University of Jijel - Faculty of exact sciences and computer science - Mathematics department

Series No. **04** *2023/2024*

Exercise 01: Suppose that $(u_n)_{n\in\mathbb{N}}$converges to l and $u_n \geq 0$ for all n. Then $(\sqrt{u_n})_{n\in\mathbb{N}}$ converges to $\sqrt{\mathrm{l}}$.

Exercise 02: Let $(u_n)_{n\in\mathbb{N}}$ be a bounded and decreasing sequence.

Prove that $(u_n)_{n\in\mathbb{N}}$ is convergent and that the greatest lower bounds of the set $\{u_n, n \in \mathbb{N}\}$ is the limit of $(u_n)_{n\in\mathbb{N}}$.

Exercise 03:

1- Using the mathematical definition of the convergence, prove that:

$$\lim_{\mathrm{n}\to+\infty} \frac{1}{n^2+1} = 0، \lim_{\mathrm{n}\to+\infty} \frac{2^n-1}{2^{n-1}} = 2، \lim_{\mathrm{n}\to+\infty} \frac{3n+1}{2n+3} = \frac{3}{2}،$$

$$\lim_{\mathrm{n}\to+\infty} (1-n) = -\infty.$$

2- Study the nature of the sequences below whose general terms are given by:

$$\forall n \in \mathbb{N}^*: u_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}.$$

$$\forall n \in \mathbb{N}^*: v_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}.$$

$$\forall n \in \mathbb{N}: w_n = \frac{1+3+9+\cdots+3^n}{3^{n+1}}.$$

Exercise 04: Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers defined as :

$$u_1 = \frac{1}{2}, and\ u_{n+1} = {u_n}^2 + \frac{3}{16}\ , for\ all\ n \in \mathbb{N}^*.$$

1- Prove that : $\forall n \in \mathbb{N}^*: \frac{1}{4} \leq u_n \leq \frac{3}{4}$.

2- Show by induction that $(u_n)_{n\in\mathbb{N}}$ is a decreasing sequence.

3- Deduce the convergence of this sequence and compute it limit.

Exercise 05: Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers defined as :

$$u_0 = 1, and\ u_{n+1} = \frac{2u_n}{2+3u_n}\ , for\ all\ n \in \mathbb{N}.$$

1- Calculate u_1 and u_2.
2- Is the sequence $(u_n)_{n\in\mathbb{N}}$ arithmetic? (Justify your response).
3- We assume that for every natural number of $\mathbb{N}$ such us: $u_n \neq 0$, we define the sequence $(v_n)_{n\in\mathbb{N}}$ as follows : $\forall n \in \mathbb{N}: v_n = \frac{1}{u_n}$.
4- Prove that the sequence $(v_n)_{n\in\mathbb{N}}$ is arithmetic, then determine its first term and its base.
5- Find the value of v_n in terms of n, then deduce the value of u_n in terms of n.
6- Study the variations of the sequence $(u_n)_{n\in\mathbb{N}}$ (i.e. the monotonicity of $(u_n)_{n\in\mathbb{N}}$).
7- Prove that: $\forall n \in \mathbb{N}: 0 < u_n \leq 1$.

Exercise 06: Study the nature of the sequences below whose general terms are given by :

$$u_{n+1} = \frac{1}{u_n},\ \ u_0 \neq 0\ is\ given\ , v_n = \frac{2^n}{n!}\ and$$

$$w_{n+1} = \left(1 + \frac{(-1)^n}{n+2}\right) w_n, w_0\ is\ given.$$

Indication: Prove that: $u_{2k} = u_0$ and that $u_{2k+1} = \frac{1}{u_0}$.

Prove that: $w_{2k} = w_0$ and that $w_{2k+1} = \frac{2k+3}{2k+2} w_0$.

<u>Exercise 07:</u> Let (u_n) and (v_n) be two sequences defined as :

$$u_0 = 1, and\ u_{n+1} = \frac{u_n}{1+u_n}\ , and\ v_n = \frac{1}{u_n},\qquad for\ all\ n \in \mathbb{N}.$$

1- Calculate u_1 and v_0.
2- Prove that (v_n) is arithmetic, then determine its first term and its base.
3- Write v_n in terms of n, then deduce the value of u_n in terms of n.
4- Calculate $S_n = \sum_{k=0}^{n} v_k = v_0 + v_1 + \cdots + v_n$.

Exercise 08 : Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers defined as :

$$u_0 = 5, and\ 3u_{n+1} = u_n + 4\ , for\ all\ n \in \mathbb{N}.$$

1- Calculate u_1 and u_2.
2- Prove that : $\forall n \in \mathbb{N}: u_n \geq 2$.
3- Show that (u_n) is decreasing.

4- Conclude that (u_n) is convergent and calculate its limit.

5- Put : $v_n = u_n - 2, \forall n \in \mathbb{N}$.

a. Prove that: (v_n) is a geometric sequence, then determine its first term and its base.

b. Conclude the value of v_n in terms of n.

6- Let: $T_n = \sum_{k=0}^{n} u_k = u_0 + u_1 + \cdots + u_n$ and

$S_n = \sum_{k=0}^{n} v_k = v_0 + v_1 + \cdots + v_n$.

Find the expression of S_n , deduce then the value of T_n in terms of n.

Exercise 09: Let (u_n) be a numerical sequence defined as: $\forall n \in \mathbb{N}^*: u_n = \frac{2^n}{n^2}$.

1- Calculate u_1 , u_2, u_3, u_4 and u_5.

2- Is the sequence $(u_n)_{n\in\mathbb{N}}$ monotonic?

3- Solve in $\mathbb{N}$ inequality : $n^2 - 2n - 1 \geq 0$.

4- Study the variation sens of the sequence $(u_n)_{n\in\mathbb{N}}$ starting from order 3.

5- Calculate u_{181} and u_{182}, then derive the natural number n_0 that satisfies that: $u_{n_0} \geq 10^{50}$.

6- Prove that : $\forall n \geq n_0: u_n \geq 10^{50}$.

Exercise 10 : Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two numerical sequences defined as: $\forall n \in \mathbb{N}: u_n = \sum_{k=0}^{n} \frac{1}{k!}$ ، $v_n = u_n + \frac{1}{n!}$.

1- Prove that $(u_n)_{n\in\mathbb{N}}$ is strictly increasing and that $(v_n)_{n\in\mathbb{N}}$ is strictly decreasing starting from a certain order n.

2- Conclude that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to the same limit l.

3- Prove that this limit is a an irrational limit ($\mathrm{l} \notin \mathbb{Q}$).

Exercise 11: Let $(u_n)_{n\in\mathbb{N}}$ be a numerical sequence defined as

$$\forall n \in \mathbb{N}^*: u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}.$$

1- Prove that the extracted sequences $(u_{2n})$and (u_{2n+1}) are adjacent.

2- Conclude that (u_n) is convergent.

Exercise 12: Using Cauchy's criterion, study the nature of the following sequences, all of which are defined on $\mathbb{N}^*$.

$$.u_n = \sum_{k=1}^{n} \frac{1}{k}, v_n = \cos\frac{1}{n}, w_n = \sum_{k=1}^{n} \frac{\sin k}{k}$$