Exercise 03 :

We consider the sequence (u_n) defined for $n \geq 1$ by :

$$u_n = \sqrt{n + \sqrt{(n-1) + \cdots + \sqrt{2 + \sqrt{1}}}}.$$

1. Show that (u_n) diverges to $+\infty$.
2. Express u_{n+1} as a function of u_n.
3. Show that $u_n \leq n$, then that $u_n = o(n)$.
4. Give a simple equivalent (asymptotic) of (u_n).
5. Determine $\lim\limits_{n \to +\infty} (u_n - \sqrt{n})$.

Exercise 04 :

We study the sequence (S_n) with general term

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

1. Justify that
$$\frac{1}{\sqrt{n+1}} \leq 2(\sqrt{n+1} - \sqrt{n}) \leq \frac{1}{\sqrt{n}}.$$
2. Determine the limit of (S_n).
3. Let $u_n = S_n - 2\sqrt{n}$. Show that the sequence (S_n) is convergent.
4. Give a simple asymptotic equivalent of (S_n).

Exercise 05 :

We study the sequence (S_n) with general term

$$S_n = \sum_{k=1}^{n} \frac{1}{k}.$$

1. Establish that for all $t > -1$, $\ln(1+t) \leq t$ and deduce that
$$ln(1+t) \geq \frac{t}{t+1}$$
2. Observe that
$$\ln(n+1) \leq S_n \leq \ln n + 1.$$
and deduce a simple asymptotic equivalent of S_n.
3. Show that the sequence $u_n = S_n - \ln n$ is convergent. Its limit is called Euler's constant and is usually denoted by γ.

Exercise 06 :

Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

1. Show that the equation $x + e^x = n$ admits a unique solution $x_n \in \mathbb{R}$.
2. Determine the limit of the sequence $(x_n)_n$.
3. Find a simple asymptotic equivalent of (x_n) as $n \to \infty$.
4. Determine two terms of the asymptotic expansion of x_n.