Chapter 07: usual functions (or Regular functions)

1/ Neberian logarithm function:

$\ln : \mathbb{R}_+^* \longrightarrow \mathbb{R}$

$x \longmapsto \ln x = y$

$\ln$ is defined and continuous on $\mathbb{R}_+^*$, strictly increasing and differentiable on $\mathbb{R}_+^*$, $\ln \in C^\infty(\mathbb{R}_+^*)$

and we have:

$\forall x \in \mathbb{R}_+^* : (\ln x)' = \frac{1}{x}$, also it has the following properties:

1) $(\ln(x))^n = \ln^n(x)$; 2/ $\ln(x^n) = n \ln(x)$; 3/ $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$;

4/ $\ln(xy) = \ln(x) + \ln(y)$.

$\forall x > 0 : \ln x \leq x - 1$; $\lim\limits_{x \to 0} \frac{\ln(1+x)}{x} = 1$, $\lim\limits_{x \to 0} \ln x = -\infty$, $\lim\limits_{x \to +\infty} \ln x = +\infty$

2) exponential function:

$e : \mathbb{R} \longrightarrow \mathbb{R}_+^*$

$x \longmapsto e^x = y$

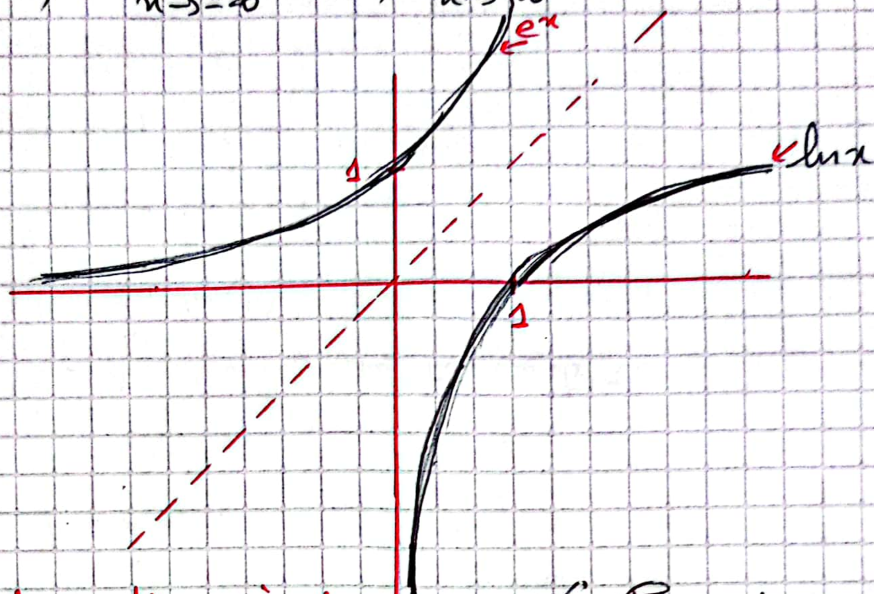
function exp is the inverse function of $\ln$, its has the same properties of $\ln$, that is, exp is strictly increasing, continuous and differentiable on $\mathbb{R}$ ($e \in C^\infty(\mathbb{R})$) and we have:

$\forall x \in \mathbb{R}: (e^x)' = e^x$, also it satisfies the following properties:

$e^0 = 1$; $\forall x \in \mathbb{R}: e^x \geq 1+x$; $e^{nx} = (e^x)^n$, $e^{x-y} = \dfrac{e^x}{e^y}$

$e^{x+y} = e^x \cdot e^y$,

$\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$; $\lim\limits_{x \to -\infty} e^x = 0$, $\lim\limits_{x \to +\infty} e^x = +\infty$



3) Logarithm function in base a: (where $a > 0$ and $a \neq 1$)

$$\log_a : \mathbb{R}_+^* \longrightarrow \mathbb{R}$$
$$x \longmapsto \log_a(x) = \frac{\ln x}{\ln a}$$

we have: $\log_a \in C^\infty(\mathbb{R}_+^*)$ and $\forall x \in \mathbb{R}_+^*: \left(\log_a(x)\right)' = \dfrac{1}{x \ln a}$

also it has the following properties:

$\log_a(xy) = \log_a(x) + \log_a(y)$,

$\log_b(x) = \log_b(a) \cdot \log_a(x)$,

$\log_{\frac{1}{a}}(x) = -\log_a(x)$.

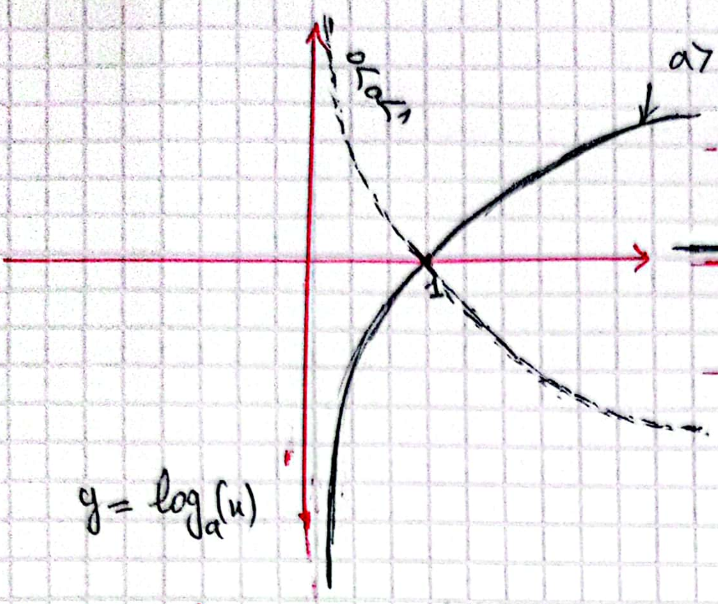
its inverse function is $\exp_a = e^{y \ln a}$, since:

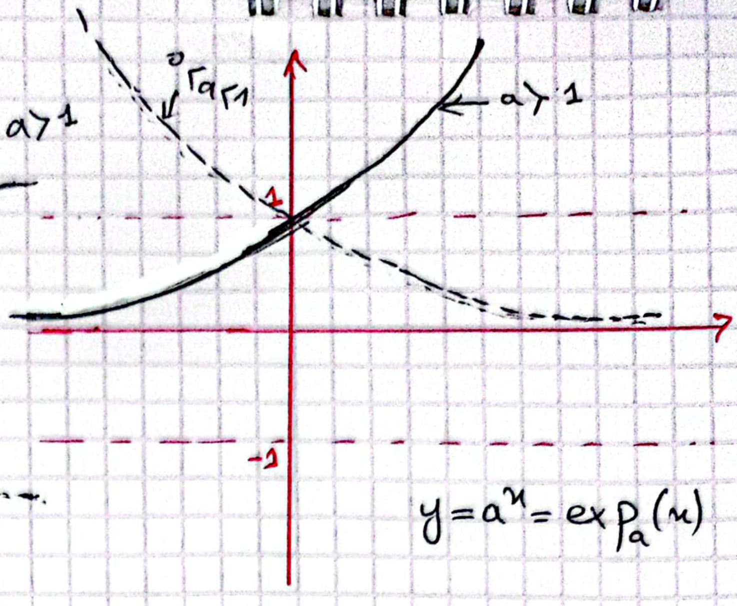
$y = \log_a x \iff y = \dfrac{\ln x}{\ln a} \iff \ln x = y \ln a \iff x = e^{y \ln a} = a^y$

and we have: $\exp_a : \mathbb{R} \longrightarrow \mathbb{R}_+^*$
$$x \longmapsto \exp_a(x) = e^{x \ln a} = a^x.$$

Note that: $\exp_a \in C^\infty(\mathbb{R})$, $(\exp_a)'(x) = \ln a \exp_a x = \ln a \cdot a^x$

$\exp_a(x+y) = (\exp_a(x))(\exp_a(y))$; $\exp_a(-x) = \dfrac{1}{\exp_a(x)}$, $\exp_{\frac{1}{a}}(x) = \dfrac{1}{\exp_a(x)} = \exp_a(-x)$





4/ Power functions is any function defined as:

$\forall x \in \mathbb{R}_+^* : \forall \alpha \in \mathbb{R} : P_\alpha(x) = x^\alpha = e^{\alpha \ln x}$ / $P_\alpha : \mathbb{R}_+^* \longrightarrow \mathbb{R}$, $x \longmapsto P_\alpha(x) = x^\alpha$

we have:

P_α: is a bijection (continuous and strictly increasing) on $\mathbb{R}_+^*$ for $\alpha > 0$.

P_α: is a bijection (" " " decreasing) on $\mathbb{R}_+^*$ for $\alpha < 0$.

P_α is differentiable on $\mathbb{R}_+^*$ and $(P_\alpha(x))' = \alpha x^{\alpha-1}$, $\forall x \in \mathbb{R}_+^*$

P_α has the following properties:

$\forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}_+^* : \lim\limits_{x \to 0^+} x^\beta |\ln x|^\alpha = 0$, $\lim\limits_{x \to +\infty} \frac{(\ln x)^\alpha}{x^\beta} = 0$. $(\forall \beta \in \mathbb{R}_+^*)$

$\forall \alpha \in \mathbb{R}, \forall a > 1 : \lim\limits_{x \to -\infty} a^x |x|^\alpha = 0$, $\lim\limits_{x \to +\infty} \frac{x^\alpha}{a^x} = 0$.

in particular:

$\lim\limits_{x \to +\infty} \frac{\ln x}{x^\alpha} = 0 \ (\alpha > 0)$, $\lim\limits_{x \to +\infty} \frac{e^x}{x^\alpha} = +\infty$.

$\lim\limits_{x \to 0} x^\alpha \ln x = 0 \ (\alpha > 0)$, $\lim\limits_{x \to +\infty} \frac{x^\alpha}{\log_a(x)} = +\infty \ (\alpha > 0, a > 1)$