Chapter 03: first-order ordinary differential equations

I - Genaralities:

1/ Definitions:

1/ Definition ①: Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be an open and let $f : U \to \mathbb{R}$ continuous function. We call ordinary differential equation (or simply **ODE**) of order n, where $n \in \mathbb{N}^*$, any equation written as

$$f(x, y, y', y^{(2)}, \dots, y^{(n)}) = 0 \quad \text{--- } (E_n)$$

connecting the variable $x \in \mathbb{R}$, the unknown function y of x and its derivatives $y', y^{(2)}, \dots, y^{(n)}$.

Here "n" represents the degree of the largest derivative of the functio[n] y in equation (E_n).

Remark:

1) If the function y has one variable x, then we are talking about ordinary differential equations (ODE).
2) If the function y has several variables x_i ($x_i \in \mathbb{R}$), then we are talking about partial differential equations (P.D.E).

• In this chapter we will be intersted in studying ordinary differential equations, that is, equations that have the form:

$$f(x, y, y', y^{(2)}, \dots, y^{(n)}) = 0,$$

where $y(x)$ is the unknown function to be found.

Definition ②: The equation $f(x, y(x), y'(x)) = 0$ or $y' = f(x, y(x))$ — (E_1) is called an ordinar[y] differential equation of the first-order (or siply 1^{st} order ODE).

its solution $y : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ is an open interval, satisfies the following two conditions:

a/ y is differentiable over I for all $x \in I$ and $(x, y(x)) \in U \subset \mathbb{R} \times \mathbb{R}$

b/ $\forall x \in I$: $y'(x) = f(x, y(x))$.

Cauchy problem's (Definition ③):

Let (x_0, y_0) be a point of U, the Cauchy problem consists of finding the solution $y : I \to \mathbb{R}$ of the equation (E_1) that satisfies:

$$(E_1)\begin{cases} y' = f(x, y(x)) \\ y(x_0) = y_0, \forall x \in I. \end{cases}$$

Theorem: we say that the function $y : I \to \mathbb{R}$ is a solution ~~to~~ of the Cauchy problem with the initial condition (x_0, y_0) if it's satisfied:

1/ y is continuous on I, $\forall x \in I : (x, y(x)) \in U$: y is continuous on I.

2/ $\forall x \in I : y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds$.

proof: $\Rightarrow$) Suppose that y is a solution of the Cauchy problem associated with the equation (E_1) and the initial condition (x_0, y_0) and we prove that y satisfies conditions ① and ② of the theorem.

from ⓐ of definition ②, Condition ① is a direct consequence (since y is diff $\Rightarrow$ y is continuous)

from ⓑ of definition ②, we have.

$\forall x \in I$: $y' = f(x, y(x))$, which equivalent to:

$$dy(x) = f(x, y(x))\, dx \quad / \quad y' = \frac{dy(x)}{dx},$$

this implies

$$\int_{x_0}^{x} dy(x) = \int_{x_0}^{x} f(s, y(s))\, ds$$

or equivalently:

$$y(x) - y(x_0) = y(x)\Big]_{x_0}^{x} = \int_{x_0}^{x} f(s, y(s))\, ds,$$

$y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s))\, ds$, which gives the desired result.

$\Leftarrow$) assume that y satisfies conditions ① and ② and we prove that y is a solution of the Cauchy problem associated with the equation (E_1) and the initial condition $(x_0, y(x_0)) = (x_0, y_0)$.

from ②, we have: $y(x_0) = y_0 + \int_{x_0}^{x_0} f(s, y(s))\, ds = y_0 \Rightarrow y(x_0) = y_0$

also from ②, we have: $y'(x) = y_0' + \left(\int_{x_0}^{x} f(s, y(s))\, ds\right)'$

$$= 0 + x' \cdot f(x, y(x)) - (x_0')\, f(x_0, y(x_0))$$

$$= f(x, y(x))$$

Hence $y'(x) = f(x, y(x))$, Therefore y is a solution of the Cauchy problem with $y' = f(x, y(x))$ and $y(x_0) = y_0$, $\forall x \in I$.

General solution and particular solution of 1^{st}-order ODE:

a/ We call a general solution of the 1^{st}-order ODE: $y' = f(x, y(x))$, any solution y of the form: $y = \varphi(x, c)$.

b/ We call a general solution of the 1^{st}-order ODE: $f(x, y(x), y'(x)) = 0$, any solution of the form: $\varphi(x, y(x), c) = 0$.

c/ We call a particular solution of the 1^{st}-order ODE: $y' = f(x, y(x))$ or $f(x, y(x), y'(x)) = 0$, any solution that results from the general solution of the given diff equation when we set certain conditions for the 1^{st}-order ODE.

II - Classification of 1^{st}-order ODE:

There are mainly three major classes of 1^{st} order ODE are:

1)- Differential equations with separate variables.

2)- Homogeneous differential equations where its terms are the same degree.

3)- Differential linear equations where y and y' are the 1^{st} degree.

- These classes are the most useful, where they are constantly encountered in physics, mechanics, electricity, thermodynamics, biology, etc.
- We will also study some special classes (or types) such as differential equations which are transform to homogeneous differential equation or to that of linear differential equation (such as the Bernoulli and Riccati equations).

III - Methods for solving 1^{st}-order ODE:

Remark: There is no general method for solving 1^{st}-order ODEs. The solution to each equation depends on its type.

III - 1. Differential equations with separate variables:

Let I and J be two intervals of $\mathbb{R}$ ($I, J \subseteq \mathbb{R}$) and let $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ be two continuous functions. We call differential equation with separate variables any equation of the form:

$$y' = f(x)\, g(y) \quad \dots (E_1)$$

Method of solution:

a/ If $g(y) = 0$, then $y = c$ is a singular (an obvious) solution of the eq (E_1)

b/ If $g(y) \neq 0$, then:

$$y' = f(x) g(y) \Leftrightarrow \frac{dy}{dx} = f(x) g(y)$$

$$\Leftrightarrow \frac{dy}{g(y)} = f(x) dx.$$

by integrating, we get:

$$\int \frac{dy}{g(y)} = \int f(x) dx.$$

which implies:

$$G(y) = F(x) + C, \quad C \in \mathbb{R},$$

where G and F are the primitives (or anti-derivatives) of $\frac{1}{g}$ and f, respectively.

Illustrative example: solve the following differential equation:

$$y' = 4x \sqrt{y-1} \quad \ldots (E_1)$$

(E_1) is a 1^{st} order ODE with separate variables.

method of solution: from (E_1) we have: $f(x) = 4x$ and $g(y) = \sqrt{y-1}$

a/ If $g(y) = 0$, then $\sqrt{y-1} = 0$ or equivalently $y = 1$ is an obvious solution of (E_1).

b/ If $g(y) \neq 0$, that is $y \neq 1$, then

$$y' = 4x\sqrt{y-1} \Leftrightarrow \frac{dy}{dx} = 4x\sqrt{y-1}$$

$$\Leftrightarrow \frac{dy}{\sqrt{y-1}} = 4x dx.$$

by integrating, we get:

$$\int \frac{dy}{\sqrt{y-1}} = \int 4x dx \Leftrightarrow 2\int \frac{dy}{2\sqrt{y-1}} = 2\int 2x dx$$

$$\Leftrightarrow 2\sqrt{y-1} = 2x^2 + C$$

$$\Leftrightarrow \sqrt{y-1} = x^2 + C_1, \quad C_1 = C/2.$$

which implies:

$y-1=(x^2+C_1)^2$, or equivalently $\boxed{y=1+(x^2+C_1)^2,\ C_1\in\mathbb{R}}$

which is the general solution of equation (E_1).

II-1: Homogeneous differential equations

Definition: (Homogeneous function) we say that the function f is homogeneous of degree "n" with respect to the variables x and y, if it satisfies the following relationship:

$\forall n\in\mathbb{N},\ \forall \lambda\in\mathbb{R}:\ f(\lambda x,\lambda y)=\lambda^n f(x,y)$

Example: $f(x,y)=x^2y^2+y^4+x^4$ is a homogeneous function of degree "4" since: $f(\lambda x,\lambda y)=\lambda^2x^2\lambda^2y^2+\lambda^4y^4+\lambda^4x^4=\lambda^4(x^2y^2+y^4+x^4)=\lambda^4 f(x,y)$

Definition: (Homogeneous equation) Let $y'=f(x,y)$ (E) be a differential equation of the 1^{st}-order. We call that $[y'=f(x,y)\ ---](E_H)$ is a homogeneous differential equation with respect to x and y if the function $f(x,y)$ is homogeneous of the degree "0" with respect to x and y, that is,

$\forall\lambda\in\mathbb{R}:\ f(\lambda x,\lambda y)=\lambda^0 f(x,y)=f(x,y)$.

Example: $y'=\frac{y}{x}-1$ is a homogeneous differential equation

Method of solution:

1- Solving using change of variable: $y'=f(x,y)\ ---\ (E_H)$.

put: $t=\frac{y}{x}\Rightarrow y=tx\Rightarrow y'=t'x+t$ / $t'=\frac{dt}{dx}$.

replacing the values of y and y' in (E_H), we get:

$t'x+t=f(1,t)\Leftrightarrow t'x=f(1,t)-t$

$$\Leftrightarrow t'=\frac{f(1,t)-t}{x}=[f(1,t)-t]\times\frac{1}{x}$$

$$\Leftrightarrow \frac{dt}{f(1,t)-t}=\frac{dx}{x}$$

which a differential equation with separate variables.

Hence, by integrating, we obtain $F(t)=\ln|x|+C$, $C\in\mathbb{R}$, [when $f(1,t)\neq t$]

where F is the anti-derivative (primitive) of the function $\frac{1}{f(1,t)-t}$

$F(t)=\ln|x|+C\Leftrightarrow |x|=e^{F(t)}\cdot e^{-C}\Leftrightarrow x=k\,e^{F(t)},\ k=\pm e^{-C}$

which equivalent to $x = k e^{f(\frac{y}{x})}$, which is the general solution of (E_H).

Illustrative example: $y' = \frac{y}{x} - 1$ $(\tilde{E}_H)$ is a homogeneous differential equation

[since $f(x,y) = \frac{y}{x} - 1 \Rightarrow f(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - 1 = \frac{y}{x} - 1 = f(x,y)$]

method of solution:

put $t = \frac{y}{x} \Rightarrow y = tx \Rightarrow y' = t'x + t$, $t' = \frac{dt}{dx}$

replacing by the values of y and y' in (E_H), we get:

$$t'x + t = \frac{tx}{x} - 1 \Leftrightarrow t'x + t = t - 1$$
$$\Leftrightarrow t'x = -1$$
$$\Leftrightarrow t' = -\frac{1}{x}$$
$$\Leftrightarrow \frac{dt}{dx} = \frac{-1}{x}$$
$$\Leftrightarrow dt = \frac{-dx}{x}.$$
$$\Leftrightarrow \int dt = -\int \frac{dx}{x}.$$
$$\Leftrightarrow t = -\ln|x| + c$$
$$\Leftrightarrow \frac{y}{x} = -\ln|x| + c$$
$$\Leftrightarrow \boxed{y = x(-\ln|x| + c)} \Leftrightarrow \boxed{y = x\left(\ln\left|\frac{1}{x}\right| + c\right)}$$

which is the general solution of equation $(\tilde{E}_H)$.

2- solving using polar coordinates:

the homogeneous equation $y' = f(x,y)$ can be written as $y' = f(\frac{y}{x})$ (E_H')

put: $\begin{cases} x = r\cos\theta, \ r > 0, \quad \theta \in R \\ y = r\sin\theta \end{cases}$

Hence $\begin{cases} dx = \cos\theta\, dr - r\sin\theta\, d\theta \\ dy = \sin\theta\, dr + r\cos\theta\, d\theta \end{cases}$

Then y' can be written as.

$$y' = \frac{dy}{dx} = \frac{\sin\theta\, dr + r\cos\theta\, d\theta}{\cos\theta\, dr - r\sin\theta\, d\theta} = \frac{\cos\theta\left(\frac{\sin\theta}{\cos\theta} dr + r\, d\theta\right)}{\cos\theta\left(dr - r\frac{\sin\theta}{\cos\theta} d\theta\right)} = \frac{tg\theta\, dr + r\, d\theta}{dr - r\, tg\theta\, d\theta}$$

Replacing by the value of y' in (E_H'), we get:

$$\frac{tg\,\theta\, dr + r\, d\theta}{dr - r\, tg\,\theta\, d\theta} = f\left(\frac{r\sin\theta}{r\cos\theta}\right) = f(tg\,\theta).$$

which equivalent to:

$$tg\,\theta\, dr + r\, d\theta = f(tg\,\theta)\left[dr - r\, tg\,\theta\, d\theta\right]$$

which implies:

$$tg\,\theta\, dr + r\, d\theta = f(tg\,\theta)\, dr - r f(tg\,\theta)\, tg\,\theta\, d\theta$$

Hence,

$$\left[tg\,\theta - f(tg\,\theta)\right] dr = -r\left[1 + f(tg\,\theta)\, tg\,\theta\right] d\theta$$

$\Leftrightarrow \dfrac{dr}{r} = \dfrac{1 + f(tg\,\theta)\, tg\,\theta}{f(tg\,\theta) - tg\,\theta}\, d\theta$ / where $f(tg\,\theta) \neq tg\,\theta$. (singular solution)

which is a differential equation with separate variables with respect to r and θ.

Example: Solve the following 1^{st}-order ODE:

$$(x^2+y^2)(x\,dx + y\,dy) - \frac{y}{x}(x\,dx - y\,dy) = 0 \quad --- (E_H')$$

put $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r>0$ and $\theta \in \mathbb{R}$?

by derivating $\Rightarrow \begin{cases} dx = \cos\theta\, dr - r\sin\theta\, d\theta \\ dy = \sin\theta\, dr + r\cos\theta\, d\theta \end{cases}$

Hence $\begin{cases} x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \\ \dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = tg\,\theta \\ x\,dx + y\,dy = r\cos^2\theta\, dr + r\sin^2\theta\, dr = r\, dr \\ x\,dx - y\,dy = r^2\cos^2\theta\, d\theta + r^2\sin^2\theta\, d\theta = r^2\, d\theta \end{cases}$

replacing by these values in (E_H'), we get:

$$r^2(r\,dr) - r^2\, tg\,\theta\, d\theta = 0 \Leftrightarrow r\, dr - tg\,\theta\, d\theta = 0 \;,\; r>0$$

which a differential equation with separate variables with respect to r and θ

hence: $\int r\, dr = \int tg\,\theta\, d\theta \Leftrightarrow \dfrac{r^2}{2} = -\ln|\cos\theta| + C \quad -(1)$

we have: $r^2 = x^2+y^2$ and $\cos^2\theta = \dfrac{x^2}{x^2+y^2}$, replacing in (1), we get:

$$\frac{x^2+y^2}{2} = -\ln\left(\frac{x^2}{x^2+y^2}\right) + C \Leftrightarrow \boxed{x^2+y^2 = -2\ln\left(\frac{x^2}{x^2+y^2}\right) + 2C}$$

which is the general solution of equation (E_H').

III-2: Differential equations that are reduced (transformed) to homogeneous:

any differential equation written as the form: $y' = \frac{ax+by+c}{dx+ey+f}$ (E_T)

where a, b, c, d, e and f are real constants.

Can be reduced (transformed) to homogeneous differential equation.

method of solution:

we put $\begin{cases} x = u + s \\ y = v + t \end{cases} \Rightarrow y' = v'$

s and t are real constants that satisfy:

$\begin{cases} as + bt + c = 0 \\ ds + et + f = 0 \end{cases}$

replacing by the values of x, y and y' in (E_T), we get the following homogeneous differential equation with respect to u and v.

$$v' = \frac{au + bv}{du + ev}.$$

Example: solve the 1^{st}-order ODE : $y' = \frac{x+2y+5}{2x+3y+1}$ --- (E_T)

(E_T) is a 1^{st}-order ODE which can be reduced (transformed) to homogeneous differential equation.

method of solution:

put $\begin{cases} x = u + s \\ y = v + t \end{cases} \Rightarrow y' = v'$

where s and t are solutions of the following linear system:

$\begin{cases} s + 2t + 5 = 0 \\ 2s + 3t + 1 = 0 \end{cases} \Rightarrow \begin{cases} s = 13 \\ t = -9 \end{cases} \Rightarrow \begin{cases} x = u + 13 \\ y = v - 9 \end{cases} \Rightarrow y' = v'$

replacing by the values of x, y and y' in (E_T), we obtain.

$v' = \frac{u+2v}{2u+3v}$ -- (E_H) which is a homogeneous differential equation

Let: $z = \frac{v}{u} \Rightarrow v = zu \Rightarrow v' = z'u + z$

Replacing by the values of v and v' in (E_H), we get:

$$z'u + z = \frac{u+2zu}{2u+3zu} \Leftrightarrow z'u + z = \frac{1+2z}{2+3z}$$

$$\Leftrightarrow z'u = \frac{1-3z^2}{2+3z} \quad \text{---} (1)$$

where (1) is a 1st-order ODE with separate variables.

by integrating (1), we get:

$$\int \frac{2+3z}{1-3z^2}\,dz = \int \frac{du}{u} \qquad / \; z' = \frac{dz}{du}$$

$$\Leftrightarrow \int \frac{2}{1-3z^2}\,dz + \int \frac{3z}{1-3z^2}\,dz = \int \frac{du}{u}$$

$$\Leftrightarrow -\frac{1}{2}\int \frac{-6z}{1-3z^2}\,dz + \int \frac{dz}{1-\sqrt{3}z} + \int \frac{dz}{1+\sqrt{3}z} = \int \frac{du}{u}$$

$$\Leftrightarrow -\frac{1}{2}\ln|1-3z^2| + \frac{1}{\sqrt{3}}\ln|1+\sqrt{3}z| - \frac{1}{\sqrt{3}}\ln|1-\sqrt{3}z| = \ln|u| + c, \; c \in \mathbb{R}$$

$$\Leftrightarrow \ln\sqrt{|1-3z^2|} + \ln\left(\left|\frac{1+\sqrt{3}z}{1-\sqrt{3}z}\right|\right)^{\frac{1}{\sqrt{3}}} = \ln|u| + c$$

$$\Leftrightarrow \ln\left(\sqrt{|1-3z^2|}\cdot\left(\left|\frac{1+\sqrt{3}z}{1-\sqrt{3}z}\right|\right)^{\frac{1}{\sqrt{3}}}\right) = \ln|u| + c$$

$$\Leftrightarrow \sqrt{|1-3z^2|}\cdot\left(\left|\frac{1+\sqrt{3}z}{1-\sqrt{3}z}\right|\right)^{\frac{1}{\sqrt{3}}} = |u|\,k \quad / \; k = e^c \in \mathbb{R}$$

$$\Leftrightarrow \boxed{\sqrt{\left|1-3\left(\frac{y+9}{x-13}\right)^2\right|}\left(\left|\frac{1+\sqrt{3}\left(\frac{y+9}{x-13}\right)}{1-\sqrt{3}\left(\frac{y+9}{x-13}\right)}\right|\right)^{\frac{1}{\sqrt{3}}} = k|x-13|}$$

which is the general solution of equation (E_T)

II - 3 - linear differential equations:

A differential equation is called linear if it's linear with respect to y and its derivative y', that is, it is written in the form:

$$y' + p(x)\,y = q(x) \quad \text{---} (E_L)$$

where: $p, q : I \to \mathbb{R}$ are two continuous functions on I. $(I \subseteq \mathbb{R})$.

methode of solution: to solve linear differential equations, we follow the following steps:

step 1: (Solution without second member): in this case we solve (E_L) with $q(x)=0$, that is,

$$y' + p(x)\, y = 0 \Leftrightarrow y' = -p(x)\, y$$
$$\Leftrightarrow \frac{dy}{y} = -p(x)\, dx.$$
$$\Leftrightarrow \int \frac{dy}{y} = -\int p(x)\, dx.$$
$$\Leftrightarrow \ln|y| = -\int p(x)\, dx + C$$
$$\Leftrightarrow |y| = e^{C}\, e^{-\int p(x)\, dx}$$
$$\Leftrightarrow y = \pm e^{C}\, e^{-\int p(x)\, dx}.$$
$$\Leftrightarrow \boxed{y = K\, e^{-\int p(x)\, dx} \quad / \; K = \pm e^{C}} \text{ — } (*)$$

which is the solution of (E_L) without second member (i.e. with $q(x)=0$)

step 2: (solution with second member) called Method of Lagrange: in this case we put: $y(x) = K(x)\, e^{-\int p(x)\, dx}$, hence

$$y' = K'(x)\, e^{-\int p(x)\, dx} - p(x)\, K(x)\, e^{-\int p(x)\, dx}.$$

Replacing y and y' in (E_L), we get:

$$K'(x)\, e^{-\int p(x)\, dx} \underbrace{- p(x)\, K(x)\, e^{-\int p(x)\, dx}} + \underbrace{K(x)\, p(x)\, e^{-\int p(x)\, dx}} = q(x)$$

$$\Rightarrow K'(x)\, e^{-\int p(x)\, dx} = q(x)$$

$$\Rightarrow K'(x) = q(x)\, e^{\int p(x)\, dx}.$$

which is a differential equation with separate variables. Integrating

$$K(x) = \int q(x)\, e^{\int p(x)\, dx} + C_1$$

Now, replacing the value of $K(x)$ in $(*)$, we get the general solution of the equation (E_L), which is

$$y(x) = \left(\int q(x)\, e^{\int p(x)\, dx} + C_1 \right) e^{-\int p(x)\, dx}.$$

Example: solve the linear differential equation: $y' = y + x$ — (E_L)

Solution: (E_L) is a linear 1^{st}-order ODE which can be written as:

$$y' - y = x \quad — (E_L')$$

step 1: (solution without second member):

$$y' - y = 0 \Leftrightarrow y' = y$$
$$\Leftrightarrow \frac{dy}{dx} = y$$
$$\Leftrightarrow \frac{dy}{y} = dx$$
$$\Leftrightarrow \int \frac{dy}{y} = \int dx$$
$$\Leftrightarrow \ln|y| = x + c$$
$$\Leftrightarrow |y| = e^c e^x$$
$$\Leftrightarrow y = \pm e^c e^x$$
$$\Leftrightarrow \boxed{y = K e^x \ / \ K = \pm e^c} \quad --- (*)$$

step 2: (Method of Lagrange)

we put: $y = K(x) e^x \Rightarrow y' = K'(x) e^x + K(x) e^x$

replacing by the values of y and y' in (E_L'), we get:

$K'(x) e^x + \underbrace{K(x) e^x} - \underbrace{K(x) e^x} = x.$

$\Rightarrow K'(x) e^x = x.$

$\Rightarrow K'(x) = x e^{-x}.$ (which is a differential equation with separate variables)

$\Rightarrow \boxed{K(x) = \int x e^{-x} dx = -(x+1) e^{-x} + C_1}$

replacing $K(x)$ in $(*)$, we get:

$$y(x) = \left(-(x+1) e^{-x} + C_1\right) e^x = \boxed{-(x+1) + C_1 e^x.}$$

which is the general solution of equation (E_L').

III_4: Bernoulli's equations: (Differential equation reduced to linear):

Any differential equation of the form:

$$y' + p(x)\, y = y^n q(x) \ , \ \forall n \in \mathbb{R}_+ \quad — (E_B)$$

is called differential equation of Bernoulli, with $p, q : I \to \mathbb{R}$ are continuous

Method of solution:

- If $n=0$, then (E_B) is linear differential equation.
- If $n=1$, then (E_B) is differential equation with separate variables.
- If $n \notin \{0,1\}$, then (E_B) is solved as follows:

step 1: we divide both sides of the equation (E_B) by y^n, we get

$$y^{-n} y' + y^{1-n} p(x) = q(x) \quad \ldots (1)$$

step 2: we put $z = y^{1-n} \Rightarrow z' = (1-n) y^{-n} y'$

Replacing by the values of y^{1-n} and $y^{-n}y'$ in (1), we obtain

$$\frac{z'}{1-n} + p(x) z = q(x) \Leftrightarrow z' + (1-n) p(x) z = (1-n) q(x) \text{ --- } (2)$$

where (2) is a linear differential equation. (we can solve it easily).

Example: solve the equation: $y' + xy = x^3 y^3 \quad - (E_B)$

(E_B) is a Bernoulli's equation.

Method of solution:

we divide both sides of (E_B) by y^3, we get

$$y' y^{-3} + x y^{-2} = x^3 \text{ --- } (1)$$

Now, we put $z = y^{-2} \Rightarrow z' = -2 y^{-3} y' \Rightarrow y^{-3} y' = \frac{z'}{-2}$

hence, replacing by the values of y^{-2} and $y^{-3}y'$ in (1), we obtain

$$\frac{z'}{-2} + x z = x^3 \Leftrightarrow z' - 2xz = -2x^3 \text{ --- } (2)$$

which is a linear equation.

firstly, we solve (2) without second member, that is,

$$z' - 2xz = 0 \Leftrightarrow \frac{z'}{z} = 2x$$

$$\Leftrightarrow \frac{dz}{z} = 2x\,dx$$

$$\Leftrightarrow \int \frac{dz}{z} = \int 2x\,dx$$

$$\Leftrightarrow \ln|z| = x^2 + C$$

$$\Leftrightarrow |z| = e^C e^{x^2}$$

$$\Leftrightarrow \boxed{z = K e^{x^2} \,/\, K = \pm e^C} \text{ --- } (*)$$

Secondly, we solve (2) with second member (there we use the Lagrange method)

we assume that $z(x) = K(x)\, e^{x^2}$, then $z'(x) = K'(x)\, e^{x^2} + 2x\, K(x)\, e^{x^2}$

replacing by the values of z and z' in (2), we get

$$K'(x)\, e^{x^2} + 2x K(x)\, e^{x^2} - 2x\, K(x)\, e^{x^2} = -2x^3$$

$$\Leftrightarrow K'(x)\, e^{x^2} = -2x^3$$

$$\Leftrightarrow K'(x) = -2x^3\, e^{-x^2}$$

$$\Leftrightarrow K(x) = \int -2x^3 e^{-x^2} dx$$

we integrate by part:

$$\begin{cases} u = x^2 \\ v' = -2x\, e^{-x^2} dx \end{cases} \Rightarrow \begin{cases} u' = 2x\, dx \\ v = e^{-x^2} \end{cases}$$

$$K(x) = x^2 e^{-x^2} + \int -2x\, e^{-x^2} dx.$$

$$\Rightarrow \boxed{K(x) = x^2 e^{-x^2} + e^{-x^2} + C_1} \text{ --- } (**)$$

Replacing $(**)$ in $(*)$, we get

$$\boxed{z(x) = \left(x^2 e^{-x^2} + e^{-x^2} + C_1\right) e^{x^2}}$$

which is the general solution of linear equation (2).

but, we have $z = y^{-2} = \frac{1}{y^2} \Rightarrow y = \frac{1}{\sqrt{z}}$

Therefore, the general solution of Bernoulli's equation (E_B) is given as:

$$\boxed{y = \frac{1}{\sqrt{z}} = \frac{1}{\sqrt{x^2 + 1 + C_1 e^{x^2}}}}$$

III-5- Riccati's equation:

Any differential equation of the form

$$y' + p(x)\, y + q(x)\, y^2 = f(x) \quad \text{--} \ (E_R)$$

where, $p, q, f : I \to \mathbb{R}$ are continuous on $I \subset \mathbb{R}$.

is called different equation of Riccati.

Method of solution

- If $q(x)=0$, then (E_R) is a linear differential equation.
- If $f(x)=0$, then (E_R) is a Bernoulli equation for $n=2$.
- If $f(x)\neq 0 \wedge q(x)\neq 0$ and let y_p is a particular solution of (E_R) (given) then (E_R) can be transformed (reduced) to

a) Bernoulli's equation when we put $y=y_p+z \Rightarrow y'=y_p'+z'$

b) Linear equation when we put $y=y_p+\frac{1}{z}, z\neq 0 \Rightarrow y'=y_p'-\frac{z'}{z^2}$

ⓐ Bernoulli's equation:

Let $y=y_p+z$, where y_p is a given particular solution, then $y'=y_p'+z'$, hence (E_R) becomes:

$$y_p'+z'+p(x)(y_p+z)+q(x)(y_p+z)^2=f(x)$$

$$\Leftrightarrow \underbrace{y_p'+p(x)y_p+q(x)y_p^2}+z'+p(x)z+2q(x)y_pz+q(x)z^2=\underbrace{f(x)}$$

$$\Leftrightarrow z'+p(x)z+2q(x)y_pz+q(x)z^2=0 \quad \text{(since } y_p \text{ satisfies } (E_R)\text{, i.e., } y_p'+p(x)y_p+q(x)y_p^2=f(x)\text{)}$$

$$\Leftrightarrow z'+(p(x)+2q(x)y_p)z+q(x)z^2=0$$

$$\Leftrightarrow \boxed{z'+(p(x)+2q(x)y_p)z=-q(x)z^2}$$

which is a Bernoulli's equation (we can solve it easily).

Example: Solve the following ODE: $(1-x^3)y'+x^2y+y^2=2x$ — (E_R) with $y_p=x^2$

(E_R) is a Riccati's equation

y_p is a particular solution, then it satisfies: $(1-x^3)y_p'+x^2y_p+y_p^2=2x$ — ①

to reduce (E_R) to an equation of Bernoulli, we put $y=y_p+z \Rightarrow y'=y_p'+z'$

replacing by y and y' in (E_R), we obtain:

$$(1-x^3)(y_p'+z')+x^2(y_p+z)+(y_p+z)^2=2x.$$

$$\Leftrightarrow \underbrace{(1-x^3)y_p'+x^2y_p+y_p^2}+(1-x^3)z'+x^2z+2y_pz+z^2=\underbrace{2x}$$

$$\Leftrightarrow (1-x^3)z'+(x^2+2y_p)z+z^2=0$$

$$\Leftrightarrow (1-x^3)z'+(x^2+2x^2)z=-z^2$$

$$\Leftrightarrow (1-x^3)z'+3x^2z=-z^2 \quad \text{---} (E_B)$$

which is a Bernoulli's equation for $n=2$.

step 1: we divide both sides of (E_B) by z^2, we get

$$(1-x^3)\, z' z^{-2} + 3x^2 z^{-1} = -1 \quad \text{— } (2)$$

step 2) we put $\boxed{t = z^{-1}} \Rightarrow t' = -z' z^{-2} \Rightarrow z' z^{-2} = -t'$

replacing by z^{-1} and $z' z^{-2}$ in (2), we get

$$(x^3-1)\, t' + 3x^2 t = -1$$

which is equivalent to

$$t' + \frac{3x^2}{x^3-1} t = -1 \quad \text{— } (E_L)$$

where (E_L) is a linear differential equation.

step 3. we solve (E_L) without second member, we get

$$t' + \frac{3x^2}{x^3-1} t = 0 \Leftrightarrow t' = \frac{3x^2}{1-x^3} t$$

$$\Leftrightarrow \frac{t'}{t} = \frac{3x^2}{1-x^3}$$

$$\Leftrightarrow \frac{dt}{t} = \frac{3x^2}{1-x^3} dx$$

$$\Leftrightarrow \int \frac{dt}{t} = \int \frac{3x^2}{1-x^3} dx$$

$$\Leftrightarrow \ln|t| = -\ln|1-x^3| + C$$

$$\Leftrightarrow |t| = \frac{e^C}{|1-x^3|}$$

$$\Leftrightarrow \boxed{t = \frac{k}{(1-x^3)}, \quad k = \pm e^C} \text{ — } (*)$$

step 4: we solve (E_L) with it second member (Method of Lagrange)

Let $t = \frac{K(x)}{1-x^3} \Rightarrow t' = \frac{K'(x)(1-x^3) + 3x^2 K(x)}{(1-x^3)^2}$

hence (E_L) becomes

$$\frac{K'(x)}{(1-x^3)} + \underbrace{\frac{3x^2 K(x)}{(1-x^3)^2}} + \underbrace{\frac{3x^2}{x^3-1} \frac{K(x)}{(1-x^3)}} = -1$$

$$\Leftrightarrow K'(x) = x^3 - 1$$

$$\Leftrightarrow K(x) = \int (x^3-1)\, dx$$

$$\Leftrightarrow \boxed{K(x) = \frac{x^4}{4} - x + C_1}, \quad C_1 \in \mathbb{R}$$

Thus, (*) becomes:

$$\boxed{t(x) = \left[\frac{x^4}{4} - x + C_1\right]\left[\frac{1}{1-x^3}\right], \; C_1 \in \mathbb{R}}$$ (general solution of (E_L))

but $t = \frac{1}{z}$ (step 2) $\Rightarrow z = \frac{1}{t}$

$$\Rightarrow \boxed{z = \frac{1-x^3}{\frac{x^4}{4} - x + C_1}}$$

Hence, the general solution of (E_R) is

$$\boxed{y = y_p + z = x^2 + \frac{1-x^3}{\frac{x^4}{4} - x + C_1}, \; C_1 \in \mathbb{R}}$$ (step 1)

ⓑ Linear equation:

Let $y = y_p + \frac{1}{z}$, where y_p is a given particular solution of (E_R).

then $y' = y_p' - \frac{z'}{z^2}$, hence (E_R) becomes

$$y_p' - \frac{z'}{z^2} + p(x)\left(y_p + \frac{1}{z}\right) + q(x)\left(y_p + \frac{1}{z}\right)^2 = f(x)$$

$$\Leftrightarrow \underbrace{y_p' + p(x)\,y_p + q(x)\,y_p^2}_{} + \left(-\frac{z'}{z^2} + p(x)\frac{1}{z} + 2\frac{y_p}{z}q(x) + \frac{q(x)}{z^2}\right) = \cancel{f(x)}$$

$$\Leftrightarrow -\frac{z'}{z^2} + \frac{1}{z}\left(p(x) + 2y_p q(x)\right) + \frac{q(x)}{z^2} = 0$$

$$\Leftrightarrow z' - \left(p(x) + 2y_p\, q(x)\right)z - q(x) = 0 \quad \left(\text{multiplying by } -z^2\right)$$

$$\Leftrightarrow z' - \left(p(x) + 2y_p q(x)\right)z = q(x) \quad \text{— } (E_L)$$

which is a linear equation.

Example: solve: $\begin{cases} 2x^2 y' = (x-1)(y^2 - x^2) + 2xy & \text{— } (E_R) \\ y_p = x \end{cases}$

(E_R) is a Riccati equation.

It is worth noting that $y_p = x$ is a particular solution of (E_R), hence it satisfies: $2x^2 y_p' = (x-1)(y_p^2 - x^2) + 2x y_p$ — (*)

Method of solution:

step 1) Let $y = y_p + \frac{1}{z} = x + \frac{1}{z} \Rightarrow y' = 1 - \frac{z'}{z^2} = y_p' - \frac{z'}{z^2}$

replacing y and y' in (E_R), we get:

$$2x^2\left(y_p' - \frac{z'}{z^2}\right) = (x-1)\left(\left(y_p + \frac{1}{z}\right)^2 - x^2\right) + 2x\left(y_p + \frac{1}{z}\right)$$

$$\Leftrightarrow \underbrace{2x^2 y_p'} - 2x^2\frac{z'}{z^2} = \underbrace{(x-1)(y_p^2 - x^2)} + (x-1)\left(\frac{2y_p}{z} + \frac{1}{z^2}\right) + \underbrace{2xy_p} + \frac{2x}{z}$$

from(*)

$$\Leftrightarrow -2x^2\frac{z'}{z^2} = (x-1)\left(\frac{2y_p}{z} + \frac{1}{z^2}\right) + \frac{2x}{z}$$

$$\Leftrightarrow -2x^2 z' = (x-1)(2y_p z + 1) + 2xz \quad [\text{multiplying by } z^2]$$

$$\Leftrightarrow -2x^2 z' = (x-1)(2xz+1) + 2xz$$

$$\Leftrightarrow -2x^2 z' = 2xz(x-1+1) + x-1$$

$$\Leftrightarrow -2x^2 z' = 2x^2 z + x - 1$$

$$\Leftrightarrow z' = -z + \frac{1-x}{2x^2}$$

$$\Leftrightarrow z' + z = \frac{1-x}{2x^2} \quad \text{---} (E_2)$$

which is a linear equation with respect to z and x.

step 2: (solution without second member)

$$z' + z = 0 \Leftrightarrow z' = -z$$

$$\Leftrightarrow \frac{z'}{z} = -1$$

$$\Leftrightarrow \frac{dz}{z} = -dx$$

$$\Leftrightarrow \int \frac{dz}{z} = -\int dx$$

$$\Leftrightarrow \ln|z| = -x + C, \quad C \in \mathbb{R}$$

$$\Leftrightarrow \boxed{z = Ke^{-x}, \quad K = \pm e^C} \text{ --- } (1)$$

step 3: (Method of Lagrange) (solution with second member)

$$z = K(x)e^{-x} \Rightarrow z' = K'(x)e^{-x} - K(x)e^{-x}$$

hence (E_2) becomes

$$K'(x)e^{-x} - K(x)e^{-x} + K(x)e^{-x} = \frac{1-x}{2x^2}$$

$$\Leftrightarrow K'(x)e^{-x} = \frac{1-x}{2x^2}$$

$$\Leftrightarrow K'(x) = \frac{1}{2x^2}e^x - \frac{1}{2x}e^x$$

$$\Leftrightarrow K(x) = \frac{1}{2}\left[\int \frac{e^x}{x^2}dx - \int \frac{e^x}{x}dx\right]$$

we integrate $\frac{e^x}{x}$ by part

$$u = \frac{1}{x} \; ; \; \Rightarrow \begin{cases} u' = \frac{-dx}{x^2}. \\ v = e^x \end{cases}$$
$$v' = e^x dx$$

Hence

$$\int \frac{e^x}{x} dx = \frac{e^x}{x} + \int \frac{e^x}{x^2} dx. + C_1$$

Thus $K(x)$ becomes

$$K(x) = \frac{1}{2}\left[\int \frac{e^x}{x^2} dx - \frac{e^x}{x} - \int \frac{e^x}{x^2} dx \right] + C_1 \quad , \; C_1 \in \mathbb{R}.$$

$$\Rightarrow \boxed{K(x) = \frac{-e^x}{2x} + C_1.} \quad , \; C_1 \in \mathbb{R}$$

Then from (1), $z = e^x\left(\frac{-e^x}{2x} + C_1\right) \Rightarrow \boxed{z = e^x C_1 - \frac{1}{2x}.}$, $C_1 \in \mathbb{R}$.

which is the general solution of (E_L).

Therefore : $y = y_p + \frac{1}{z} = x + \frac{1}{z} = x + \frac{1}{e^x C_1 - \frac{1}{2x}}.$

$$\Rightarrow \boxed{y = x + \frac{2x}{2x e^x C_1 - 1} \quad , \; C_1 \in \mathbb{R}}$$

is the general solution of (E_R).