

Analysis 02 (1st level of Mathematic)

Chapter 01: indefinite integrals

indefinite integral, some properties of the indefinite integrals
integration methods: integration by change of variable,
integration by parts, integration of regular expressions,
integration of fractional functions.

Chapter 02: definite integrals

definite integral, some properties of the definite integrals,
integral function of its upper bound, Newton-Leibniz formula,
Cauchy-Schwarz integral, Darboux sums - conditions for
the existence of the integral, properties of Darboux sums,
integrability of monotonic continuous functions.

Chapter 03: first order differential equations

generalities, classification of first order differential equations,
equation with separable variables, homogenous equation,
linear equation, Bernoulli method, method of variable of
the Lagrange constant, Bernoulli equation, total differential
equation, Riccati equation.

Chapter 04: Second order differential equations with constant coefficients.

homogenous second order differential equation with constant
coefficients, non-homogenous second order differential equation
with constant coefficients, methods for solving 2nd order
differential equation with constant coefficients.

Chapter 01: indefinite integrals and the primitive

I) The primitives:

1) Definition: Let f be a function defined on the (interval) Domain D and F be a differentiable function on D . we say that F is a primitive function of f if and only if: $\forall x \in D: F'(x) = f(x)$.

Example: $F(x) = \frac{1}{3}x^3 + C$ is the primitive of $f(x) = x^2$, with $C \in \mathbb{R}$.
• $F(x) = \ln|x|$ is the primitive of $f(x) = \frac{1}{x}$, $x \in \mathbb{R}_+^*$.

2) the set of primitive functions and indefinite integrals:

Definition: we call the undefined or indefinite integral of the function f defined on the interval $[a, b]$, which is denoted by $\int f(x) dx$, all (every) primitive function of f , and we have:
 $\int f(x) dx = F(x) + C$ with $F'(x) = f(x)$

Theorem: If the function f admits a primitive function F on the domain D , then the functions $F + C$ (with C is a constant) are all primitive functions of f on D , since:

$$\forall x \in D: (F + C)'(x) = F'(x) + 0 = f(x)$$

• the set of primitive functions of f on D is denoted by $\int f(x) dx$ where: $\int f(x) dx = F(x) + C$, $C = \text{constant} \in \mathbb{R}$.

Example:

$$I_1 = \int_D \frac{1}{x} dx, D = \mathbb{R}^* \Rightarrow I_1 = \ln|x| + C$$

$$I_2 = \int_D \cos x dx, D = \mathbb{R} \Rightarrow I_2 = \sin x + C$$

3) properties of the indefinite integrals (or of the primitives)

Let f and g be two functions defined on the Domain D , then

$$1) \int \lambda f(x) dx = \lambda \int f(x) dx, \forall \lambda \in \mathbb{R}.$$

$$2) \int (f+g)(x) dx = \int f(x) dx + \int g(x) dx.$$

4/ integration methods

1-4/ integration by parts: If f and g be two functions defined and differentiable on the same domain D , then according to the properties of differentiation, we have

$$(fg)'(x) = f'(x)g(x) + g'(x)f(x)$$

by integration, we get:

$$(fg)(x) = \int (fg)'(x) dx = \int f'(x)g(x) dx + \int g'(x)f(x) dx$$

in another meaning:

$$\boxed{\int f'(x)g(x) dx = (fg)(x) - \int g'(x)f(x) dx} \quad (*)$$

this expression is called the expression of integration by parts.

Remark: To calculate the integral or the primitive function of the function $u = f'g$ by using the integration by parts, we must choose the functions f' and g well, so that the integral $\int f'(x)g(x) dx$ is easy to calculate.

Illustrative examples

1/ $I_1 = \int x^n \ln x dx, n \in \mathbb{Z} - \{-1\}$

put: $\begin{cases} g(x) = \ln x \\ f'(x) = x^n dx \end{cases} \Rightarrow \begin{cases} g'(x) = \frac{1}{x} dx \\ f(x) = \frac{x^{n+1}}{n+1} \end{cases}$

using $(*)$, we get:

$$I_1 = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int \frac{x^{n+1}}{x} dx = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n dx$$

$$\Rightarrow \boxed{I_1 = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C}$$

2/ $I_2 = \int \arcsin x dx$

put: $\begin{cases} g(x) = \arcsin x \\ f'(x) = 1 dx \end{cases} \Rightarrow \begin{cases} g'(x) = \frac{dx}{\sqrt{1-x^2}} \\ f(x) = x \end{cases}$

Using $(*)$, we obtain

$$I_2 = x \arcsin x - \int \frac{x dx}{\sqrt{1-x^2}} = x \arcsin x - \frac{1}{2} \int 2x (1-x^2)^{-\frac{1}{2}} dx$$

$$\Rightarrow \boxed{I_2 = x \arcsin x + \frac{1}{2} (1-x^2)^{\frac{1}{2}} + C = x \arcsin x + \sqrt{1-x^2} + C}$$

1.4-4: Some integrals that allow us to use the mtd of integration by parts
a) the product of a polynomial with an exponential function
to calculate the integral $I = \int p_n(u) e^{au} du$, where p_n is a polynomial function of the degree n (i.e., $\deg(p_n(u)) = n$)
we can use the method of integration by parts, and in this case we always put:

$$\begin{cases} g(u) = p_n(u) \\ f'(u) = e^{au} du \end{cases} \Rightarrow \begin{cases} g'(u) = p_{n-1}(u) \\ f(u) = \frac{1}{a} e^{au}, a \neq 0. \end{cases}$$

using (*), I become

$$I = \left[\frac{1}{a} p_n(u) e^{au} \right] - \frac{1}{a} \int e^{au} p_{n-1}(u) du$$

Then, we integrate I_1 by parts again, and we always set g as the polynomial function and f' as the exponential function, and we repeat the process n times (i.e., we integrate by parts n times).

Illustrative example: calculate the integral:

$$I = \int (u^2 - 3u + 2) e^{-u} du$$

we have: $p_2(u) = u^2 - 3u + 2$, in this integral we integrate by parts two times (twice), where we always put:

$$\begin{cases} g(u) = u^2 - 3u + 2 \\ f'(u) = e^{-u} du \end{cases} \Rightarrow \begin{cases} g'(u) = (2u - 3) du \\ f(u) = -e^{-u}. \end{cases}$$

Using (*) we get:

$$I = -(u^2 - 3u + 2) e^{-u} + \int (2u - 3) e^{-u} du$$

Now, we integrate I_1 by parts, where

$$\begin{cases} g(u) = 2u - 3 \\ f'(u) = e^{-u} du \end{cases} \Rightarrow \begin{cases} g'(u) = 2 du \\ f(u) = -e^{-u} \end{cases}$$

using again (*), we get

$$I_1 = -(2u - 3) e^{-u} + 2 \int e^{-u} du = -(2u - 3) e^{-u} - 2e^{-u} + C,$$

replacing the value of I_1 in I , we get

$$I = -(n^2 - 3n + 2)e^{-n} - (2n - 3)e^{-n} - 2n + C$$

hence:

$$\boxed{I = e^{-n}(-n^2 + n - 1) + C} \quad \text{--- (2)}$$

2nd mtd of integration:

$$\int (n^2 - 3n + 2)e^{-n} dn = (an^2 + bn + c)e^{-n} + C_1 \quad \text{--- (1)}$$

hence by derivating (1) we get:

$$(n^2 - 3n + 2)e^{-n} = [(an^2 + bn + c)e^{-n} + C_1]' = (2an + (2a - b)n + b - c)e^{-n}$$

$$\Rightarrow \begin{cases} -a = 1 \\ 2a - b = -3 \\ b - c = 2 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 1 \\ c = -1 \end{cases}$$

replacing in (1), we get:

$$\boxed{\int (n^2 - 3n + 2)e^{-n} dn = (-n^2 + n - 1)e^{-n} + C_1}$$

which is the same as (2).

of the product of an exponential function with sinus or cos
to calculate the integral $I = \int \cos pn e^{ax} dx$ or $I = \int \sin pn e^{ax} dx$
we ~~can~~ use the integration by parts only two times.

Remark: Choosing f' and g in the first time is qualitative or arbitrary but in the second time it is related to what we chose in the first time.

Illustrative example: calculate the following integral

$$I = \int \sin 2n e^n dn$$

$$\begin{cases} f'(n) = e^n \\ g(n) = \sin 2n \end{cases} \Rightarrow \begin{cases} f(n) = e^n \\ g'(n) = 2 \cos 2n \end{cases}$$

using (*), we get:

$$I = \sin 2n e^n - 2 \int \cos 2n e^n dn$$

we integrate I_1 by parts again, there we must put:

$$\begin{cases} f'(u) = e^u \quad \Rightarrow \quad \begin{cases} f(u) = e^u \\ g'(u) = -2 \sin 2u \end{cases} \\ g(u) = \cos 2u \end{cases}$$

using (1), we get

$$I_1 = e^u \cos 2u + 2 \int \underbrace{e^u \sin 2u}_{I} du + C.$$

replacing the value I_1 in I , we obtain

$$\begin{aligned} I &= \sin 2u e^u - 2 I_1 \\ &= \sin 2u e^u - 2 [e^u \cos 2u + 2 I] + C. \end{aligned}$$

$$\Rightarrow I = \sin 2u e^u - 2 e^u \cos 2u - 4 I + C.$$

$$\Rightarrow 5 I = (\sin 2u - 2 \cos 2u) e^u + C.$$

$$\Rightarrow \boxed{I = \frac{1}{5} (\sin 2u - 2 \cos 2u) e^u + C}$$

2.4: the integrability change of variable:

Theorem: Let h be a function of the class C^1 defined from $[a, b]$ to $[a, b]$ and f be a continuous function on $[a, b]$ then for $t = h(x)$, we have:

$$\int_a^b f(h(x)) h'(x) dx = \int_{h(a)}^{h(b)} f(t) dt = \int_a^b f(t) dt \quad \text{--- (1)}$$

$h: [a, b] \rightarrow [a, b]$
 $x \mapsto h(x) = t$

in addition, if h is a bijective (function), then for $u = h(t)$ we have:

$$\int_a^b f(u) du = \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(t)) h'(t) dt = \int_a^b f(h(t)) h'(t) dt.$$

$h: [a, b] \rightarrow [a, b]$
 $t \mapsto h(t) = u$

therefore, if F is a primitive function of f , then $F(h)$ is a primitive function of $f(h) \cdot h'$, that is,

$$\int (f \circ h) h' = F \circ h + C = F(h) + C$$

proof: Since F is a primitive function of f then $F'(u) = f(u)$. Using the properties of differentiation of the compound/compos function, we obtain:

$(F \circ R)'(t) = F'(R(t)) h'(t) = f(R(t)) \cdot R'(t) = (f \circ h)(t) \cdot h'(t)$
 therefore: $(F \circ h)$ is the primitive function of $(f \circ h) R'$.
 Hence:

$$\int_a^B f(R(x)) h'(x) dx = [F \circ R]_a^B = F(R(B)) - F(R(a)) = \int_{R(a)}^{R(B)} f(t) dt.$$

which gives integral (1) of the theorem.

Illustrative examples

① by using the following change of variable $u = a \sin t$, $a > 0$ calculate the integral: $I = \int_0^a \sqrt{a^2 - u^2} du$ — (1)

$$u = 0 \Rightarrow a \sin t = 0 \Rightarrow \sin t = 0 \xrightarrow{\text{sin injection}} t = \arcsin 0 = 0 \Rightarrow \boxed{t = 0}$$

$$u = a \Rightarrow a \sin t = a \Rightarrow \sin t = 1 \xrightarrow{\text{sin injection}} t = \arcsin 1 = \arcsin(\sin \frac{\pi}{2}) = \boxed{\frac{\pi}{2}}$$

$$u = a \sin t \Rightarrow du = a \cos t dt$$

integral (1) becomes:

$$I = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2 t} (a \cos t) dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2 (1 - \sin^2 t)} a \cos t dt$$

$$= a^2 \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t} \cos t dt$$

$$= a^2 \int_0^{\frac{\pi}{2}} \cos^2 t dt = a^2 \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} dt$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2t) dt$$

$$= \frac{a^2}{2} \left[t + \frac{1}{2} \sin 2t \right]_0^{\frac{\pi}{2}} = \boxed{\frac{a^2 \pi}{2}}$$

② by using the change of variable $t = \cos u$, calculate

$$I = \int_0^\pi \sin^3 u du \quad \text{--- (2)}$$

$$u = 0 \Rightarrow t = \cos 0 = 1$$

$$u = \pi \Rightarrow t = \cos \pi = -1$$

$$t = \cos u \Rightarrow dt = -\sin u du.$$

integral (2) becomes: $I = \int_0^\pi \sin^2 u \sin u du$, thus

$$I = -\int_0^{\pi} \sin^2 u (-\sin u du) = -\int_0^{\pi} (1 - \cos^2 u) (-\sin u) du$$

$$= -\int_1^{-1} (1 - t^2) dt = \int_{-1}^1 (1 - t^2) dt = \left[t - \frac{t^3}{3} \right]_{-1}^1 = \left[\frac{4}{3} \right]$$

③ by using the change of variable $t = e^u$, calculate:

$$I = \int_1^2 \frac{du}{e^u + e^{-u}} \quad \text{--- ③}$$

$$u = 1 \Rightarrow t = e^1$$

$$u = 2 \Rightarrow t = e^2$$

$$t = e^u \Rightarrow dt = e^u du$$

hence integral ③ becomes:

$$I = \int_1^2 \frac{e^u du}{e^{2u} + 1} = \int_e^{e^2} \frac{dt}{t^2 + 1} = \operatorname{arctg} t \Big|_e^{e^2} = \boxed{\operatorname{arctg} e^2 - \operatorname{arctg} e}$$

Remark: Note that in the first example we stipulated (exig) that the function $f: \text{int}$ be bijective, while we didn't stipulate the ~~on the~~ in the last two examples.

II. Integration of regular expressions In the following table we give the integral of some usual functions.

$$1) \int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C, & \text{if } n \neq -1 \\ \ln |x| + C, & \text{if } n = -1 \end{cases}$$

$$2) \int e^{ax} dx = \frac{e^{ax}}{a} + C \quad (a \in \mathbb{R}^*)$$

$$3) \int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$

$$4) \int \frac{dx}{\sqrt{1-x^2}} = \operatorname{arcsin} x + C$$

$$5) \int \cos x dx = \sin x + C$$

$$6) \int \sin x dx = -\cos x + C$$

$$7) \int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$8) \int \frac{dx}{\sin^2 x} = -\operatorname{cotg} x + C$$

$$9) \int \frac{du}{\cos u} = \ln \left| \operatorname{tg} \left(\frac{u}{2} + \frac{\pi}{4} \right) \right| + C$$

$$10) \int \frac{du}{\sin u} = \ln \left| \operatorname{tg} \frac{u}{2} \right| + C$$

$$11) \int \operatorname{tg} u \, du = -\ln |\cos u| + C$$

$$12) \int \operatorname{ctg} u \, du = \ln |\sin u| + C$$

$$13) \int \frac{du}{1-u^2} = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C$$

$$14) \int \frac{du}{\sqrt{x^2+a}} = \ln \left| u + \sqrt{x^2+a} \right| + C, \quad a \neq 0$$

$$15) \int \operatorname{ch} u \, du = \operatorname{sh} u + C$$

$$16) \int \operatorname{sh} u \, du = \operatorname{ch} u + C$$

$$17) \int \frac{du}{\operatorname{ch}^2 u} = \operatorname{th} u + C$$

$$18) \int \frac{du}{\operatorname{sh}^2 u} = -\operatorname{cth} u + C$$

$$19) \int \frac{du}{\operatorname{ch} u} = 2 \operatorname{arctg} e^u + C$$

$$20) \int \frac{du}{\operatorname{sh} u} = \ln \left| \operatorname{th} \frac{u}{2} \right| + C$$

$$21) \int \operatorname{th} u \, du = \ln (\operatorname{ch} u) + C$$

$$22) \int \operatorname{cth} u \, du = \ln |\operatorname{sh} u| + C$$

*) particular cases:

$$1) \int f'(u) f^u(u) \, du = \begin{cases} \frac{f^{u+1}(u)}{u+1} + C, & \text{if } u \neq -1 \\ \ln |f(u)| + C, & \text{if } u = -1. \end{cases}$$

$$2) \int f'(u) e^{f(u)} \, du = e^{f(u)} + C$$

III - integration of rational functions.

1) Definitions:

Definition 0.1) we call a rational function all function of the form $\frac{p(u)}{q(u)}$, where $p(u)$ and $q(u)$ are two polynomials that it

coefficients in $\mathbb{R}$.

Example the functions $\frac{x^4 - 3x^3 + 5x^2 + x - 3}{6x^3 + 2x - 1}$ and $\frac{x+1}{2x^3 + x - 2}$ are rational functions.

Definition (2) (Simple elements)

a/ we call a simple element of the first type with degree $n \in \mathbb{N}^*$ all fractional function ^{that is} written in the form:

$$\frac{d}{(x-a)^n}, \text{ where } d \text{ and } a \text{ are constant coefficients in } \mathbb{R}.$$

its primitive is:

$$\int \frac{d}{(x-a)^n} dx = \begin{cases} d \ln|x-a| + C, & \text{if } n=+1 \\ \frac{-1}{n-1} \frac{d}{(x-a)^{n-1}} + C, & \text{if } n \neq +1 \end{cases}, \quad C, d \in \mathbb{R}.$$

Example each of the following functions is a simple element of the first type: $\frac{-1}{n+1}$, $\frac{5}{(x-3)^4}$ and $\frac{1}{(x+2)^3}$.

b/ we call a simple element of the second type with the degree $n \in \mathbb{N}^*$, all fractional function, that is written in the form $\frac{\alpha x + \beta}{(ax^2 + bx + c)^n}$, where $\alpha, \beta, a, b, c \in \mathbb{R}$.

and $b^2 - 4ac$ is always negative ($b^2 - 4ac < 0$).

example each of the following functions: $\frac{x+2}{(x^2+x+1)^2}$ and $\frac{x+1}{2x^2+x+1}$ is a simple element of the second type.

Definition (3) (pole of rational function)

Let $f(x) = \frac{p(x)}{(x-a)^\alpha q_1(x)}$, where $p(a) \neq 0$ and $q_1(a) \neq 0$

be a fractional function.

then a is called a pole of ^{the} degree α of f .

*/ If $\alpha = 1$, then a is called a simple pole.

*/ If $\alpha > 1$, then a is called a multiple pole of degree α .

Example: Let $f(u) = \frac{u^2 + 1}{(u-2)(u-1)(u-3)^2}$ be a fractional function then $u=1$ and $u=2$ are simple poles, while $u=3$ is a multiple pole of degree $n=2$.

2- Factoring a fractional function:

Theorem: Each fractional function $f(u) = \frac{p(u)}{q(u)}$ can be decomposed (factorized) into the sum of simple fractions of the first or second type.

Example 1: Let $f(u) = \frac{u^2 + 2}{(u+1)^3(u-2)}$ and $g(u) = \frac{1}{(u-1)(u-2)}$ be

two fractional functions, then

$$f(u) = \frac{a}{u+1} + \frac{b}{(u+1)^2} + \frac{c}{(u+1)^3} + \frac{d}{u-2}$$

st mtd:

To find the value of the constants a, b, c and d , we standardize the denominators and compare with $f(u)$ to obtain

$$a = -\frac{2}{9}, \quad b = \frac{1}{3}, \quad c = -1 \quad \text{and} \quad d = \frac{2}{9}$$

in the same way, we get the value of a and b of g such that

$$g(u) = \frac{a}{u-1} + \frac{b}{u-2}, \quad \text{Here } a = -1 \text{ and } b = 1$$

2nd mtd:
or we follow the following method

* for the function g , we have:

$$g(u) = \frac{a}{u-1} + \frac{b}{u-2} = \frac{1}{(u-1)(u-2)} \quad \text{--- (*)}$$

step 1: computing the value of a:

we multiply both sides of (*) by $(u-1)$ to get

$$a + \frac{b(u-1)}{u-2} = \frac{1}{u-2} \quad \text{--- (**)}$$

then, we put $u=1$ in (**) to get the value $a = -1$

step 2: computing the value of b:

Now, multiplying both sides of (*) by $(u-2)$ to get

$$\frac{a(u-2)}{u-1} + b = \frac{1}{u-1} \quad \text{--- (**')}$$

then, put $u=2$ in (**) to get the value: $b = 1$

*/ for function f , we have:

$$f(u) = \frac{u^2 + 2}{(u+1)^3(u-2)} = \frac{a}{u+1} + \frac{b}{(u+1)^2} + \frac{c}{(u+1)^3} + \frac{d}{u-2} \quad (2)$$

step 1: computing the value of c :

we multiply both sides of (2) by $(u+1)^3$ and then, we put $u=-1$

we get: $c = -1$

step 2: computing the value of d :

we multiply both sides of (2) by $(u-1)$ and then, we put $u=2$

we get: $d = \frac{2}{9}$

step 3: computing the values of a and b .

for computing the values of a and b , we put in (2), $u=0$ and $u=1$ we obtain the following system:

$$\begin{cases} a + b = \frac{1}{9} \\ 2a + b = -\frac{2}{9} \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{3} \\ b = \frac{1}{3} \end{cases}$$

Example 2: Let $f(u) = \frac{u^3 + 1}{(u+2)^4}$ be a fractional function.

Decompose the function f into the sum of simple fractions of the first or second type. we have:

$$f(u) = \frac{u^3 + 1}{(u+2)^4} = \frac{a}{u+2} + \frac{b}{(u+2)^2} + \frac{c}{(u+2)^3} + \frac{d}{(u+2)^4}$$

1st mtd: to compute the values of a, b, c and d , we standardize the denominators and compare with $f(u)$ to get:

$a = 1, b = -6, c = 12$ and $d = -7$.

2nd mtd: to decompose functions of the form $\frac{p(u)}{(u-a)^r}$ where $\deg(p) \geq r$, we perform the process of successive division of the function $p(u)$ by $(u-a)$ until a constant quotient is obtained.

for example 2, we have: $f(u) = \frac{u^3 + 1}{(u+2)^4}$, where $\deg(u^3 + 1) = 3 \geq 1$. Here, we perform the process of successive division of the function $u^3 + 1$ by $(u+2)$ until a constant quotient is obtained, that is

$$\begin{array}{r|l}
 x^3+1 & n+2 \\
 \hline
 -x^3+2n^2 & x^2-2n+4 \\
 \hline
 -2n^2+1 & -x^2+2n \\
 \hline
 -2n^2-4n & -4n+4 \\
 \hline
 4n+1 & -4n-8 \\
 \hline
 4n+8 & 12 \\
 \hline
 -7 &
 \end{array}$$

Hence:
$$\frac{x^3+1}{(n+2)^4} = \frac{-7}{(n+2)^4} + \frac{12}{(n+2)^3} + \frac{6}{(n+2)^2} + \frac{1}{n+2}.$$

Example 03 Decompose the function f into the sum of simple fractions of the first and/or second type, such that

$$f(n) = \frac{4n^3 + 11n^2 + 12n + 8}{(n^2 + 2n + 3)^2(n+1)} = \frac{an+b}{(n^2+2n+3)^2} + \frac{cn+d}{n^2+2n+3} + \frac{e}{n+1}$$

after calculations and simplifications, we find:

$a = 1, b = -1, c = d = 0$ and $e = 1$

3) integrating a fractional functions:

to integrate the fractional function $f(n) = \frac{p(n)}{q(n)}$ we follow the following steps:

Step 1: if $\deg(p) \geq \deg(q)$ we perform the Euclidean division of the function $p(n)$ by the function $q(n)$ until an irreducible fraction is obtained (i.e., until we obtain a remainder of division $R(n)$ with a degree less than $p(n)$), that is,

$$\frac{p(n)}{q(n)} = E(n) + \frac{R(n)}{q(n)}, \quad \deg(R) < \deg(q).$$

Example 1 $f(n) = \frac{n^3 + n + 1}{n^2 + 1} = n + \frac{1}{n^2 + 1}.$

If $\deg(p) < \deg(q)$, we go to the next step (step 2):

Step 2: we decompose the function $\frac{p(n)}{q(n)}$ (or $\frac{R(n)}{q(n)}$) into the sum of simple fractions of the first and/or second type.

Step 3: Now, it remains to integrate the obtained simple elements gotten in step 2.

3-1: integrating simple elements of the first type:

we previously saw that: if the function f is written as $f(u) = \frac{a}{(u-a)^n}$ then

$$\int f(u) du = \int \frac{a du}{(u-a)^n} = \begin{cases} a \ln|u-a| + C, & \text{if } n=1 \\ \frac{a}{1-n} \frac{1}{(u-a)^{n-1}} + C, & \text{if } n \neq 1 \end{cases}$$

3-2: integrating simple elements of the second type: $n \neq 0, n \neq 1$

If the function f is written in the form: $\frac{\lambda x + \beta}{(ax^2 + bx + c)^n}$ with $\Delta = b^2 - 4ac < 0$, then we try to write the derivative of the denominator in the numerator as follows:

$$\begin{aligned} \frac{\lambda x + \beta}{(ax^2 + bx + c)^n} &= \frac{\frac{\lambda}{2a} (2ax + b) + \beta - \frac{\lambda}{2a} b}{(ax^2 + bx + c)^n} \\ &= \underbrace{\frac{\lambda}{2a}}_{\text{constant}} \frac{2ax + b}{(ax^2 + bx + c)^n} + \underbrace{\left(\beta - \frac{\lambda}{2a} b\right)}_{\text{constant}} \frac{1}{(ax^2 + bx + c)^n} \end{aligned}$$

we remark that $\frac{2ax + b}{(ax^2 + bx + c)^n}$ is of the form $\frac{U'(x)}{[U(x)]^n}$ and its primitive is:

$$\int \frac{2ax + b}{(ax^2 + bx + c)^n} dx = \begin{cases} \ln|ax^2 + bx + c| + K, & \text{if } n=1 \\ \frac{1}{1-n} \frac{1}{(ax^2 + bx + c)^{n-1}} + K, & \text{if } n \neq 1 \end{cases}$$

Hence, it remains to integrate the following term: $\frac{1}{(ax^2 + bx + c)^n}$. For this, we try to write $\frac{1}{ax^2 + bx + c}$ in the form $\frac{1}{1+t^2}$ as follows:

$$\begin{aligned} ax^2 + bx + c &= a \left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2} \right] \\ &= a \frac{(4ac - b^2)}{4a^2} \left[1 + \frac{\left(x + \frac{b}{2a}\right)^2}{\frac{4ac - b^2}{4a^2}} \right] \\ &= \frac{4ac - b^2}{4a} \left[1 + \left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right)^2 \right] \end{aligned}$$

we put now, $t = \frac{2ax + b}{\sqrt{4ac - b^2}}$, hence:

$$\int \frac{dx}{(ax^2 + bx + c)^n} = a \int \frac{dt}{(1+t^2)^n} = a I_n, \text{ with } I_n = \int \frac{dt}{(1+t^2)^n}$$

• if $n = 1$, then $I_n = I_1 = \int \frac{dt}{1+t^2} = \operatorname{arctg} t + C_1 = \operatorname{arctg} \left(\frac{2ax+b}{\sqrt{4ac-b^2}} \right) + C$

• if $n > 1$, then I_n is computed by the following recurrence relation
step:

$$(2n-1) I_n = \frac{t}{(1+t^2)^{n-1}} + (2n-3) I_{n-1}, \text{ with } I_1 = \operatorname{arctg} t + C$$

example: compute $I = \int \frac{n+1}{(n^2+n+1)^2} dn$

$$\frac{n+1}{(n^2+n+1)^2} = \frac{\frac{1}{2}(2n+1) + \frac{1}{2}}{(n^2+n+1)^2} = \frac{1}{2} \frac{2n+1}{(n^2+n+1)^2} + \frac{1}{2} \cdot \frac{1}{(n^2+n+1)^2} \quad (*)$$

Hence: $\int \frac{2n+1}{(n^2+n+1)^2} dn = \int (2n+1) (n^2+n+1)^{-2} dn =$

$$= - \left[(n^2+n+1)^{-1} \right] + C = \boxed{\frac{-1}{n^2+n+1} + C} \quad \text{--- (I)}$$

Now, it remains to integrate the term: $\frac{1}{(n^2+n+1)^2}$

$$\begin{aligned} n^2+n+1 &= \left(n+\frac{1}{2}\right)^2 + \frac{3}{4} \\ &= \frac{3}{4} \left(\frac{4}{3} \left(n+\frac{1}{2}\right)^2 + 1 \right) \\ &= \frac{3}{4} \left(\left[\frac{2}{\sqrt{3}} \left(n+\frac{1}{2}\right) \right]^2 + 1 \right) \\ &= \frac{3}{4} \left(\left(\frac{2n+1}{\sqrt{3}} \right)^2 + 1 \right) \end{aligned}$$

we put: $t = \frac{2n+1}{\sqrt{3}} \Rightarrow dt = \frac{2}{\sqrt{3}} dn \Rightarrow dn = \frac{\sqrt{3}}{2} dt$

Hence: $\int \frac{dn}{(n^2+n+1)^2} = \int \frac{\frac{dt}{2}}{\frac{3}{4}(t^2+1)} = \frac{4}{3} \cdot \frac{1}{2} \int \frac{dt}{(1+t^2)^2} = \frac{2}{3} \int \frac{dt}{(1+t^2)^2}$

thus: $\int \frac{dn}{(n^2+n+1)^2} = \frac{2}{3} I_2 \quad \text{--- (II)}$

from the relationship of recurrence, we have:

$$(2n-1) I_n = \frac{t}{(1+t^2)^{n-1}} + (2n-3) I_{n-1}, \text{ with } I_1 = \operatorname{arctg} t + C$$

for $n=2$: we get the value of I_2 as follows:

$$(2 \times 2 - 1) I_2 = \frac{t}{(1+t^2)^1} + (2 \times 2 - 3) I_1$$

$$\Rightarrow I_2 = \frac{1}{3} \left(\frac{t}{1+t^2} + \operatorname{arctg} t \right) + C \quad \text{--- (4*)}$$

replacing (**) in (I), we get:

$$\int \frac{dn}{(n^2+n+1)^2} = \frac{2}{3} \cdot \frac{1}{3} \left(\frac{1}{1+t^2} + \operatorname{arctg} t \right) + C \quad \text{--- (III)}$$

Now, replacing (I) and (III) in (*), we obtain the value of I, i.e.

$$\begin{aligned} \int \frac{n+1}{(n^2+n+1)^2} dn &= \frac{-1}{2(n^2+n+1)} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} \left[\frac{1}{1+t^2} + \operatorname{arctg} t \right] + C \\ &= -\frac{1}{2(n^2+n+1)} + \frac{1}{9} \left(\frac{1}{t^2+1} + \operatorname{arctg} t \right) + C / 6 = C + R \end{aligned}$$

$$\Rightarrow \int \frac{n+1}{(n^2+n+1)^2} dn = -\frac{1}{2(n^2+n+1)} + \frac{1}{9} \left[\frac{\frac{2n+1}{\sqrt{3}}}{1 + \left(\frac{2n+1}{\sqrt{3}} \right)^2} + \operatorname{arctg} \left(\frac{2n+1}{\sqrt{3}} \right) \right] + C$$

4) integrating the functions of the form $R(n, n^{\frac{m_1}{n_1}}, n^{\frac{m_2}{n_2}}, \dots, n^{\frac{m_i}{n_i}})$ where R is a fractional function written in terms of $n, n^{\frac{m_1}{n_1}}, \dots, n^{\frac{m_i}{n_i}}$ and $i \in \mathbb{N}^*$ [R is not a polynomial function].

In this case, to integrate the function R , we search the least common multiple of the numbers $n_1, n_2, \dots, n_i$ and let it be k , that is $\text{ppcm}(n_1, n_2, \dots, n_i) = k$;

then, we put $n = t^k$, in this case, the function R becomes a fractional function of the polynomials.

illustrative example: compute the following integral $I = \int \frac{n^{\frac{1}{2}}}{n^{\frac{3}{4}}+2} dn$

note that: $\text{ppcm}(2, 4) = 4$, hence $n = t^4 \Rightarrow dn = 4t^3 dt$.

$$\begin{aligned} \text{therefore: } I &= \int \frac{t^2}{t^3+2} \cdot 4t^3 dt = 4 \int \frac{t^5}{t^3+2} dt \\ &= 4 \int \left(t^2 - \frac{2t^2}{t^3+2} \right) dt \\ &= 4 \int t^2 dt - 8 \int \frac{t^2}{t^3+2} dt = 4 \int t^2 dt - \frac{8}{3} \int \frac{3t^2}{t^3+2} dt \\ &= 4 \frac{t^3}{3} - \frac{8}{3} \ln |t^3+2| + C \end{aligned}$$

$$\Rightarrow \int \frac{n^{\frac{1}{2}}}{n^{\frac{3}{4}}+2} dn = \frac{4}{3} \left[n^{\frac{3}{4}} - 2 \ln(n^{\frac{3}{4}}+2) \right] + C, \quad t = n^{\frac{1}{4}}$$

5/ integrating the functions of the form $\int R(x, \sqrt[n]{\frac{ax+b}{cx+d}}) dx$
 where $ad-bc \neq 0$

in this case, we put $t = \sqrt[n]{\frac{ax+b}{cx+d}}$ to obtain a rational function in terms of t that is easy to integrate by following the following steps:

$$t = \sqrt[n]{\frac{ax+b}{cx+d}} \Rightarrow t^n = \frac{ax+b}{cx+d} \Rightarrow x = \frac{dt^n - b}{a - ct^n} \Rightarrow dx = \frac{n(ad-bc)t^{n-1}}{(a-ct^n)^2} dt$$

$$\left[\text{since } dx = \frac{ndt^n + \cancel{t^{n-1}}(a-ct^n) + nct^{n-1}(dt^n-b)}{(a-ct^n)^2} dt \right]$$

$$= \frac{nt^{n-1}(ad - dt^n + dt^n - bc)}{(a-ct^n)^2} dt$$

therefore, the integration becomes as follows:

$$\int R(x, \sqrt[n]{\frac{ax+b}{cx+d}}) dx = \int R\left(\frac{dt^n - b}{a - ct^n}, t\right) \frac{n(ad-bc)t^{n-1}}{(a-ct^n)^2} dt$$

$$= \int f(t) dt$$

where f is a rational function in terms of t . that easy to integrate:

illustrative example: compute $\int \frac{dx}{1+\sqrt{1+x}}$

note that $a=b=d=n=1$ and $c=0$

we put: $t = \sqrt{1+x} \Rightarrow x = t^2 - 1 \Rightarrow dx = 2t dt$

$$\int \frac{dx}{1+\sqrt{1+x}} = \int \frac{2t dt}{1+t} = 2 \int \frac{t}{1+t} dt = 2 \left[\int \left(1 - \frac{1}{1+t}\right) dt \right]$$

$$= 2t - 2 \ln|1+t| + C$$

substituting the value of t , we find:

$$\boxed{\int \frac{dx}{1+\sqrt{1+x}} = 2\sqrt{1+x} - 2 \ln[1+\sqrt{1+x}] + C}$$

6/ integrating the functions of the form: $\int R(x, \sqrt{ax^2+bx+c}) dx$
 where $a, b, c \in \mathbb{R}$ and $a \neq 0$

• If $a=0$, we get a special case of integration of functions of

the previous form (form ⑤ i.e., $\int R(u, \sqrt{\frac{au+b}{cu+d}}) du$)

• If $a \neq 0$, the integral $\int R(u, \sqrt{ax^2+bx+c}) du$ can be transformed into an integral of fractional function by using a change in the Euler variables

a) First Euler substitution:

• If $\Delta = b^2 - 4ac > 0$, then the equation $ax^2+bx+c=0$ admits two different real roots, denoted by x_1 and x_2 .

in this case, we put:

$$\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)(x-x_2)} = (x-x_1)t$$

$$\text{hence: } a(x-x_1)(x-x_2) = (x-x_1)^2 t^2 \Leftrightarrow a(x-x_2) = (x-x_1)t^2$$

$$\Rightarrow x(a-t^2) = ax_2 - x_1 t^2 \Rightarrow x = \frac{ax_2 - x_1 t^2}{a-t^2} = \boxed{\frac{x_1 t^2 - ax_2}{t^2 - a}}$$

$$\text{therefore: } \boxed{x = \frac{x_1 t^2 - ax_2}{t^2 - a}} \Rightarrow \boxed{dx = \frac{2at(x_2 - x_1)}{(t^2 - a)^2} dt}$$

$$[\text{since: } dx = \frac{2x_1 t(t^2 - a) - 2t(x_1 t^2 - ax_2)}{(t^2 - a)^2} dt]$$

in this case, the function R becomes a fractional function in terms of t that easy to integrate.

Illustrative example: compute the value of the following integrals

$$I = \int \frac{dx}{\sqrt{x^2 - x - 2}}$$

$$\text{we have: } x^2 - x - 2 = (x+1)(x-2)$$

then, we put $\sqrt{(x+1)(x-2)} = (x+1)t$, hence:

$$x-2 = (x+1)t^2 \Rightarrow \boxed{x = \frac{2+t^2}{1-t^2}} \quad (*)$$

$$\Rightarrow dx = \frac{6t}{(1-t^2)^2} dt \quad [\text{since: } dx = \frac{2t(1-t^2) + 2t(2+t^2)}{(1-t^2)^2} = \frac{6t}{(1-t^2)^2} dt]$$

Thus, integral I becomes:

$$I = \int \frac{dx}{\sqrt{x^2 - x - 2}} = \int \frac{1}{(x+1)t} \cdot \frac{6t}{(1-t^2)^2} dt \stackrel{(*)}{=} \int \frac{1}{\left(\frac{2+t^2}{1-t^2} + 1\right)t} \cdot \frac{6t}{(1-t^2)^2} dt$$

$$I = \int \frac{6 dt}{(1-t^2)^2 \cdot \left(\frac{2+t^2+1-t^2}{1-t^2} \right)} = \int \frac{6 dt}{3(1-t^2)} = 2 \int \frac{dt}{(1-t^2)} = 2 \int \frac{dt}{(1-t)(1+t)}$$

$$\frac{2}{1-t^2} = \frac{a}{1-t} + \frac{b}{1+t} \quad \text{--- (I)}$$

• to obtain "a" we multiply both sides of (I) by $(1-t)$ and then we put $t=1$, we get: $a=1$

• to obtain "b" we multiply both sides of (I) by $(1+t)$ and then we put $t=-1$, we get: $b=1$

$$\text{hence: } I = \int \frac{2 dt}{1-t^2} = \int \frac{dt}{1-t} + \int \frac{dt}{1+t} = -\ln|1-t| + \ln|1+t| + C$$

$$\boxed{I = \ln \left| \frac{1+t}{1-t} \right| + C}$$

$$\text{but: } \sqrt{(n+1)(n-2)} = (n+1)t \Rightarrow t = \sqrt{\frac{n-2}{n+1}}$$

hence I becomes:

$$\boxed{I = \ln \left| \frac{1 + \sqrt{\frac{n-2}{n+1}}}{1 - \sqrt{\frac{n-2}{n+1}}} \right| + C}$$

Remark: we can put: $\sqrt{ax^2+bx+c} = (x-x_2)t \Rightarrow t = \sqrt{\frac{a(x-x_1)}{x-x_2}}$ and we obtain the same result. [since $\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)(x-x_2)}$]

2nd. mtd: we can put in the previous example:

$$\sqrt{(n+1)(n-2)} = (n-2)t \Rightarrow t = \sqrt{\frac{n+1}{n-2}}$$

$$\text{hence: } n+1 = (n-2)t^2 \Rightarrow n = \frac{1+2t^2}{t^2-1}$$

$$\Rightarrow dx = \frac{-6t}{(t^2-1)^2} dt$$

$$\begin{aligned} I \text{ becomes: } I &= \int \frac{dx}{\sqrt{x^2-x-2}} = \int \frac{-6t}{(t^2-1)^2} \cdot \frac{1}{\left(\frac{1+2t^2}{t^2-1} - 2 \right)t} dt \\ &= \int \frac{-6 dt}{(t^2-1)^2 \cdot \left(\frac{1+2t^2-2t^2+2}{t^2-1} \right)} = \int \frac{-6 dt}{3(t^2-1)} \\ &= -2 \int \frac{dt}{(t^2-1)} = \int \frac{2 dt}{1-t^2} \end{aligned}$$

which is the same as the previous integral.

b) Second Euler substitution:

If $\Delta < 0$ and $a > 0$ (of the equation ax^2+bx+c), then we put

$$\sqrt{ax^2+bx+c} = \pm \sqrt{a} x + t \quad (1)$$

Taking the sign + before $\sqrt{a}$, we have:

$$\sqrt{ax^2+bx+c} = (\sqrt{a} x + t)^2 = ax^2 + t^2 + 2\sqrt{a} xt$$

$$\Rightarrow bx + c = t^2 + 2\sqrt{a} xt \Rightarrow x = \frac{t^2 - c}{b - 2\sqrt{a} t} \quad (2)$$

replacing (2) in (1), we obtain

$$\sqrt{ax^2+bx+c} = \sqrt{a} x + t = \sqrt{a} \frac{t^2 - c}{b - 2\sqrt{a} t} + t$$

from (2), we have:

$$dx = \frac{2t(b - 2\sqrt{a}t) + 2\sqrt{a}t(t^2 - c)}{(b - 2\sqrt{a}t)^2} dt = \frac{2bt - 2\sqrt{a}t^2 - 2\sqrt{a}c}{(b - 2\sqrt{a}t)^2} dt$$

Hence, the function R becomes a fractional function in terms of t that easy to integrate it.

Example: compute $I = \int \frac{dx}{\sqrt{x^2+5}}$

we have: $\Delta < 0$ and $a = 1 > 0$, then we put

$$\sqrt{x^2+5} = x + t \quad (*) \Rightarrow t = \sqrt{x^2+5} - x \quad (2)$$

$$\Rightarrow x^2+5 = x^2+2xt+t^2 \Rightarrow x = \frac{5-t^2}{2t} \Rightarrow dx = \frac{-5-t^2}{2t^2} dt \quad (**)$$

replacing (*) in (*), we get:

$$\sqrt{x^2+5} = x + t = \frac{5-t^2}{2t} + t = \frac{5+t^2}{2t} \quad (3*)$$

Now, replacing (**) and (3*) in I , we get:

$$\begin{aligned} I &= \int \frac{\frac{-5-t^2}{2t^2} dt}{\frac{5+t^2}{2t}} = \int \frac{2t}{5+t^2} \cdot \frac{-5-t^2}{2t^2} dt = - \int \frac{5+t^2}{5+t^2} dt \\ &= - \int \frac{dt}{t} = - \ln|t| + C = - \ln|\sqrt{x^2+5} - x| + C \end{aligned}$$

hence:

$$I = \int \frac{dx}{\sqrt{x^2+5}} = - \ln|\sqrt{x^2+5} - x| + C$$

c/ Third Euler substitution

If $\Delta < 0$ and $c > 0$ of the equation $ax^2 + bx + c = 0$, then we put

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c} \Rightarrow x = \frac{2\sqrt{c}t - b}{a - t^2} \quad (\text{by taking the sign + before } \sqrt{c})$$

hence: $dx = \frac{2\sqrt{c}(a - t^2) + 2t(2\sqrt{c}t - b)}{(a - t^2)^2} dt = \frac{2\sqrt{c}a - 2bt}{(a - t^2)^2} dt$

$$\Rightarrow dx = \frac{2(a\sqrt{c} - bt)}{(a - t^2)^2} dt$$

from (*), we have: $\sqrt{ax^2 + bx + c} = xt + \sqrt{c} \Rightarrow$

$$\sqrt{ax^2 + bx + c} = \frac{(2\sqrt{c}t - b)t + \sqrt{c}}{a - t^2}$$

Therefore, the function R becomes a fractional function in terms of t that easy to integrate.

Example: compute the integral: $I = \int \frac{1 - \sqrt{x^2 + x + 1}}{x^2 \sqrt{x^2 + x + 1}} dx$

here, we have: $\Delta < 0$ and $c > 0$, then we put:

$$\sqrt{x^2 + x + 1} = xt + 1 \Leftrightarrow x^2 + x + 1 = x^2t^2 + 2xt + 1$$

$$\Leftrightarrow x^2(1 - t^2) + x(1 - 2t) = 0$$

$$\Leftrightarrow x((1 - t^2)x + (1 - 2t)) = 0$$

$$\Rightarrow \begin{cases} x = 0 \\ x = \frac{2t - 1}{1 - t^2} \end{cases} \quad (\text{rejected, } \Delta < 0, x \notin \mathbb{R}, x \in \mathbb{C})$$

$$x = \frac{2t - 1}{1 - t^2} \quad (\text{accepted})$$

$$\Rightarrow x = \frac{2t - 1}{1 - t^2} \Rightarrow dx = \frac{2t^2 - 2t + 2}{(1 - t^2)^2} dt$$

hence the integral becomes:

$$I = \int \frac{1 - \sqrt{x^2 + x + 1}}{x^2 \sqrt{x^2 + x + 1}} dx = \int \frac{1 - \frac{2t - 1}{1 - t^2}t + 1}{\left(\frac{2t - 1}{1 - t^2}\right)^2 \left(\frac{2t - 1}{1 - t^2}t + 1\right)} \cdot \frac{2t^2 - 2t + 2}{(1 - t^2)^2} dt$$

$$= \int \frac{(2t - 1)}{1 - t^2} \cdot \frac{2t^2 - 2t + 2}{(1 - t^2)^2} \cdot \frac{1}{\left(\frac{2t - 1}{1 - t^2}\right)^2 \left(\frac{2t^2 - t + 1 - t^2}{1 - t^2}\right)} dt$$

$$= \int \frac{(2t - 1)t(2t^2 - 2t + 2)(1 - t^2)^3}{(1 - t^2)^3(2t - 1)^2(t^2 - t + 1)} dt = \int \frac{t(2t^2 - 2t + 2)}{(2t - 1)(t^2 - t + 1)} dt$$

$$\Rightarrow I = \int \frac{2t}{2t-1} dt = \int \frac{2t-1+1}{2t-1} dt = \int dt + \int \frac{dt}{2t-1}$$

$$\Rightarrow I = t + \frac{1}{2} \ln|2t-1| + C.$$

$$\text{Hence : } I = \int \frac{1 - \sqrt{n^2+n+1}}{n^2 \sqrt{n^2+n+1}} dn = \frac{\sqrt{n^2+n+1} - 1}{n} + \ln \left| \frac{2\sqrt{n^2+n+1} - 2 - n}{n} \right|$$

$$\left[\text{Since : } \sqrt{n^2+n+1} = nt + 1 \Rightarrow t = \frac{\sqrt{n^2+n+1} - 1}{n} \right]$$

7/ Some other forms of integration:

a/ integration of the form $\int \cos^n x \sin^m x dx$, w. $n, m \in \mathbb{N}$

a-1/ If n is odd and m is even:

in this case, we put $t = \sin x \Rightarrow dt = \cos x dx$.

$$\text{Example: } I = \int \sin^2 x \cos^3 x dx.$$

we put $t = \sin x \Rightarrow dt = \cos x dx$, hence I becomes

$$I = \int \frac{\sin^2 x}{t^2} \frac{\cos^2 x}{1-t^2} \frac{\cos x dx}{dt} = \int t^2(1-t^2) dt = \int (t^2 - t^4) dt$$

$$\Rightarrow I = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

a-2/ If n is even and m is odd:

In this case, we put: $t = \cos x \Rightarrow dt = -\sin x dx$.

$$\text{Example: compute: } I = \int \cos^2 x \sin^3 x dx.$$

we put $t = \cos x \Rightarrow dt = -\sin x dx$, then I becomes

$$I = \int \frac{\cos^2 x}{t^2} \frac{\sin^2 x}{1-t^2} \frac{\sin x dx}{-dt} = - \int t^2(1-t^2) dt = - \int (t^2 - t^4) dt$$

$$= -\frac{t^3}{3} + \frac{t^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

a-3/ If n and m are odd together:

In this case, we can put $t = \sin x$ or $t = \cos x$ and follow the same previous steps.

$$\text{Example: compute } I = \int \cos^3 x \sin^3 x dx$$

we put: $t = \sin u \Rightarrow dt = \cos u du$, hence I becomes

$$I = \int \frac{\cos^2 u}{1-t^2} \frac{\sin^3 u}{t^3} \underbrace{\cos u du}_{dt} = \int (1-t^2) t^3 dt = \int (t^3 - t^5) dt$$

$$\Rightarrow I = \frac{t^4}{4} - \frac{t^6}{6} + C = \frac{\sin^4 u}{4} - \frac{\sin^6 u}{6} + C.$$

a- 4/9 n and m are even together:

In this case, we use the following trigonometric equalities: $\cos^2 u = \frac{1+\cos 2u}{2}$ and $\sin^2 u = \frac{1-\cos 2u}{2}$.

Example: compute: $I = \int \sin^2 u \cos^2 u du$

using the above equations, we get:

$$I = \int \sin^2 u \cos^2 u du = \int \left(\frac{1-\cos 2u}{2} \right) \left(\frac{1+\cos 2u}{2} \right) du \quad (*)$$

$$I = \frac{1}{4} \int (1 - \cos^2 2u) du = \frac{1}{4} \int du - \frac{1}{4} \int \underbrace{\cos^2 2u}_{I'} du = \frac{1}{4} u - \frac{1}{4} I'$$

using again the relation: $\cos^2 x = \frac{1+\cos 2x}{2}$, hence, we get

$$\int \cos^2 2u du = \int \left(\frac{1+\cos 4u}{2} \right) du = \frac{1}{2} \left[\int du + \int \cos 4u du \right]$$

$$\Rightarrow I' = \frac{1}{2} u + \frac{1}{8} \sin 4u + C$$

replacing I' in $(*)$, we get:

$$I = \frac{1}{4} u - \frac{1}{4} \left[\frac{1}{2} u + \frac{1}{8} \sin 4u \right] + C.$$

$$\Rightarrow \boxed{I = \frac{1}{8} u - \frac{1}{32} \sin 4u + C}$$

b) integration of the form: $\int \cos \alpha u \sin \beta u du$, $\alpha, \beta \in \mathbb{R}^*$
in this case, we use the following relationship

$$\cos \alpha u \sin \beta u = \frac{1}{2} [\sin(\alpha+\beta)u - \sin(\alpha-\beta)u] \quad \text{--- (I)}$$

Example: compute the integral: $I = \int \cos 3u \sin 5u du$
from (I), we have:

$$\cos 3u \sin 5u = \frac{1}{2} [\sin 8u - \sin 2u], \text{ hence } I \text{ becomes}$$

$$I = \frac{1}{2} \int (\sin 8u - \sin 2u) du = \left(-\frac{1}{16} \right) \cos 8u + \left(-\frac{1}{4} \right) \cos 2u + C$$

$$\boxed{I = -\frac{1}{4} \left[\frac{1}{4} \cos 8u + \cos 2u \right] + C}$$

C/ integration of the form $\int f(\cos u, \sin u) du$, where f is a fractional function.

in this case, we put generally $t = \tan \frac{u}{2}$

$$\Rightarrow dt = \frac{1}{2} \frac{1}{\cos^2 \frac{u}{2}} du = \frac{1}{2} \frac{\cos^2 \frac{u}{2} + \sin^2 \frac{u}{2}}{\cos^2 \frac{u}{2}} du = \frac{1}{2} (1 + \tan^2 \frac{u}{2}) du$$

$$\Rightarrow dt = \frac{1+t^2}{2} du \Rightarrow \boxed{du = \frac{2 dt}{1+t^2}}$$

in the same way, we get:

$$\cos u = \cos\left(\frac{u}{2} + \frac{u}{2}\right) = \cos^2 \frac{u}{2} - \sin^2 \frac{u}{2} = \frac{\cos^2 \frac{u}{2} - \sin^2 \frac{u}{2}}{\cos^2 \frac{u}{2} + \sin^2 \frac{u}{2}}$$

$$\Rightarrow \cos u = \frac{\cos^2 \frac{u}{2} (1 - \tan^2 \frac{u}{2})}{\cos^2 \frac{u}{2} (1 + \tan^2 \frac{u}{2})} = \frac{1-t^2}{1+t^2} \Rightarrow \boxed{\cos u = \frac{1-t^2}{1+t^2}}$$

$$\text{and } \sin u = \sin\left(\frac{u}{2} + \frac{u}{2}\right) = 2 \sin \frac{u}{2} \cos \frac{u}{2} = 2 \tan \frac{u}{2} \cos^2 \frac{u}{2}$$

$$\Rightarrow \sin u = \frac{2 \tan \frac{u}{2}}{\frac{1}{\cos^2 \frac{u}{2}}} = \frac{2 \tan \frac{u}{2}}{\frac{\cos^2 \frac{u}{2} + \sin^2 \frac{u}{2}}{\cos^2 \frac{u}{2}}} = \frac{2 \tan \frac{u}{2}}{1 + \tan^2 \frac{u}{2}} = \frac{2t}{1+t^2}$$

$$\Rightarrow \boxed{\sin u = \frac{2t}{1+t^2}}$$

Example: compute the following integrals: $I_1 = \int \frac{du}{\sin u}$ and $I_2 = \int \frac{du}{\cos u}$

we put $t = \tan \frac{u}{2} \Rightarrow dt = \frac{2 dt}{1+t^2}$, $\sin u = \frac{2t}{1+t^2}$ and $\cos u = \frac{1-t^2}{1+t^2}$

$$\text{Hence: } I_1 = \int \frac{2 dt}{1+t^2} \cdot \frac{1+t^2}{2t} = \int \frac{dt}{t} = \ln|t| + C = \boxed{\ln|\tan \frac{u}{2}| + C}$$

$$I_2 = \int \frac{2 dt}{1+t^2} \cdot \frac{1+t^2}{1-t^2} = \int \frac{2 dt}{1-t^2} = \int \frac{dt}{1-t} + \int \frac{dt}{1+t} = \ln \left| \frac{1+t}{1-t} \right| + C$$

$$\text{Hence: } \boxed{I_2 = \ln \left| \frac{1 + \tan \frac{u}{2}}{1 - \tan \frac{u}{2}} \right| + C}$$