University of Jijel - Faculty of exact sciences and computer science - Mathematics department

Series No. 07 **2024/2025**

Exercise 01: Let $f, g: [a, b] \to \mathbb{R}$ be two continuous functions, differentiable on $]a, b[$, such that $g'(x) \neq 0,\ \forall x \in]a, b[$, and let $h(x) = f(x) - mg(x)$, with $m = \frac{f(b)-f(a)}{g(b)-g(a)}$ and $(a) \neq g(b)$.

1- Show that there exists c in $]a, b[$ such that : $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

2- Assume that $f(a) = g(a) = 0$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = l$. Prove that $\lim_{x\to a} \frac{f(x)}{g(x)} = l$.

Exercise 02: Compute the following limits by using l'Hôpital's Rule:

$$1)\ \lim_{x\to -\infty} \left(\frac{x^2}{e^{1-x}}\right), 2)\ \lim_{x\to 1^+} (x-1)\tan\left(\frac{\pi}{2}x\right), 3)\ \lim_{x\to +\infty} \frac{2x+4e^x}{2-e^x}, 4)\ \lim_{x\to 0^+} [\cos(2x)]^{\frac{1}{x^2}}.$$

Exercise 03: Prove the following inequality using the Mean Value Theorem.

$$\forall x > y: \frac{x-y}{1+x^2} < \arctan x - \arctan y < \frac{x-y}{1+y^2}.$$

Conclude that :

$$\forall x > 0: \frac{x}{1+x^2} < \arctan x < x.$$

Exercise 04: Compute the fourth derivative of the following functions in two different ways:

$$f(x) = \frac{1}{1-x^2}, g(x) = \arcsin x\ sh\ x.$$

Exercise 05: Are the following functions differentiable? If so, provide their derivatives:

$$f(x) = \begin{cases} 2x+1 & , if\ x \leq 1 \\ x^2+2 & , if\ x > 1 \end{cases}, \quad g(x) = \begin{cases} x^2+1 & , if\ x \neq 2 \\ 6 & , if\ x = 2 \end{cases}$$

$$and\ \ h(x) = \begin{cases} x\cos\frac{1}{x} & , if\ x \neq 0 \\ 0 & , if\ x = 0 \end{cases}.$$

Exercise 06: 1) By using the development of Macc-Laurin with remainder of Lagrange for $n = 2$, prove that, for all $x > 0$, we have :

- ❖ $x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$.
- ❖ $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} < e^x < 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}e^x$.

2) Deduce the value of the limit : $\lim_{x\to 0} \frac{e^x - \ln(1+x) - 1}{x^2}$.

3) Check your result by applying the l'Hôpital's Rule.

Exercise 07: Let h be a function defined as : $h(x) = 2\sqrt{x} - \ln x$, we want to compute the following limit $\lim_{x\to+\infty} \frac{\ln x}{x}$.

1) Find the definition domain of h , denoted D_h .
2) Prove that h has an extreme point $\bar{x}$ on D_h . Give its type and then deduce the sign of h .
3) Show that : $\forall x \geq 1: 0 \leq \frac{\ln x}{x} \leq \frac{2}{\sqrt{x}}$.
4) Conclude the value of the limit : $\lim_{x\to+\infty} \frac{\ln x}{x}$.