Mohammed Seddik Benyahia University - JIJEL

Faculty of Exact Sciences and Computer Science

Department of Mathematics

Master 2 : Functional Analysis

Asymptotic Analysis

2025 October

tutorial number 1

Exercise 01 :

Let D be a domain of $\mathbb{R}$ $(\mathbb{C})$; $x_0 \in \overline{D}$ and two functions φ and ψ defined on D with real or complex values.

1. Demonstrate in the neighborhood of the point x_0 the following operations
   (a) If $\varphi = O(\psi)$ and $\alpha > 0$ then $|\varphi|^\alpha = O(|\psi|^\alpha)$.
   (b) If $\varphi_i = O(\psi_i)$, $i = 1, 2, \ldots, n$ and a_i constants then $\sum\limits_{i=1}^{n} a_i\varphi_i = O(\sum\limits_{i=1}^{n} |a_i||\psi_i|)$.
   (c) If $\varphi_i = O(\psi_i)$, $i = 1, 2, \ldots, n$ and if it existsψ such that $\psi_i \leq \psi$, and a_i constants then $\sum\limits_{i=1}^{n} a_i\varphi_i = O(\psi)$.
   (d) If $\varphi_i = O(\psi_i)$, $i = 1, 2, \ldots, n$, then $\prod\limits_{i=1}^{n} a_i\varphi_i = O(\prod\limits_{i=1}^{n} \psi_i)$.
   (e) Let $x \in \mathbb{R}$, $D =]a, b[\subset \mathbb{R}$, if $\varphi(x) = O(\psi(x))$; when $x \to b$, and if φ and ψ are integrable in D, then $\int\limits_x^b \varphi(t)dt = O(\int\limits_x^b |\psi(t)|dt)$, $x \to b$.
   (f) Let a parameter $\xi \in]\alpha, \beta[$ and let $\varphi(x, \xi)$ and $\psi(x, \xi)$ two functions depend on the parameterξ, and $\varphi(x, \xi) = O(\psi(x, \xi))$ uniformly when $x \to x_0$, then $\int\limits_\alpha^\beta \varphi(x, \xi)d\xi = O(\int\limits_\alpha^\beta |\psi(x, \xi)|d\xi)$, $x \to x_0$.
2. Demonstrate all operations (a − f) for (o) and $(\sim)$.

Exercise 02 :

Let $\gamma > 0$, the aim of the exercise is to prove that $e^{\gamma n} = o(n!)$. $(n \in \mathbb{N}^*)$

For this, we pose, for $n \geq 1, u_n = e^{\gamma n}$ and $v_n = n!$.

1. Prove that there exists an integer $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,
$$\frac{u_{n+1}}{u_n} \leq \frac{1}{2}\frac{v_{n+1}}{v_n}.$$
2. Deduce that there exists a constant $c > 0$ such that, for all $n \geq n_0$,
$$u_n \leq c\left(\frac{1}{2}\right)^{n-n_0} v_n.$$
3. Conclude that
$$e^{\gamma n} = o(n!).$$