Series No. 06 2023/2024

Exercise 01: Let $f:\mathbb{R}\to\mathbb{R}$ be the function given by :

$$f(x)=\begin{cases} x^2+a & ,if \quad x>2 \\ ax-1 & ,if \quad x\leq 2 \end{cases}$$

Find the value of a such that f is continuous.

Exercise 02: Prove, using the definition, that each of the following function is continuous on the given domain:

1- $f(x)=ax+b, a,b\in\mathbb{R}, a\neq 0, on\ \mathbb{R}$.
2- $g(x)=\sqrt{x}, on\ [0,+\infty[$.
3- $h(x)=\frac{1}{x}, on\ \mathbb{R}^*$.

Exercise 03: Let $f,g:[0,1]\to\mathbb{R}$ be continuous functions and define

$$h(x)=\begin{cases} f(x) & ,if \quad x\in\mathbb{Q}\cap[0,1], \\ g(x) & ,if \quad x\in\mathbb{Q}^c\cap[0,1]. \end{cases}$$

a- Prove that if $f(a)=g(a)$, for some $a\in[0,1]$, then h is continuous at a.
b- **N.A:** Find all the points on $[0,1]$ at which the function h is continuous such that

$$h(x)=\begin{cases} x & ,if \quad x\in\mathbb{Q}\cap[0,1], \\ 1-x & ,if \quad x\in\mathbb{Q}^c\cap[0,1]. \end{cases}$$

Exercise 04: Prove the extension by continuity of the following functions:

$$f(x)=\begin{cases} x^2-1 & ,if\ x<2 \\ \frac{3}{2}x & ,if\ x>2 \end{cases}, \qquad g(x)=\frac{1}{1+\mathrm{e}^{\frac{1}{x-1}}}, \qquad h(x)=\frac{\tan ax}{\sin bx}, a,b\in\mathbb{R}^*.$$

Exercise 05: 1- Prove, using the definition, that if $f,g:I\to\mathbb{R}$ are continuous at $x_0\in I$, then the function $f-g$ is continuous at x_0.

2- Deduce by absurd reasoning that, if a function h is not continuous at point $x_0\in I$, then the function h-g is not continuous at x_0.

Exercise 06: 1- Let f,g be two functions continuous on $[a,b]$. Suppose $f(a)<g(a)$ and $f(b)>g(b)$. Prove that there exists $x_0\in]a,b[$ such that $f(x_0)=g(x_0)$.

2- Prove that the equation $\mathrm{e}^x=-x$ has at least one solution in $\mathbb{R}$.

Exercise 07: Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=\frac{x^2}{1+x^2}$.

Prove that f is uniformly continuous on $\mathbb{R}$.

Exercise 08: Let E be an open subset of $\mathbb{R}$ and let f be defined on E.

1- If f is differentiable at $x_0\in E$, then f is continuous at this point.
2- Prove that the absolute value function is continuous at 0, but it is not differentiable at this point.
3- What we conclude?

Exercise 09: Let $f,g:[a,b]\to\mathbb{R}$ be two continuous functions, differentiable on $]a,b[$.

1- Show that there exists c in $]a,b[$ such that:

$$\big(f(b)-f(a)\big)g'(c)=\big(g(b)-g(a)\big)f'(c).$$

2− Assume that $f(a)=g(a)=0$, and $\lim_{x\to a}\frac{f'(x)}{g'(x)}=l$. Prove that $\lim_{x\to a}\frac{f(x)}{g(x)}=l$.

Exercise 10: Compute the following limits by using l'Hôpital's Rule:

$$1)\ \lim_{x\to 0}\left(\frac{1+x}{x}-\frac{1}{\ln(1+x)}\right),\ 2)\ \lim_{x\to 0}\frac{sh\,x+\sin x-2x}{ch\,x+\cos x-2}.$$

Exercise 11: Prove the following inequality using the Mean Value Theorem.

$$\forall x>y>0: \frac{x-y}{x}<\ln x-\ln y<\frac{x-y}{y}.$$

Conclude that :

$$\forall x>1: \frac{x-1}{x}<\ln x<x-1.$$

Exercise 12: Compute the third derivative of the following function in two different ways:

$$f(x)=\frac{1}{1-x^2},\ g(x)=\mathrm{e}^{-x}sh\,x.$$

Exercise 13: Are the following functions differentiable?

$$f(x)=\begin{cases}x^2\mathrm{e}^{-x^2} & ,if\ |x|\le 1\\ 1 & ,if\ |x|>1\end{cases},\qquad g(x)=\begin{cases}x^2+2 & ,if\ \ x\ne 2\\ 6 & ,if\ \ x=2\end{cases}$$

$$and\ \ h(x)=\begin{cases}x\sin\frac{1}{x} & ,if\ \ x\ne 0\\ 0 & ,if\ \ x=0\end{cases}.$$