

Série de TD N1
Intégrales doubles et triples

Exercices 1 : Domaines d'intégration doubles et triples

1- Tracer les domaines suivants

- $D_1 = \{(x, y) | x \geq 0, x^2 \leq y \leq 1\}$
- $D_2 = \{(x, y) | a^2 < x^2 + y^2 < b^2, y \geq 0\}$ avec $b > a > 0$
- $D_3 = \{(x, y) | x \geq 0, y \geq 0, x + y \leq 1\}$
- $D_4 = \{(x, y) | x \geq 0, y \geq 0, y \leq 2 - x^2\}$
- $D_5 = \{(x, y) | x \geq 0, y \geq 0, x \leq \sqrt{2 - y}\}$
- R surface fermée limitée par $y = x + 1, y = x - 3, y = -\frac{x}{3} + 2, y = -\frac{x}{3} + 4$
- $D_6 = \{(x, y, z) | 0 \leq x \leq 2, x \leq y \leq 2, 0 \leq z \leq y^2\}$
- $D_7 = \{(x, y, z) | 0 \leq x \leq 2, -x^2 \leq y \leq 2, 0 \leq z \leq 2\}$

Exercices 2 : Intégration doubles et triples

Exprimer et calculer chaque intégrale définie dans son propre domaine (exercice précédents) :

e^{x-y}	→	$\{D_3\}$
xy	→	$\{D_4\}$
xy	→	$\{D_5\}$
$4x^2 + y + z$	→	$\{D_6\}$
$x - y$	→	$\{D_7\}$
$ x - y $	→	$\{D_1\}$

Exercices 3 : Jacobien (transformation)

1- En utilisant le jacobien, transformez le domaine $\{D_2\}$ en un domaine $\{T\}$ respectant les coordonnées polaire.

$$x = \rho \cos \theta, \quad y = \rho \sin \theta$$

Résoudre dans ce domaine $\{T\}$, les intégrales doubles des fonctions suivantes :

$$f(x, y) = (x^2 + y^2)$$

$$f(x, y) = \frac{y}{(x^2 + y^2)}$$

2- Calculer le jacobien et tracer les limites du nouveau domaine $\{S\}$ issu du domaine $\{R\}$ en tenant compte des nouvelles variables suivantes $u = y - x$ et $v = y + \frac{x}{3}$

3- Calculer les intégrales suivantes sur un volume d'une sphère de rayon R=1.5.

$$f(x, y) = \sqrt{(x^2 + y^2 + z^2)^3}$$

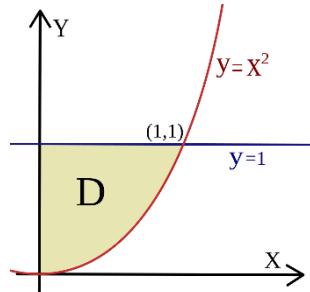
$$f(x, y) = \frac{x}{\sqrt{(x^2 + y^2 + z^2)}}$$

Solution Série de TD1

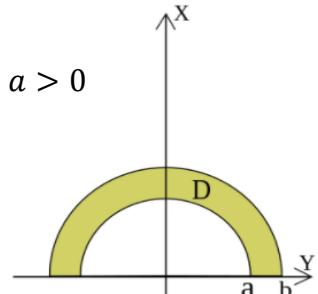
Intégrales doubles et triples

Exercice 1 :

- $D_1 = \{(x, y) | x \geq 0, x^2 \leq y \leq 1\}$

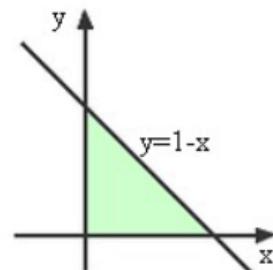


- $D_2 = \{(x, y) | a^2 < x^2 + y^2 < b^2, y \geq 0\}$ avec $b > a > 0$

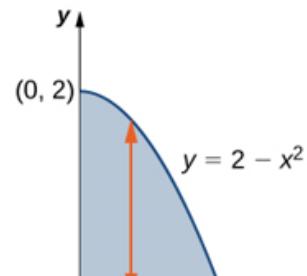


- $D_3 = \{(x, y) | x \geq 0, y \geq 0, x + y \leq 1\}$

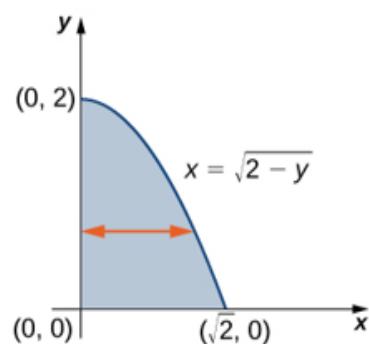
$$x + y \leq 1 \quad \text{d'où} \quad y \leq 1 - x$$



- $D_4 = \{(x, y) | x \geq 0, y \geq 0, y \leq 2 - x^2\}$

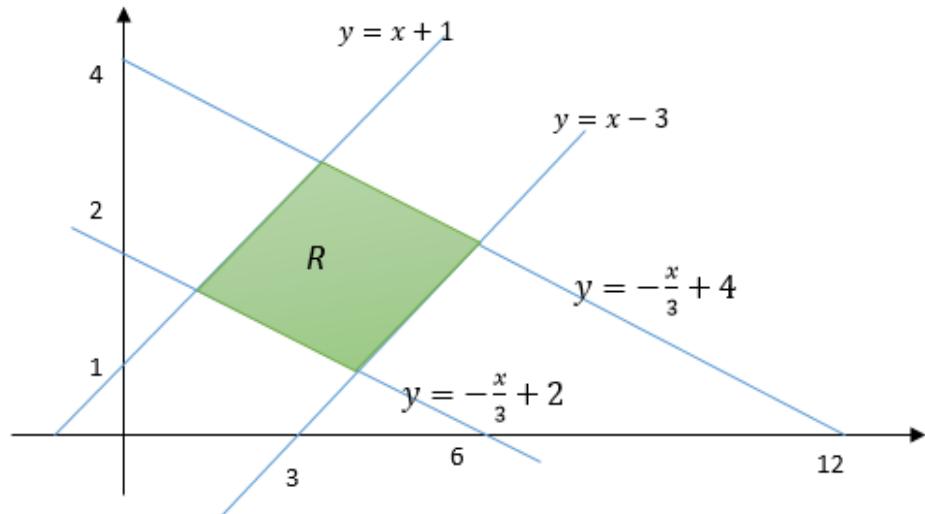


- $D_5 = \{(x, y) | x \geq 0, y \geq 0, x \leq \sqrt{2 - y}\}$

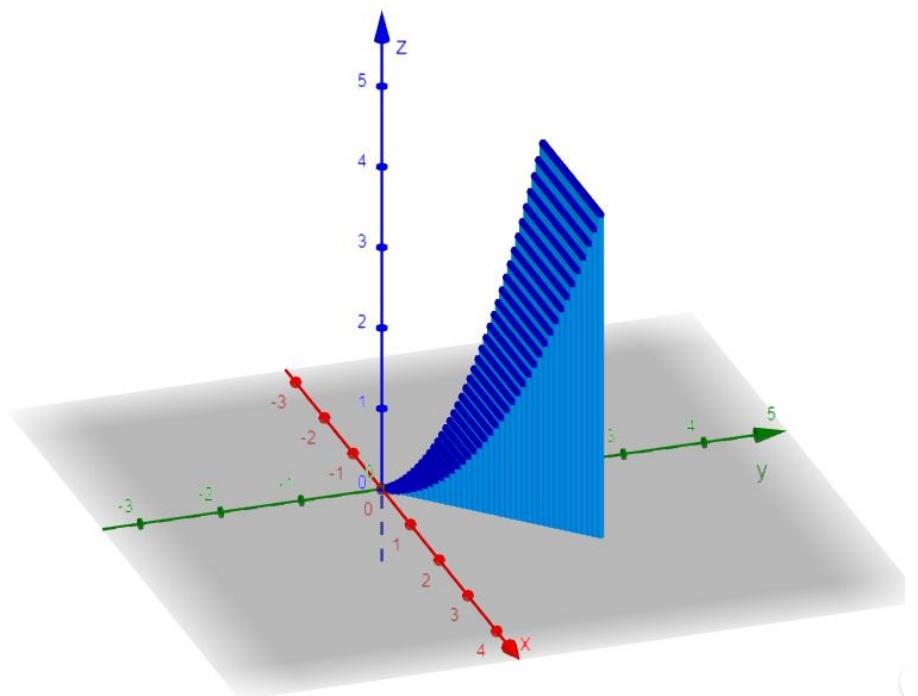


Les domaines $\{D_4\}$ et $\{D_5\}$ sont identique.

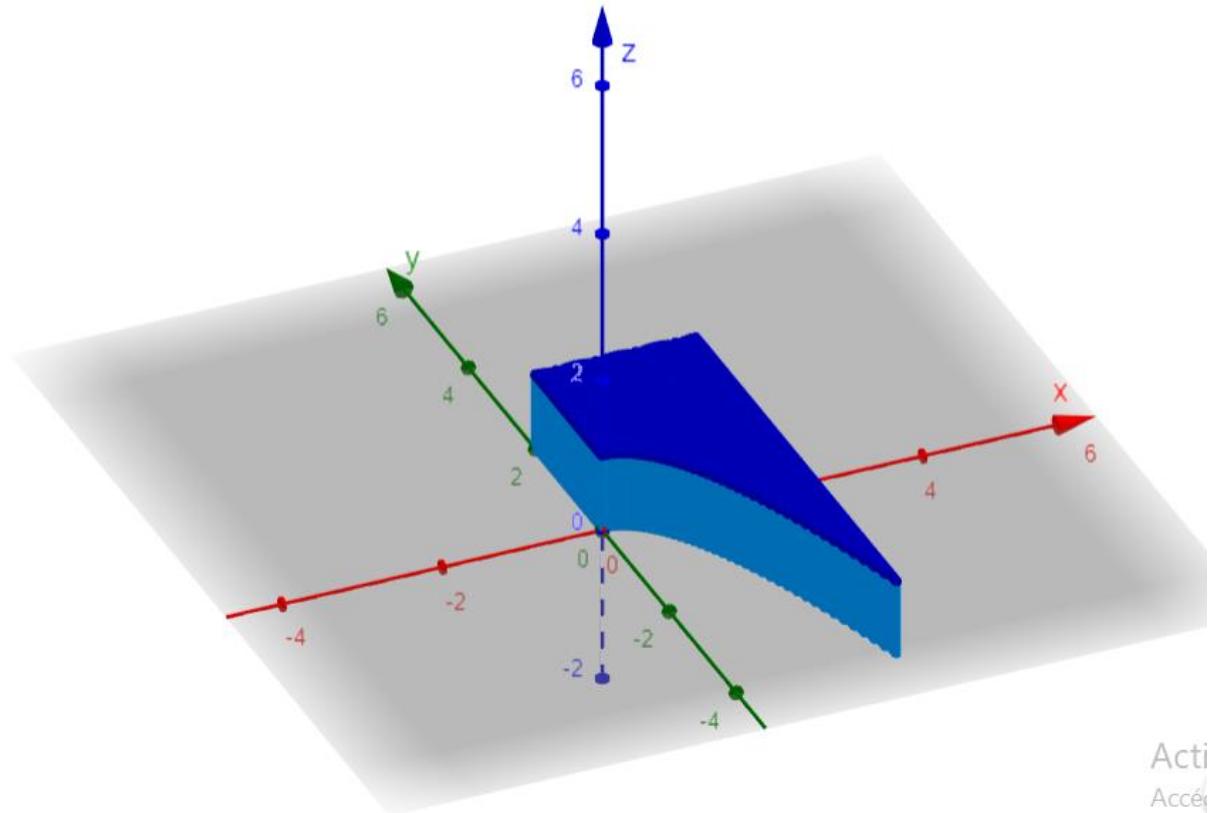
R surface fermée limitée par $y = x + 1$, $y = x - 3$, $y = -\frac{x}{3} + 2$, $y = -\frac{x}{3} + 4$



- $D_6 = \{(x, y, z) | 0 \leq x \leq 2, x \leq y \leq 2, 0 \leq z \leq y^2\}$



- $D_7 = \{(x, y, z) | 0 \leq x \leq 2, -x^2 \leq y \leq 2, 0 \leq z \leq 2\}$



Acti
Accé

Exercices 2 : Intégration doubles et triples

$$e^{x-y} \longrightarrow \{D_3\}$$

$$I_1 = \int_0^1 \int_0^{1-x} e^{x-y} dy dx = \int_0^1 [-e^{x-y}]_0^{1-x} dx = \int_0^1 e^x [-e^{-y}]_0^{1-x} dx$$

$$I_1 = \int_0^1 (1 - e^{x-1}) e^x dx = \int_0^1 (e^x - e^{2x-1}) dx = \left(e^x - \frac{1}{2} e^{2x-1} \right)_0^1$$

$$I_1 = \left(e^1 - \frac{1}{2} e^1 - 1 + \frac{1}{2} e^{-1} \right) = \left(\frac{1}{2} e^1 + \frac{1}{2} e^{-1} - 1 \right)$$

$$xy \longrightarrow \{D_4\}$$

$$I_2 = \int_0^{\sqrt{2}} \int_0^{2-x^2} xy \, dy \, dx = \int_0^{\sqrt{2}} x \left[\frac{y^2}{2} \right]_0^{2-x^2} \, dx = \int_0^{\sqrt{2}} \frac{x(2-x^2)^2}{2} \, dx$$

$$I_2 = \frac{1}{2} \int_0^{\sqrt{2}} (x^5 - 4x^3 + 4x) \, dx = \frac{1}{2} \left[\frac{x^6}{6} - \frac{4x^4}{4} + 4 \frac{x^2}{2} \right]_0^{\sqrt{2}} = \frac{2}{3}$$

Vérification (Fubini) $xy \longrightarrow \{D_5\}$

$$I_3 = \int_0^2 \int_0^{\sqrt{2-y}} xy \, dx \, dy = \int_0^2 y \left[\frac{x^2}{2} \right]_0^{\sqrt{2-y}} \, dx = \frac{1}{2} \int_0^2 y(2-y) \, dy$$

$$I_3 = \frac{1}{2} \left(y^2 - \frac{y^3}{3} \right)_0^2 = \frac{1}{2} \left(4 - \frac{8}{3} \right) = \frac{2}{3} = I_2$$

$4x^2 + y + z \longrightarrow \{D_6\}$
 $\bullet \quad D_6 = \{(x, y, z) | 0 \leq x \leq 2, x \leq y \leq 2, 0 \leq z \leq y^2\}$

$$I_4 = \int_0^2 \int_x^2 \int_0^{y^2} (4x^2 + y + z) \, dz \, dy \, dx = \int_0^2 \int_x^2 \left[4x^2 z + yz + \frac{z^2}{2} \right]_0^{y^2} \, dy \, dx$$

$$\begin{aligned} I_4 &= \int_0^2 \int_x^2 \left(4x^2 y^2 + y^3 + \frac{y^4}{2} \right) \, dy \, dx = \int_0^2 \left[\frac{4x^2 y^3}{3} + \frac{y^4}{4} + \frac{y^5}{10} \right]_x^2 \, dx \\ I_4 &= \int_0^2 \left[\frac{32x^2}{3} + 4 + \frac{16}{5} - \frac{4x^5}{3} - \frac{x^4}{4} - \frac{x^5}{10} \right] \, dx = \int_0^2 \left[-\frac{43x^5}{30} - \frac{x^4}{4} + \frac{32x^2}{3} + \frac{36}{5} \right] \, dx \\ I_4 &= \left[-\frac{43x^6}{180} - \frac{x^5}{20} + \frac{32x^3}{9} + \frac{36x}{5} \right]_0^2 = \left[-\frac{43 \times 64}{180} - \frac{32}{20} + \frac{256}{9} + \frac{72}{5} \right] \\ &= \left[-\frac{688}{45} - \frac{72}{45} + \frac{1280}{45} + \frac{648}{45} \right] = \frac{1168}{45} \end{aligned}$$

$$x - y \longrightarrow \{D_7\}$$

- $D_7 = \{(x, y, z) | 0 \leq x \leq 2, -x^2 \leq y \leq 2, 0 \leq z \leq 2\}$

$$I_5 = \int_0^2 \int_{-x^2}^2 \int_0^2 (x - y) dz dy dx = \int_0^2 \int_{-x^2}^2 [(x - y)z]_0^2 dy dx$$

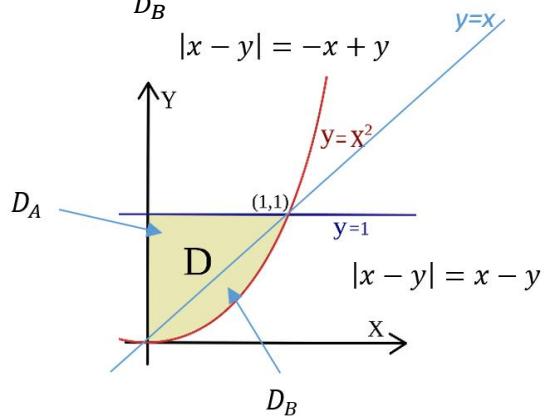
$$I_5 = \int_0^2 \left[2 \left(xy - \frac{y^2}{2} \right) \right]_{-x^2}^2 dx = 2 \int_0^2 \left(2x - 2 + x^3 + \frac{x^4}{2} \right) dx$$

$$I_5 = \left[2x^2 - 4x + \frac{x^4}{2} + \frac{x^5}{5} \right]_0^2 = \frac{72}{5}$$

$$|x - y| \longrightarrow \{D_1\}$$

- $D_1 = \{(x, y) | x \geq 0, x^2 \leq y \leq 1\}$

$$I_6 = \iint_{D_1} |x - y| dx dy = \iint_{D_A} (-x + y) dx dy + \iint_{D_B} (x - y) dx dy$$



$$I_6 = \int_0^1 \int_x^1 (-x + y) dy dx + \int_0^1 \int_{x^2}^x (x - y) dy dx$$

$$I_{6A} = \int_0^1 \int_x^1 (-x + y) dy dx = \int_0^1 \left[-xy + \frac{y^2}{2} \right]_x^1 dx = \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) dx$$

$$I_{6A} = \left[\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2} \right]_0^1 = \frac{1}{6}$$

$$I_{6B} = \int_0^1 \int_{x^2}^x (x-y) dy dx = \int_0^1 \left[xy - \frac{y^2}{2} \right]_{x^2}^x dx = \int_0^1 \left(\frac{x^2}{2} - x^3 + \frac{x^4}{2} \right) dx$$

$$I_{6B} = \left[\frac{x^5}{10} - \frac{x^4}{4} + \frac{x^3}{6} \right]_0^1 = \frac{1}{60}$$

$$I_6 = I_{6A} + I_{6B} = \frac{1}{6} + \frac{1}{60} = \frac{11}{60}$$

Exercices 3 : Jacobien (transformation)

- $D_2 = \{(x, y) | a^2 < x^2 + y^2 < b^2, y \geq 0\}$ avec $b > a > 0$

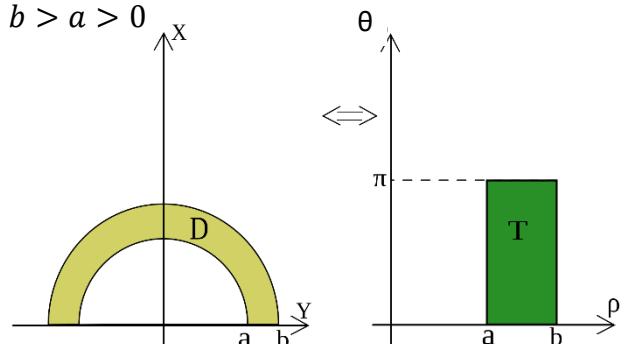
$$x = \rho \cos \theta, \quad y = \rho \sin \theta$$

On a

$$x^2 + y^2 = \rho^2$$

$$a < \rho < b$$

$$y \geq 0 \rightarrow \rho \sin \theta \geq 0$$



$$\text{Tant que } \rho \geq 0 \text{ alors } \sin \theta \geq 0 \text{ et } 0 \leq \theta \leq \pi$$

Donc

- $T = \{(\rho, \theta) | a < \rho < b, 0 \leq \theta \leq \pi\}$

Le jacobien dans ce cas s'écrira sous cette forme

$$\frac{D(x, y)}{D(\rho, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{vmatrix} = \rho \geq 0$$

$$\iint_{D_2} f(x, y) \, dx dy = \iint_T g(\rho, \theta) \left| \frac{D(x, y)}{D(u, v)} \right| d\rho d\theta$$

$$1- f(x, y) = (x^2 + y^2)$$

$$J_1 = \iint_{D_2} (x^2 + y^2) \, dx dy = \iint_T \rho^2 \rho \, d\rho d\theta$$

$$J_1 = \int_0^{\pi} \int_a^b \rho^3 \, d\rho d\theta = \pi \frac{\rho^4}{4} \Big|_a^b = \frac{\pi}{4} (b^4 - a^4)$$

$$2- f(x, y) = \frac{x}{(x^2 + y^2)}$$

$$J_1 = \iint_{D_2} \frac{y}{(x^2 + y^2)} \, dx dy = \iint_T \frac{\rho \sin \theta}{\rho^2} \rho \, d\rho d\theta$$

$$J_1 = \int_0^{\pi} \int_a^b \sin \theta \, d\rho d\theta = -\cos \theta \Big|_0^{\pi} \rho \Big|_a^b = 2(b - a)$$

1- Calculer le jacobien et tracer les limites du nouveau domaine $\{S\}$ issu du domaine $\{R\}$ en tenant compte des nouvelles variables suivantes $u = y - x$ et $v = y + \frac{x}{3}$

$$\text{Pour le demi-plan } y \leq x + 1 \quad \rightarrow \quad u = y - x \leq 1$$

$$\text{Pour le demi-plan } y \geq x - 3 \quad \rightarrow \quad u = y - x \geq -3$$

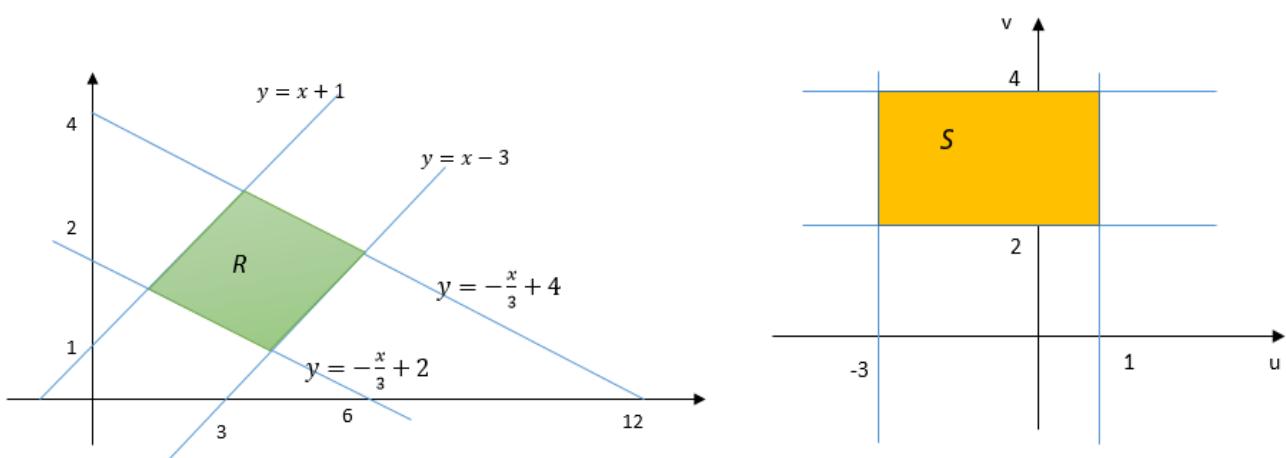
$$\text{Pour le demi-plan } y \leq \frac{-x}{3} + 4 \quad \rightarrow \quad v = y + \frac{x}{3} \leq 4$$

$$\text{Pour le demi-plan } y \geq \frac{-x}{3} + 2 \quad \rightarrow \quad v = y + \frac{x}{3} \geq 2$$

En écrivant les équations de x et y , nous aurons :

$$x = -\frac{3}{4}u + \frac{3}{4}v \quad y = \frac{1}{4}u + \frac{3}{4}v$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{vmatrix} = \left| \frac{-12}{16} \right| = \frac{3}{4}$$



Pour tout calcul d'intégrales de fonction $f(u, v)$ seront plus faciles avec des bornes indépendantes dans le domaine (S)

$$\int_{-3}^1 \int_2^4 f(u, v) \frac{3}{4} du dv$$

3- Calcul de $f(x, y) = \sqrt{(x^2 + y^2 + z^2)^3}$ sur un volume de rayon $R=1.5$

$$\int_{Sphere} \int \int \sqrt{(x^2 + y^2 + z^2)^3} dx dy dz = \int_0^\pi \int_0^{2\pi} \int_0^R f(\rho, \theta, \varphi) |\rho^2 \sin \varphi| d\rho d\theta d\varphi$$

$$\begin{aligned} & \int_0^\pi \int_0^{2\pi} \int_0^R \rho^3 |\rho^2 \sin \varphi| d\rho d\theta d\varphi \\ &= \int_0^\pi \int_0^{2\pi} \int_0^R \rho^5 \sin \varphi d\rho d\theta d\varphi = \frac{R^6}{6} 2\pi (-\cos \varphi|_0^\pi) = \frac{2R^6 \pi}{3} \end{aligned}$$

3- Calcul de $f(x, y) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$ sur un volume de rayon R=1.5

$$x = \rho \cos \theta \sin \varphi$$

$$\int_{Sphere} \int \int \frac{x}{\sqrt{x^2 + y^2 + z^2}} dx dy dz = \int_0^{\pi} \int_0^{2\pi} \int_0^R f(\rho, \theta, \varphi) |\rho^2 \sin \varphi| d\rho d\theta d\varphi$$

$$\int_0^{\pi} \int_0^{2\pi} \int_0^R \frac{\rho \cos \theta \sin \varphi}{\rho} |\rho^2 \sin \varphi| d\rho d\theta d\varphi = \int_0^{\pi} \int_0^{2\pi} \int_0^R \rho^2 \sin^2 \varphi \cos \theta d\rho d\theta d\varphi$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^R \rho^2 d\rho \cos \theta d\theta \frac{1}{2} (d\varphi + \cos 2\varphi d\varphi) = \frac{1}{2} \left(\frac{R^3}{3} \right) (-\sin \theta)_0^{2\pi} \left(\varphi - \frac{1}{2} \sin 2\varphi \right)_0^{\pi} = 0$$