

Chapter 04, 2nd-order ordinary differential equations with constant coefficients (2nd-order ODE)

Definition: we call second-order differential equation with constant coefficients any equation of the form:

$$ay'' + by' + cy = f(x), \quad a \neq 0 \quad \dots (E)$$

where a, b and c are real constants and f is a continuous function.

Remark: If $f(x) = 0$, then (E) is called homogeneous equation. else (E) is a non-homogeneous equation.

Theorem: Let y_p is a particular solution of non-homogeneous equation (E) and y_h is a general solution of homogeneous equation of (E), then $y = y_p + y_h$ is the general solution of non-homogeneous equation (E).

Method of solution of equation: $ay'' + by' + cy = f(x), \quad a \neq 0$

1/ homogeneous solution (y_h) or (solution without second member)

$$\text{Let } ay'' + by' + cy = 0 \quad \dots (H)$$

we assume that $y = e^{rx}$, then $y' = r e^{rx}$ and $y'' = r^2 e^{rx}$, $r \in \mathbb{R}$

(H) becomes:

$$(ar^2 + br + c)e^{rx} = 0 \Rightarrow ar^2 + br + c = 0 \quad (\text{since } e^{rx} > 0, \forall r \in \mathbb{R})$$

Now, finding a solution of (H) is equivalent to find a solution of characteristic equation $ar^2 + br + c = 0 \quad \dots (1)$

by computing the value of $\Delta = b^2 - 4ac$, we distinguish three cases:

a/ If $\Delta > 0$, hence equation (1) has two real solutions r_1 and r_2 where $r_1 \neq r_2$. In this case the homogeneous solution of equation (H) is written as follows:

$$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \text{ with } C_1, C_2 \text{ are two real constants.}$$

b/ If $\Delta = 0$, hence equation (1) has a real multiple solution

r_1 . In this case, the homogeneous solution of equation (H) is written as follows:

$$y_h = (C_1 x + C_2) e^{rx}, \text{ where } C_1 \text{ and } C_2 \text{ are two real constants.}$$

c/ If $\Delta < 0$, then equation (H) has two complex solutions r_1 and r_2 where $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R}$.

In this case the homogeneous solution of equation (H) is written as follows:

$$y_h = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x), C_1, C_2 \in \mathbb{R} \text{ are constants.}$$

2/ particular solution (y_p) (or nonhomogeneous solution or solution with second member)

The solution to differential equations with a second side (member) depends on the nature of its second side (member) $f(x)$.

a/ The second side is the product of a polynomial with an exponential function:

Theorem 1: Let (E) be the following differential equation

$$ay'' + by' + cy = p_n(x) e^{\lambda x} \dots (E).$$

The solution to the equation (E) with the second side is as follows:

$$y_p = p_m(x) e^{\lambda x}, \text{ where } p_m \text{ is a polynomial of degree:}$$

a/ $\deg(p_m) = \deg(p_n)$, if λ isn't a root of the characteristic equation.

b/ $\deg(p_m) = \deg(p_n) + 1$, if λ is a simple root of the characteristic equation.

c/ $\deg(p_m) = \deg(p_n) + 2$, if λ is a double root of the characteristic equation.

Example: $y'' - y' - 2y = 3x e^{2x} \dots (E)$

1/ Homogeneous solution (y_h):

Firstly, we solve the homogeneous equation: $y'' - y' - 2y = 0 \dots (H)$

Assume that: $y = e^{rx} \Rightarrow y' = r e^{rx} \Rightarrow y'' = r^2 e^{rx}$

Hence, (H) becomes:

$$(r^2 - r - 2) e^{rx} = 0 \Leftrightarrow r^2 - r - 2 = 0. \quad (1) \text{ (since } e^{rx} > 0, \forall r \in \mathbb{R}).$$

Hence Δ of (1) equals to: $\Delta = 1+8 = 9 > 0$,

then equation (1) has two real solutions:

$$r_1 = \frac{1-3}{2} = -1 \text{ and } r_2 = \frac{1+3}{2} = 2.$$

Therefore, the general solution of homogeneous equation (H) is

$$y_H = C_1 e^{-x} + C_2 e^{2x}, \text{ where } C_1 \text{ and } C_2 \text{ are real constants.}$$

2/ particular solution (y_p): There, we have: $p_n(x) = 3x \wedge e^{\alpha x} = e^{2x}$.

$\Rightarrow y_p = P_m(x) e^{2x}$, where P_m is a polynomial of degree:

• $\deg(P_m) = \deg(P_n) + 1 = 1 + 1 = 2$ (since $\alpha = 2$ is a simple root of characteristic equation (1)), hence:

$P_m(x) = ax^2 + bx + c$, this implies

$$y_p = P_m(x) e^{2x} = (ax^2 + bx + c) e^{2x}$$

$$\Rightarrow y_p' = ((2a+2b)x + 2ax^2 + 2c + b) e^{2x}$$

$$\Rightarrow y_p'' = [4ax^2 + (8a+4b)x + (2a+4b+4c)] e^{2x}.$$

Replacing y_p , y_p' and y_p'' in (E), we obtain:

$$[4ax^2 + (8a+4b)x + (2a+4b+4c)] e^{2x} - [(2a+2b)x + 2ax^2 + 2c + b] e^{2x} - 2[ax^2 + bx + c] e^{2x} = 3x e^{2x}$$

by simplifying, we get:

$$6ax + 2a + 3b = 3x$$

In comparison, we find: $\begin{cases} 6a = 3 \Rightarrow a = \frac{1}{2} \\ 2a + 3b = 0 \Rightarrow b = -\frac{1}{3} \end{cases}$, we can take $c = 0$.

$$\text{Hence: } P_m(x) = \frac{1}{2}x^2 - \frac{1}{3}x$$

$$\text{and then: } y_p = \left(\frac{1}{2}x^2 - \frac{1}{3}x \right) e^{2x}.$$

Hence, the general solution of the equation (E) is:

$$y = y_p + y_H = C_1 e^{-x} + C_2 e^{2x} + \left(\frac{1}{2}x^2 - \frac{1}{3}x \right) e^{2x}, \quad C_1, C_2 \in \mathbb{R}$$

b/ the second side is the product of a polynomial with ^{and} sinus/cosinus functions.

Theorem 2: Let (E) be the following differential equation:

$$ay'' + by' + cy = A_n(x) \cos nx + B_m(x) \sin nx \dots (E)$$

where A_m and B_m are polynomials of degree n and m , respectively.
The solution y_p to the equation (E) is as follows:

$$y_p = p_k(x) \cos nx + q_k(x) \sin nx$$

where p_k and q_k are polynomials of degree k such that:

$k = \max(n, m)$, if $i\beta$ isn't a root of the characteristic equation.

$k = \max(n, m) + 1$, if $i\beta$ is a simple root of the characteristic equation.

Example: $y'' + y = \cos x \dots (E) \Leftrightarrow y'' + y = 1 \cos x + 0 \sin x.$

Solution:

1/ y_h : we solve the homogeneous equation: $y'' + y = 0 \dots (H)$

Assume that: $y = e^{rx} \Rightarrow y' = r e^{rx} \Rightarrow y'' = r^2 e^{rx}.$

Hence (H) becomes: $r^2 + 1 = 0 \Leftrightarrow r = \pm i$

$$\Rightarrow y_h = (c_1 \cos x + c_2 \sin x) e^{0x}, \quad c_1, c_2 \in \mathbb{R}.$$

2/ y_p : $y_p = p_k(x) \cos x + q_k(x) \sin x$, where p_k and q_k are polynomials of degree k such that:

$k = \max(0, 0) + 1$ (since i is a simple root of equation ①).

Hence: $y_p = (a_1 x + b_1) \cos x + (a_2 x + b_2) \sin x.$

$$\Rightarrow y_p' = (a_1 + a_2 x + b_2) \cos x + (-a_1 x + b_1 + a_2) \sin x.$$

$$\Rightarrow y_p'' = (-a_1 x - b_1 + 2a_2) \cos x + (-a_2 x - 2a_1 - b_2) \sin x.$$

Replacing by y_p , y_p' and y_p'' in (E), we obtain:

$$(-a_1 x - b_1 + 2a_2) \cos x + (-a_2 x - 2a_1 - b_2) \sin x + (a_1 x + b_1) \cos x + (a_2 x + b_2) \sin x = \cos x + 0 \sin x$$

$$\Rightarrow 2a_2 \cos x - 2a_1 \sin x = \cos x + 0 \sin x$$

in comparison, we find:

$$\begin{cases} 2a_2 = 1 & \Rightarrow a_2 = \frac{1}{2} \\ -2a_1 = 0 & \Rightarrow a_1 = 0 \end{cases}$$

we can take $b_1 = b_2 = 0$, hence:

$y_p = \frac{1}{2} n \sin n$ is the particular solution of (E).

As conclusion: the general solution of equation (E) is as follows:

$$y = y_p + y_R = C_1 \cos n + C_2 \sin n + \frac{1}{2} n \sin n, \quad C_1, C_2 \in \mathbb{R}$$

c) The second side is the product of a polynomial with an exponential and sinus and/or cosinus functions:

Theorem 3: Let (E) be the following equation:

$$a y'' + b y' + c y = e^{\alpha x} (p_n(x) \cos \beta x + p_m(x) \sin \beta x) \quad \text{--- (E)}$$

where p_n and p_m are polynomials of degree n and m , respectively

The solution (y_p) of (E) is as follows:

$y_p = e^{\alpha x} (p_k(x) \cos \beta x + q_k(x) \sin \beta x)$, where p_k and q_k are polynomials of degree k such that:

$k = \max(n, m)$, if $\alpha \pm i\beta$ isn't a root of characteristic equation

$k = \max(n, m) + 1$, if $\alpha \pm i\beta$ is a simple root of characteristic equation

Example: Let: $y'' + y' + y = 2x \cos \sqrt{3} x + e^x \sin \sqrt{3} x$ --- (E)

1/ y_R : (Homogeneous solution): for finding y_R , we solve the equation: $y'' + y' + y = 0$. --- (H)

assume that: $y = e^{rx} \Rightarrow y' = r e^{rx} \Rightarrow y'' = r^2 e^{rx}$

Replacing in (H), we get:

$$(r^2 + r + 1) e^{rx} = 0 \Leftrightarrow r^2 + r + 1 = 0 \quad \text{--- (1)} \quad (\text{since } e^{rx} > 0, \forall x \in \mathbb{R})$$

$\Delta = 1 - 4 \times 1 = -3 < 0 \Rightarrow$ (1) has two complex solutions:

$$r_1 = \frac{-1 - i\sqrt{3}}{2} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \quad \text{and} \quad r_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\text{hence } y_R = e^{-\frac{1}{2}x} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right), \quad C_1, C_2 \in \mathbb{R}$$

2/ y_p : (particular solution) $y_p = y_{p1} + y_{p2}$, where

y_{p1} and y_{p2} are particular solutions of equations:

$$\frac{1}{2} y'' + y' + y = (2x \cos \sqrt{3} x + 0 \sin \sqrt{3} x) e^{0x} \quad \text{--- (E}_1) \quad (\because y_{p1})$$

$$\text{and } y'' + y' + y = (\sin \sqrt{3} x + 0 \cos \sqrt{3} x) e^x \quad \text{--- (E}_2) \quad (\because y_{p2})$$

respectively.

•) finding (y_{p_1}) of (E_1) : we have:

$$y_{p_1} = e^{0x} (p_k(x) \cos \sqrt{3}x + q_k(x) \sin \sqrt{3}x)$$

where p_k and q_k are polynomials of degree:

$$k = \max(\deg(p_k), \deg(q_k)) = \max(n, m) = \max(1, 0) = 1 \text{ (since } i\sqrt{3} \text{ isn't a root of equation characteristic (1))}$$

Hence: $p_k(x) = a_1x + b_1$ and $q_k(x) = a_2x + b_2$, thus:

$$y_{p_1} = (a_1x + b_1) \cos \sqrt{3}x + (a_2x + b_2) \sin \sqrt{3}x.$$

$$\Rightarrow y'_{p_1} = (\sqrt{3}a_2x + a_1 + \sqrt{3}b_2) \cos \sqrt{3}x + (-\sqrt{3}a_1x + a_2 + \sqrt{3}b_1) \sin \sqrt{3}x.$$

$$\Rightarrow y''_{p_1} = (-3a_1x + 2\sqrt{3}a_2 - 3b_1) \cos \sqrt{3}x - (3a_2x + 2\sqrt{3}a_1 + 3b_2) \sin \sqrt{3}x.$$

Replacing by y_{p_1} , y'_{p_1} and y''_{p_1} in (E_1) , we obtain:

$$[(-2a_1 + \sqrt{3}a_2)x + 2\sqrt{3}a_2 - 2b_1 + \sqrt{3}b_2] \cos \sqrt{3}x + [(-2a_2 + \sqrt{3}a_1)x + a_2 - 2\sqrt{3}a_1 - 2b_2 - \sqrt{3}b_1] \sin \sqrt{3}x = 2x \cos \sqrt{3}x$$

by matching:

$$\begin{cases} -2a_1 + \sqrt{3}a_2 = 2 \\ 2\sqrt{3}a_2 - 2b_1 + \sqrt{3}b_2 = 0 \\ -2a_2 + \sqrt{3}a_1 = 0 \\ -2\sqrt{3}a_1 - 2b_2 + a_2 - \sqrt{3}b_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -\frac{4}{7} \\ a_2 = \frac{2\sqrt{3}}{7} \\ b_1 = \frac{46}{49} \\ b_2 = \frac{12\sqrt{3}}{49} \end{cases}$$

$$\text{Hence: } \boxed{y_{p_1}(x) = \left(-\frac{4}{7}x + \frac{46}{49}\right) \cos \sqrt{3}x + \left(-\frac{2\sqrt{3}}{7}x + \frac{12\sqrt{3}}{49}\right) \sin \sqrt{3}x}$$

•) finding (y_{p_2}) of (E_2) , we have:

$y_{p_2} = e^x (p_k(x) \cos \sqrt{3}x + q_k(x) \sin \sqrt{3}x)$, where p_k and q_k are polynomials of degree:

$$k = \max(\deg(p_k), \deg(q_k)) = \max(0, 0) = 0 \text{ (since } 1 \pm i\sqrt{3} \text{ isn't a root of characteristic equation (1))}$$

Hence : $p_u(u) = a$ and $q_u(u) = b$, thus :

$$y_{p_2}(u) = e^u (a \cos \sqrt{3} u + b \sin \sqrt{3} u)$$

$$\Rightarrow y'_{p_2}(u) = [(a + b\sqrt{3}) \cos \sqrt{3} u + (b - a\sqrt{3}) \sin \sqrt{3} u] e^u.$$

$$\Rightarrow y''_{p_2}(u) = [(-2a + 2b\sqrt{3}) \cos \sqrt{3} u + (-2b - 2a\sqrt{3}) \sin \sqrt{3} u] e^u.$$

Replacing by this values in (E_2) , we obtain :

$$[(a + b\sqrt{3}) \cos \sqrt{3} u + (b - a\sqrt{3}) \sin \sqrt{3} u + (-2a + 2b\sqrt{3}) \cos \sqrt{3} u + (-2b - 2a\sqrt{3}) \sin \sqrt{3} u] e^u + [a \cos \sqrt{3} u + b \sin \sqrt{3} u] e^u = \sin \sqrt{3} u e^u$$

in comparison, we get :

$$\begin{cases} 3b\sqrt{3} = 0 \\ -3a\sqrt{3} = 1 \end{cases} \Rightarrow \begin{cases} b = 0 \\ a = -\frac{1}{3\sqrt{3}} \end{cases}$$

hence $y_{p_2} = e^u \left(-\frac{1}{3\sqrt{3}} \cos \sqrt{3} u \right) \Rightarrow y_p = y_{p_1} + y_{p_2}$

Therefore, the general solution of equation (E) is :

$$y = y_p + y_h = y_{p_1} + y_{p_2} + y_h$$

$$\Rightarrow y = \left(\frac{-4}{7} u + \frac{46}{49} \right) \cos \sqrt{3} u + \left(\frac{-2\sqrt{3}}{7} u + \frac{12\sqrt{3}}{49} \right) \sin \sqrt{3} u + \left(\frac{-1}{3\sqrt{3}} \cos \sqrt{3} u \right) e^u + e^{\frac{1}{2}u} \left(c_1 \cos \frac{\sqrt{3}}{2} u + c_2 \sin \frac{\sqrt{3}}{2} u \right), c_1, c_2 \in \mathbb{R}.$$

Exa p 1 : solve the following equations :

1/ $y'' + y' - 2y = 0 \Rightarrow y = y_h = c_1 e^{-2u} + c_2 e^u, c_1, c_2 \in \mathbb{R}.$

2/ $y'' + 9y = 0 \Rightarrow y = y_h = c_1 \cos 3u + c_2 \sin 3u, c_1, c_2 \in \mathbb{R}.$

3/ $y'' - 4y' + 4y = 0 \Rightarrow y = y_h = (c_1 + c_2 u) e^{2u}, c_1, c_2 \in \mathbb{R}.$

4/ $y'' - 5y' + 6y = (2u+1)e^u \Rightarrow y = y_h + y_p = c_1 e^{2u} + c_2 e^{3u} + (u+2)e^u, c_1, c_2 \in \mathbb{R}.$

5/ $y'' - 5y' + 6y = 3e^{2u} + e^{3u} \Rightarrow y = ?$

6/ $y'' - 2y' + y = e^u \Rightarrow y = ?$

7/ $y'' + y' + y = \cos u + \sin 3u \Rightarrow y = ?$

8/ $y'' + y' + y = e^{\frac{1}{2}u} \cos \frac{\sqrt{3}}{2} u \Rightarrow y = ?$

$$9/ y'' + y' + y = e^{\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + \sin x.$$

$$10/ y'' + y' = (x-1) \sin x$$

$$11/ y'' + y = \cos x + 2.$$

We can summarize the particular solutions of 2nd-order differential equation with constant coefficients (E) in the following table:
where $(ay'' + by' + cy = f(x)) \dots (E)$, with $a \neq 0$

form of the function $f(x)$	root of the characteristic eqn	form of y_p of (E)
$P_m(x)$	"0" isn't a root of characteristic equation $[ar^2 + br + c = 0]$	$q_m(x)$
	"0" is a root of degree "s" of characteristic equation	$x^s q_m(x)$
$P_m(x) e^{ax}$	"a" isn't a root of characteristic equation.	$q_m(x) e^{ax}$
	"a" is a root of degree "s" of characteristic equation	$x^s q_m(x) e^{ax}$
$P_m(x) \cos \beta x + q_m(x) \sin \beta x$	" $\pm i\beta$ " isn't a root of characteristic equation	$P_k(x) \cos \beta x + q_k(x) \sin \beta x$
	" $\pm i\beta$ " is a simple root of characteristic equation	$x (P_k(x) \cos \beta x + q_k(x) \sin \beta x)$
$e^{\alpha x} (P_m(x) \cos \beta x + q_m(x) \sin \beta x)$	" $\alpha \pm i\beta$ " isn't a root of characteristic equation	$e^{\alpha x} (P_k(x) \cos \beta x + q_k(x) \sin \beta x)$
	" $\alpha \pm i\beta$ " is a simple root of characteristic equation	$x e^{\alpha x} (P_k(x) \cos \beta x + q_k(x) \sin \beta x)$

where P_n and q_m are polynomials of degree n and m , respectively.
and P_k and q_k are polynomials of degree $k = \max(n, m)$.