omega_{\rho\nu} \leftarrow$$

$\mathcal{J}^{\mu,\nu\rho} = T^{\mu\nu}x^{\rho} - T^{\mu\rho}x^{\nu}$ est la densité du moment angulaire du champ.

L'invariance de l'action par rotation implique l'invariance du moment angulaire résidant dans :

$$\partial_\mu \mathcal{J}^{\mu,\nu\rho} = 0 .$$

Il est important que $T^{\mu\nu}$ la densité énergie-impulsion soit symétrique $(T^{\mu\nu} = T^{\nu\mu})$ pour que le moment angulaire se conserve.

Le moment angulaire généralisé est :

$$\mathcal{J}^{\nu\rho} = \int d^3x \, \mathcal{J}^{0,\nu\rho}(x,t)$$

et nous avons alors :

$$\frac{d\mathcal{J}^{\nu\rho}}{dt} = \int_V d^3x \, \partial_0 \mathcal{J}^{0,\nu\rho} = -\int_V d^3x \, \partial_i \mathcal{J}^{i,\nu\rho}$$

$$= -\oint_{S(V)} ds_i \, \mathcal{J}^{i,\nu\rho} = 0 .$$

La conservation du moment angulaire.

Formalisme Hamiltonien :

On essaye d'introduire le formalisme Hamiltonien, le temps sera alors privilégié, En effet, on associe au champ $\varphi(x) \equiv \varphi(\vec{x}, t)$ un champ conjugué.

$$\pi(\vec{x}, t) = \frac{\partial L(t)}{\partial(\partial_0 \varphi)} = \frac{\delta}{\delta(\partial_0 \varphi(\vec{x}, t))} \int d^3y\, \mathcal{L}(\vec{y}, t).$$

et d'une manière similaire on définit le crochet de Poisson de deux fonctionnelles L_1 et L_2 par :

$$\{L_1, L_2\} = -\{L_2, L_1\}$$
$$= \int d^3x \left(\frac{\delta L_1}{\delta \pi(\vec{x}, t)} \frac{\delta L_2}{\delta \varphi(\vec{x}, t)} - \frac{\delta L_1}{\delta \varphi(\vec{x}, t)} \frac{\delta L_2}{\delta \pi(\vec{x}, t)} \right).$$

En particulier, on a :

$$\{\pi(\vec{x}, t), \varphi(\vec{y}, t)\} = \int d^3x' \left(\frac{\delta \pi(\vec{x}, t)}{\delta \pi(\vec{x}', t)} \frac{\delta \varphi(\vec{y}, t)}{\delta \varphi(\vec{x}', t)} - \frac{\delta \pi(\vec{x}, t)}{\delta \varphi(\vec{x}', t)} \frac{\delta \varphi(\vec{y}, t)}{\delta \pi(\vec{x}', t)} \right)$$
$$= \int d^3x' \left[\delta(\vec{x} - \vec{x}') \cdot \frac{\delta \varphi(\vec{y}, t)}{\delta \varphi(\vec{x}', t)} - 0 \right].$$
$$= \int d^3x' \, \delta(\vec{x} - \vec{x}') \, \delta(\vec{y} - \vec{x}').$$

$$\{\pi(\vec{x}, t), \varphi(\vec{y}, t)\} = \delta(\vec{x} - \vec{y}).$$

$$\{\pi(\vec{x},t),\pi(\vec{y},t)\} = \int d^3x' \left(\underbrace{\frac{\delta\pi(\vec{x},t)}{\delta\pi(\vec{x}',t)}}_{\delta(\vec{x}-\vec{x}')} \underbrace{\frac{\delta\pi(\vec{y},t)}{\delta\varphi(\vec{x}',t)}}_{0} - \underbrace{\frac{\delta\pi(\vec{x},t)}{\delta\varphi(\vec{x}',t)}}_{0} \underbrace{\frac{\delta\pi(\vec{y},t)}{\delta\pi(\vec{x}',t)}}_{\delta(\vec{y}-\vec{x}')} \right)$$

$$\{\pi(\vec{x},t),\pi(\vec{y},t)\} = 0$$

$$\{\varphi(\vec{x},t),\varphi(\vec{y},t)\} = \int d^3x' \left(\frac{\delta\varphi(\vec{x},t)}{\delta\pi'(\vec{x}',t)} \frac{\delta\varphi(\vec{y},t)}{\delta\varphi(\vec{x}',t)} - \frac{\delta\varphi(\vec{x},t)}{\delta\varphi(\vec{x}',t)} \frac{\delta\varphi(\vec{y},t)}{\delta\pi(\vec{x}',t)} \right)$$

$$= \int d^3x' \left(0 . \delta(\vec{y}-\vec{x}') - \delta(\vec{x}-\vec{x}') . 0 \right).$$

$$\{\varphi(\vec{x},t),\varphi(\vec{y},t)\} = 0$$

Les équations de mouvement sont.

$$\partial_0\varphi(\vec{x},t) = \{H,\varphi(\vec{x},t)\} = \int d^3x' \left(\frac{\delta H}{\delta\pi(\vec{x}',t)} \frac{\delta\varphi(\vec{x},t)}{\delta\varphi(\vec{x}',t)} - \frac{\delta H}{\delta\varphi(\vec{x}',t)} \underbrace{\frac{\delta\varphi(\vec{x},t)}{\delta\pi(\vec{x}',t)}}_{0} \right)$$

$$= \int d^3x' \frac{\delta H}{\delta\pi(x',t)} \delta(\vec{x}-\vec{x}')$$

$$\partial_0\varphi(\vec{x},t) = \frac{\delta H}{\delta\pi(x,t)}$$

$$\partial_0\pi(\vec{x},t) = \{H,\pi(\vec{x},t)\} = \int d^3x' \left(\frac{\delta H}{\delta\pi(\vec{x}',t)} \frac{\delta\pi(\vec{x},t)}{\delta\varphi(\vec{x}',t)} - \frac{\delta H}{\delta\varphi(\vec{x}',t)} \frac{\delta\pi(\vec{x},t)}{\delta\pi(\vec{x}',t)} \right)$$

avec : $H(t) = \int d^3x\, T^{00}(\vec{x},t)$.

- $T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \partial^\nu\varphi - g^{\mu\nu}\mathcal{L}$.

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)} \partial^0\varphi - \mathcal{L} = \pi\dot{\varphi} - \mathcal{L}.$$

avec : $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}$

* Symétries internes :

Soient T^s $(s=1,2,\dots,r)$ des générateurs infinitésimaux associés à la transformation :

$$\delta\varphi(x) = \delta\alpha_s(x)\, T^s\, \varphi(x).$$

avec $(T^s)^+ = -T^s$ et vérifiant :

$$[T^{s_1}, T^{s_2}] = C^{s_1 s_2}{}_{s_3}\, T^{s_3}.$$

Sous cette transformation on a :

$$\mathcal{L}(\varphi) \longrightarrow \tilde{\mathcal{L}}(\varphi+\delta\varphi).$$

$\hat{\mathcal{L}}(\varphi+\delta\varphi)$ est fonction de $\varphi(x)$, $\partial_\mu\varphi(x)$, $\alpha_s(x)$ et $\partial_\mu\alpha_s(x)$.

La variation de l'action donne :

$$\delta I = \int d^4x\, \delta\hat{\mathcal{L}}.$$

$$\delta\mathcal{L} = \hat{\mathcal{L}} - \mathcal{L} = \hat{\mathcal{L}}(\varphi+\delta\varphi) - \mathcal{L}(\varphi+\delta\varphi-\delta\varphi).$$

$$\delta\mathcal{L} = \frac{\partial\hat{\mathcal{L}}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) + \frac{\partial\mathcal{L}}{\partial(\delta\alpha_s)}\delta\alpha_s + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\alpha_s)}\delta(\partial_\mu\alpha_s)$$

$$\frac{\partial\hat{\mathcal{L}}}{\partial\varphi} - \partial_\mu\frac{\partial\hat{\mathcal{L}}}{\partial(\partial_\mu\varphi)} = 0$$

$$\delta\mathcal{L} = \partial_\mu\frac{\partial\hat{\mathcal{L}}}{\partial(\partial_\mu\varphi)}\delta\varphi + \frac{\partial\hat{\mathcal{L}}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) + \frac{\partial\mathcal{L}}{\partial(\delta\alpha_s)}\delta\alpha_s + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\alpha_s)}\delta(\partial_\mu\alpha_s).$$

$$\delta\mathcal{L} = \underbrace{\partial_\mu\left(\frac{\partial\hat{\mathcal{L}}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)}_{\text{divergence}} + \frac{\partial\mathcal{L}}{\partial(\delta\alpha_s)}\delta\alpha_s + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\alpha_s)}\delta(\partial_\mu\alpha_s)$$

Ce qui donne pour δI:

$$\delta I = \int d^4x \left(\frac{\partial \hat{\mathcal{L}}}{\delta(\delta\alpha_s)} \delta\alpha_s + \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\nu \delta\alpha_s)} \partial_\nu(\delta\alpha_s) \right)$$

$$= \int d^4x \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\delta\alpha_s)} - \partial_\nu \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\nu \delta\alpha_s)} \right) \right) \delta\alpha_s$$

L'invariance de I par la transformation, implique

$$\partial_\nu \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\nu \delta\alpha_s)} \right) = \frac{\partial \hat{\mathcal{L}}}{\partial(\delta\alpha_s)} .$$

on appelle courant de Noether associé à la transformation précédente

$$j^\nu_s(x) = \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\nu(\delta\alpha_s))}$$

La conservation du courant n'est satisfaite que si :

$\dfrac{\partial \hat{\mathcal{L}}}{\partial(\partial_\nu(\delta\alpha))} = 0$, Donc que si $\mathcal{L}$ est invariant par la transformation générée par T_s.

La charge de Noether est dans ce cas :

$$Q_s = \int d^3x \, j^0_s(\vec{x}, t) .$$

et on a : $\dfrac{dQ_s}{dt} = 0$ si $\mathcal{L}$ est invariant par la transformation T_s.

on a : $$j^0_s(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \delta\alpha_s)} = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\,\frac{\partial(\partial_0\varphi)}{\partial(\partial_0\delta\alpha_s)}.$$

on a : $\delta(\partial_0\varphi) = \partial_0(\delta\alpha_s)\,T^s\varphi(x)$, i.e :

$$\frac{\delta(\partial_0\varphi)}{\delta(\partial_0\delta\alpha)} = T^s\varphi(x).$$

ce qui donne :

$$j^0_s = \Pi(x)\,T^s\varphi(x).$$

Alors : $$Q_s = \int d^3x\, \Pi(x)\,T^s\varphi(x).$$

et nous avons :

$$\{Q_s, F\} = \int d^3x \left(\frac{\delta Q_s(y)}{\delta\pi(x)}\frac{\delta F(y)}{\delta\varphi(x)} - \frac{\delta Q_s(y)}{\partial\varphi(x)}\frac{\delta F(y)}{\delta\pi(x)}\right).$$

$$\{Q_s, F\} = \int d^3x \left(\frac{\delta F(y)}{\delta\varphi(x)}\,T^s\varphi(x) - \Pi(x)\,T^s\frac{\delta F}{\delta\pi(x)}\right)$$

et en particulier :

$$\{\varphi^s, \phi(x) = T^s\varphi(x).$$

$$\{\varphi^s, \pi(x)\} = -\pi(x)\,T^s = T^s\pi(x).$$

ou bien autrement

$$\{\delta\alpha_s\,\varphi^s, \phi(x)\} = \delta\alpha_s\,\{\varphi^s, \phi(x)\}$$
$$= \delta\alpha_s\, T^s \phi(x) = \delta\phi(x) .$$

$$\{\delta\alpha_s\,\varphi^s, \pi(x)\} = \delta\alpha_s\,\{\varphi^s, \pi(x)\}$$
$$= \delta\alpha_s\, T^s \pi(x) = \delta\pi(x)$$

on dit que φ^s génère la variation via les crochet de Poisson de la même manière que H génère la variation dans les charges vérifiant

$$\{\varphi^{s_1}, \varphi^{s_2}\} = \int d^3x \left(\frac{\delta\varphi^{s_1}(x)}{\delta\pi(x,t)} \frac{\delta\varphi^{s_2}(y)}{\delta\varphi(x,t)} - \frac{\delta\varphi^{s_1}(x)}{\delta\varphi(x,t)} \frac{\delta\varphi^{s_2}(y)}{\delta\pi(x,t)} \right) .$$

$$= \int d^3x \Big(T^{s_1}\phi(x')\,\delta(\vec{x}-\vec{x}')\,\pi(y)\,T^{s_2}\,\delta(\vec{x}-\vec{y}')$$
$$- \pi(\vec{x}')\,T^{s_1}\delta(\vec{x}-\vec{x}').\,T^{s_2}\,\phi(y)\,\delta(\vec{x}-\vec{y}')\Big)$$

$$= -\int d^3x \left(\pi(x)\,T^{s_1}T^{s_2}\,\phi(x) - \pi(x)\,T^{s_1}T^{s_2}\,\phi(x)\right)$$

$$/ \; T^{s_1}\,\pi = -\pi\, T^{s_1} .$$

$$= -\int d^3x\, \pi(x)\,[T^{s_1}, T^{s_2}]\,\phi(x)$$

$$\{\varphi^{s_1}, \varphi^{s_2}\} = -C^{s_1 s_2}{}_{s_3}\,\varphi^{s_3}$$

$$\frac{d\varphi^s}{dt} = \{H(t), \varphi^s(t)\} = \int d^3x\, \frac{\partial\mathcal{L}}{\partial(\delta\alpha_s)} .$$

$$\{ J^0_{s_1}(\vec{x},t), J^0_{s_2}(\vec{y},t)\} = -C_{s_1 s_2}{}^{s_3}\, J^0_{s_3}(\vec{x},t)\,\delta(\vec{x}-\vec{y}) .$$

Champ de Klein-Gordon:

a) champ réel à une composante.

Dans ce cas la densité lagrangienne est donnée par :

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} \mu^2 \phi^2$$

L'équation du champ est :

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0 \quad -(*)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\mu^2 \phi \quad \ldots (1)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi \quad \ldots (2)$$

remplaçons (1) et (2) dans (*) :

$$-\mu^2 \phi - \partial_\mu \partial^\mu \phi = 0 \Rightarrow \Box \phi + \mu^2 \phi = 0.$$

$$\Rightarrow (\Box + \mu^2) \phi = 0.$$

qui est l'éq de Klein-Gordon, si on fixe $\mu = \frac{mc}{\hbar}$ ou bien $\mu = m$.

Le tenseur impulsion-énergie est :

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}.$$

$$T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2} g^{\mu\nu} \partial_\alpha \varphi \partial^\alpha \varphi + \frac{1}{2} \mu^2 g^{\mu\nu} \phi^2.$$

$$T^{\mu\nu} = \frac{1}{2} g^{\mu\nu} \left(\mu^2 \phi^2 - \partial_\alpha \varphi \partial^\alpha \varphi \right) + \partial^\mu \varphi \partial^\nu \varphi.$$

qui est symétrique. $T^{\mu\nu} = T^{\nu\mu}$.

La densité d'énergie,

$$T^{00} = \frac{1}{2}\left(\mu^2\phi^2 - \partial_0\phi\partial^0\phi\right) + \partial^0\phi\,\partial^0\phi - \frac{1}{2}\partial_i\phi\,\partial^i\phi.$$

$$= \frac{1}{2}\left(\mu^2\phi^2 - \partial_0\phi\,\partial^0\phi\right) + \partial^0\phi\,\partial^0\phi + \frac{1}{2}\partial^i\phi\,\partial^i\phi.$$

$$T^{00} = \frac{1}{2}\left(\partial_0\phi\partial^0\phi + \mu^2\phi^2\right) + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2$$

$\vec{\nabla} = \partial i.$

$(+,-,-,-).$

qui peut aussi s'écrire avec l'introduction:

$$\Pi = \frac{\partial L}{\partial(\partial_0\phi)} = \partial^0\phi.$$

Comme:

$$\mathcal{H} = T^{00} = \frac{1}{2}\Pi^2 + \frac{\mu^2}{2}\phi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2.$$

$$H(t) = \int_V d^3x\,\mathcal{H} = \int T^{00}(x)\,d^3x.$$

↳ L'énergie du champ.

si $L(t)$ est invariant par translation on a: $\delta\left(\int L(t)\,dt\right) = 0$

qui donne: $\frac{d}{dt}H(t) = 0.$

L'impulsion du champ est:

$$P^i = \int_V d^3x\;T^{0i}$$

$$T^{0i} = \partial^0\phi\,\partial^i\phi = \Pi\,\partial^i\phi$$

Alors l'impulsion du champ est :

$$P^i(t) = \int_V d^3x \; \Pi(\vec{x},t)\, \partial^i \phi(\vec{x},t) .$$

Le tenseur moment-angulaire du champ :

$$\mathcal{J}^{\mu,\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu .$$

$$= \frac{1}{2} g^{\mu\nu} \left(m^2\phi^2 - \partial_\alpha \phi\, \partial^\alpha \phi\right) x^\rho + \partial^\mu \phi\, \partial^\nu \phi\, x^\rho .$$

$$- \frac{1}{2} g^{\mu\rho} \left(m^2\phi^2 - \partial_\alpha \phi\, \partial^\alpha \phi\right) x^\nu - \partial^\mu \phi\, \partial^\rho \phi\, x^\nu .$$

$$\mathcal{J}^{\mu,\nu\rho} = \frac{1}{2}\left(m^2\phi^2 - \partial_\alpha \phi\, \partial^\alpha \phi\right)\left(g^{\mu\nu} x^\rho - g^{\mu\rho} x^\nu\right)$$

$$+ \partial^\mu \phi \left(\partial^\nu \phi\, x^\rho - \partial^\rho \phi\, x^\nu\right)$$

② Champ de Klein Gordon complexe :

L'extension du champ précédent au cas complexe est immédiate :

$$\phi \longrightarrow \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2).$$

où ϕ_1 et ϕ_2 sont des champs de K.G réel.

Il n'est pas difficile de s'assurer alors que :

$$\mathcal{L}(\phi_1, \phi_2, \partial_\mu \phi_1, \partial_\mu \phi_2) \longrightarrow \mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*)$$

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2.$$

$$= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2}\mu^2 \phi_1^2 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2}\mu^2 \phi_2^2.$$

on a :

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad - ①$$

$$\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \quad - ②$$

$$① + ② \Rightarrow \phi_1 = \frac{1}{\sqrt{2}}(\phi + \phi^*)$$

$$① - ② \Rightarrow \phi_2 = \frac{-i}{\sqrt{2}}(\phi - \phi^*).$$

Alors,

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu\left(\frac{\phi+\phi^*}{\sqrt{2}}\right)\partial^\mu\left(\frac{\phi+\phi^*}{\sqrt{2}}\right) - \mu^2\left(\frac{\phi+\phi^*}{\sqrt{2}}\right)^2 - \partial_\mu\left(\frac{\phi-\phi^*}{\sqrt{2}}\right)\partial^\mu\left(\frac{\phi-\phi^*}{\sqrt{2}}\right) + \mu^2\left(\frac{\phi-\phi^*}{\sqrt{2}}\right)^2\right\},$$

$$\mathcal{L} = \frac{1}{4}\left\{\partial_\mu(\phi+\phi^*)\partial^\mu(\phi+\phi^*) - \partial_\mu(\phi-\phi^*)\partial^\mu(\phi-\phi^*) - \mu^2\left[(\phi+\phi^*)^2 - (\phi-\phi^*)^2\right]\right\}.$$

$$\mathcal{L} = \frac{1}{4}\Big\{ \cancel{\partial_\mu \phi \partial^\mu \phi} + \cancel{\partial_\mu \phi^* \partial^\mu \phi^*} + \partial_\mu \phi^* \partial^\mu \phi + \partial_\mu \phi^* \partial^\mu \phi^*$$

$$- \cancel{\partial_\mu \phi \partial^\mu \phi} + \partial_\mu \phi \partial^\mu \phi^* + \partial_\mu \phi^* \partial^\mu \phi - \cancel{\partial_\mu \phi^* \partial^\mu \phi^*}$$

$$- \mu^2 \big[\cancel{\phi^2} + \cancel{\phi^{*2}} + 2\phi\phi^* - \cancel{\phi^2} - \cancel{\phi^{*2}} + 2\phi\phi^* \big] \Big\}$$

$$\mathcal{L} = \frac{1}{4}\left\{ 4\partial_\mu \phi \partial^\mu \phi^* - 4\mu^2 \phi \phi^* \right\}$$

$$\mathcal{L} = \left\{ \partial_\mu \phi \partial^\mu \phi^* - \mu^2 \phi \phi^* \right\}.$$

Les équations de champs seront :

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0 \\[2ex] \dfrac{\partial \mathcal{L}}{\partial \varphi^*} - \partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \right) = 0 \end{cases}$$

ce qui donne :

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \varphi} = -\mu^2 \phi^*, \quad \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^*. \\[2ex] \dfrac{\partial \mathcal{L}}{\partial \varphi^*} = -\mu^2 \phi, \quad \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi. \end{cases}$$

$$\begin{cases} \partial_\mu \partial^\mu \phi^* + \mu^2 \phi^* = 0. \\ \partial_\mu \partial^\mu \phi + \mu^2 \phi = 0 \end{cases}$$

Dans ce cas on a alors deux moments conjugués :

$$\begin{cases} \pi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\phi}^* \\[2ex] \pi^* = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}^*} = \dot{\phi} \end{cases}$$

La densité Hamiltonienne est :

$$\mathcal{H} = T^{00}$$

nous devons d'abord chercher $T^{\mu\nu}$.

$$T^{\mu\nu} = \partial^\nu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial^\nu \phi^* \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} - g^{\mu\nu} \mathcal{L}$$

$$= \partial^\nu \phi \, \partial^\mu \phi^* + \partial^\nu \phi^* \partial^\mu \phi - g^{\mu\nu} \left(\partial_\alpha \phi^* \partial^\alpha \phi - \mu^2 \phi^* \phi \right)$$

$$T^{\mu\nu} = \partial^\nu \phi \, \partial^\mu \phi^* + \partial^\nu \phi^* \partial^\mu \phi - g^{\mu\nu} \partial_\alpha \phi^* \partial^\alpha \phi + g^{\mu\nu} \mu^2 \phi^* \phi \quad (*)$$

$g^{00} = 1$

$$T^{00} = \partial^0 \phi \, \partial^0 \phi^* + \cancel{\partial^0 \phi^* \partial^0 \phi} - \cancel{\partial_0 \phi^* \partial^0 \phi} - \partial_i \phi^* \partial^i \phi + \mu^2 \phi^* \phi$$

$$= \partial^0 \phi \, \partial^0 \phi^* + (\vec{\nabla} \phi^*)(\vec{\nabla} \phi) + \mu^2 \phi^* \phi$$

$$= \dot{\phi} \, \dot{\phi}^* + (\vec{\nabla} \phi^*)(\vec{\nabla} \phi) + \mu^2 \phi^* \phi$$

$$T^{00} = \pi^* \pi + (\vec{\nabla} \phi^*)(\vec{\nabla} \phi) + \mu^2 \phi^* \phi$$

et donc $\mathcal{H} = \pi^* \pi + (\vec{\nabla} \phi^*)(\vec{\nabla} \phi) + \mu^2 \phi^* \phi$ → La densité-énergie.

. La densité impulsion :

$$T^{0i} = \partial^0 \phi \, \partial^i \phi^* + \partial^0 \phi^* \partial^i \phi$$

$$= \dot{\phi} \, \vec{\nabla} \phi^* + \dot{\phi}^* \vec{\nabla} \phi$$

$$T^{0i} = \pi^* \vec{\nabla} \phi^* + \pi \vec{\nabla} \phi$$