University of Jijel - Faculty of exact sciences and computer science - Mathematics department

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Exercise 01: Let A and B be two nonempty subsets of $\mathbb{R}$ such that $A \subset B$.

Prove that if B is bounded then A is bounded and that:

$$\sup A \leq \sup B \ \wedge \inf A \geq \inf B.$$

Exercise 02: Suppose that A and B are two nonempty bounded sets of real numbers.

1) Show that $A \cup B$ and $A \cap B$ are also bounded. Further prove the following statements:
$$\sup(A \cup B) = \max(\sup A, \sup B) \ \wedge \inf(A \cup B) = \min(\inf A, \inf B).$$
$$\sup(A \cap B) \leq \min\,(\sup A, \sup B) \ \wedge \inf(A \cap B) \geq \max(\inf A, \inf B).$$
2) Let C be the set of all sums $x + y$ where $x \in A$ and $y \in B$, i.e.,
$$C = A + B = \{z = x + y / x \in A \wedge y \in B\}.$$

Prove that C is bounded and that:

$$\sup C = \sup A + \sup B \ \wedge \inf C = \inf A + \inf B.$$

Exercise 03: Let A be a nonempty subset of real numbers, which is bounded. Let $(-A)$ denote the set of all real numbers $(-x)$, where x belongs to A.

$$-A = \{-x / x \in A\}$$

Prove that if A is bounded then $-A$ is also bounded and that:

$$\sup(-A) = -\ \inf A \wedge \inf(-A) = -\sup A.$$

Exercise 04: Let A be a nonempty bounded subset of $\mathbb{R}_+^*$ and define:

$$B = \left\{\frac{1}{x} / x \in A\right\}$$

1) Show that B is bounded if $\inf A \neq 0$ and that:
$$\sup B = \frac{1}{\inf \mathrm{A}}.$$
2) Prove that if $\inf A = 0$, then $\sup B$ can not exists.

Exercise 05: Let $a_i \in \mathbb{R}, b_i \in \mathbb{R}_+^*: \forall\, 1 \leq i \leq n$. And let :
$m = \min\{a_i, 1 \leq i \leq n\}$ and $\mathrm{M} = \max\{a_i, 1 \leq i \leq n\}$.

1) Prove that: $m \leq \frac{\sum_{i=1}^{n} a_i\, b_i}{\sum_{i=1}^{n}\ b_i} \leq M$.
2) Prove that: $\min_{1\leq i\leq n}(\frac{a_i}{b_i}) \leq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n}\ b_i} \leq \max_{1\leq i\leq n}(\frac{a_i}{b_i})$.

Exercise 06: Prove that the following inequalities hold for all real numbers x and y such that $x > 0$:

1) $\forall (x, y) \in \mathbb{R}_+^{*2}, \exists n \in \mathbb{N}: x < ny$.
2) $\forall (x, y) \in \mathbb{R}_+^{*2}, \exists r \in \mathbb{Q}: x < ry$.

Exercise 07: Let A and B be two sets defined as follows:

$$A = \left\{2 - \frac{1}{2n} / n \in \mathbb{N}^*\right\}$$

$$B = \left\{\frac{3n+1}{n+1} / n \in \mathbb{N}\right\}$$

1) Prove that A and B are bounded.
2) Find the least upper bounds and the greatest lower bounds of these sets (Justify and rigorously prove your claim).
3) Find if there exists the maximum and the minimum of each set (Justify that).