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Functions of a complex variable

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CHAPTER 1

HOLOMORPHIC FUNCTIONS

1.1 Complex plane

Definition 1.1.1 *A complex number is a number z of the form $z = x + iy$ where $x, y \in \mathbb{R}$.*
If $z = x + iy$, $x, y \in \mathbb{R}$, then $x = \Re(z)$ is the real part of z and $y = \Im(z)$ is the imaginary part of z.

Remark 1.1.1 **1)** $\mathbb{R} \subset \mathbb{C}$.

2) *Two complex numbers are equal when their real and imaginary parts are equal, that is $z_1 = x_1 + iy_1 = z_2 = x_2 + iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.*

3) *Every complex number $z = x + iy$ can be considered as a point $M(x, y)$ on the cartesian plane.*

Definition 1.1.2 *Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.*

1) *The addition of* z_1 *and* z_2 *is the complex number* z *given by*

$$z = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

2) *The subtraction of* z_1 *and* z_2 *is the complex number* z *given by*

$$z = z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

3) *The multiplication of* z_1 *and* z_2 *is the complex number* z *given by*

$$z = z_1.z_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2).$$

4) *The division of* z_1 *by* z_2 *where* $z_2 \neq 0$ *is the complex number* z *given by*

$$z = \frac{z_1}{z_2} = \frac{x_1x_2 - y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 + x_1y_2}{x_2^2 + y_2^2}.$$

Proposition 1.1.1 *Let* $z_j = x_j + iy_j,\ j = 1, 2, 3$ *be complex numbers. Then*

a) $z_1 + z_2 = z_2 + z_1$ *and* $z_1.z_2 = z_2.z_1$, *the addition and the multiplication are commutative.*

b) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ *and* $(z_1.z_2).z_3 = z_1.(z_2.z_3)$, *the addition and the multiplication are associative.*

c) $z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3$, *distributivity of multiplication over addition.*

Definition 1.1.3 *Let* $z = x + iy,\ x, y \in \mathbb{R}$ *be a complex number. The conjugate of* z *is the complex number given by* $\overline{z} = x - iy$.

Proposition 1.1.2 *Let* z, z_1 *and* z_2 *be complex numbers. Then*

1) $z = \overline{z}$ *if and only if* $z \in \mathbb{R}$.

2) $z = -\overline{z}$ *if and only if* $z = iy,\ y \in \mathbb{R}$.

3) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ *and* $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

4) $\overline{z_1.z_2} = \overline{z_1}.\overline{z_2}$ *and* $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ *if* $z_2 \neq 0$.

5) $\overline{\overline{z}} = z$.

Proposition 1.1.3 *Let* $z = x + iy$, $x, y \in \mathbb{R}$ *be a complex number and let* $\overline{z}$ *be its conjugate. Then* $z + \overline{z} = 2\Re(z)$ *and* $z - \overline{z} = 2i\Im(z)$. *That is*

$$\Re(z) = \frac{z + \overline{z}}{2} \text{ and } \Im(z) = \frac{z - \overline{z}}{2i}.$$

Remark 1.1.2 *A complex number* $z = x + iy$, $x, y \in \mathbb{R}$ *can be considered as a vector from the origin* $O(0, 0)$ *to the point* $M(x, y)$.

Definition 1.1.4 *Let* $z = x + iy$, $x, y \in \mathbb{R}$ *be a complex number. The modulus of z is the real number* $|z| = \sqrt{x^2 + y^2}$.

Proposition 1.1.4 *Let* z, z_1, z_2 *be complex numbers and let* $\overline{z}$ *be the conjugate of* z. *Then*

1) $|z| \geq 0$ *and* $|z| = 0$ *if and only if* $z = 0$.

2) $|z| = \sqrt{z.\overline{z}}$.

3) $|z| = |\overline{z}|$.

4) $|z_1.z_2| = |z_1||z_2|$ *and* $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$.

5) $|z_1 + z_2| \leq |z_1| + |z_2|$.

6) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

7) $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$, $\forall z \in \mathbb{C}^\star$.

Definition 1.1.5 *Let* $z = x + iy \in \mathbb{C}^\star$. *The number z can be expressed in trigonometric (or polar) form as*

$$z = r(\cos\theta + \sin\theta),$$

where $r = |z| = \sqrt{x^2 + y^2}$ *and* θ *is a real number such that* $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, *that is* $\theta = \arctan\left(\frac{y}{x}\right)$.

- *The number θ is called an argument of z and we write $\theta = \arg z$.*
- *We denote by $Arg(z)$ the principal value of $\arg(z)$ and it is defined as the unique value of $\arg(z)$ such that $-\pi < \arg(z) \leq \pi$.*

Remark 1.1.3 **a)** *$\arg z$ is the angle measured in radians that the vector corresponds to z makes with the positive real axis.*

b) *The argument of $z = 0$ is not defined.*

Proposition 1.1.5 *Let z_1 and z_2 be two nonzero complex numbers. Then*

1) $\arg(z_1.z_2) = \arg(z_1) + \arg(z_2)$.

2) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Definition 1.1.6 *Let $z = x + iy \in \mathbb{C}$. The exponential of z is defined by*

$$e^z = e^x(\cos y + i \sin y).$$

Proposition 1.1.6 *Let $z_1, z_2 \in \mathbb{C}$. Then*

1) $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

2) $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$.

Remark 1.1.4 **1)** *If $z = iy,\ y \in \mathbb{R}$, then we have the formulas of Euler*

$$e^{iy} = \cos y + i \sin y.$$

2) *For every $y \in \mathbb{R}$, we have*

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and } \sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$

Definition 1.1.7 *Let $z = x + iy \in \mathbb{C}^\star$. The number z can be expressed in exponential form as $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$.*

Proposition 1.1.7 *Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two nonzero complex numbers. Then*

(i) $z_1.z_2 = r_1 e^{i\theta_1}.r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

(ii) $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$.

(iii) *The conjugate of $z = re^{i\theta} \in \mathbb{C}^\star$ is given by $\overline{z} = \overline{(re^{i\theta})} = re^{-i\theta}$.*

Proposition 1.1.8 (De Moivre's formula)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,\ \forall\theta \in \mathbb{R},\ \forall n \in \mathbb{N}.$$

Corollary 1.1.1 *Let $z = re^{i\theta}$ be a nonzero complex number where $r = |z|$ and $\theta = Argz$. Then the n distinct roots of z are given by*

$$z_k = (r)^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}},\ k = 0, 1, \dots, n-1,\ n \in \mathbb{N}^\star.$$

Definition 1.1.8 *Let $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}_+^\star$.*

1) *The set $B(z_0, r) = \{z \in \mathbb{C},\ |z - z_0| < r\}$ is called the open disk centered at z_0 with radius r.*

2) *The set $\overline{B}(z_0, r) = \{z \in \mathbb{C},\ |z - z_0| \leq r\}$ is called the closed disk centered at z_0 with radius r.*

3) *The set $C(z_0, r) = \{z \in \mathbb{C},\ |z - z_0| = r\}$ is called the circle centered at z_0 with radius r.*

4) *The set $B(0, 1) = \{z \in \mathbb{C},\ |z| < 1\}$ is called the open unit disk.*

5) *The set $C(0, 1) = \{z \in \mathbb{C},\ |z| = 1\}$ is called the unit circle.*

Remark 1.1.5 *$B(z_0, r)$ is also called the $\varepsilon-$neighborhood of z_0 or a neighborhood of z_0.*

Definition 1.1.9 *Let A be a subset of $\mathbb{C}$ and let $z_0 \in A$. We say that z_0 is an interior point of A if there is a neighborhood of z_0 that is completely contained in A.*

Example 1.1.1 **1)** *Let A be the right half-plane, $A = \{z \in \mathbb{C},\ Rez > 0\}$, then $z_0 = \frac{1}{2}$ is an interior point of A.*

2) *Every point z in the open disk $B(z_0, r) = A$ is an interior point of A.*

3) *Let $A = \overline{B}(0, 1)$, then every complex number z such that $|z| = 1$ is not an interior point of A and every complex number z such that $|z| < 1$ is an interior point of A.*

Definition 1.1.10 *Let $A \subset \mathbb{C}$ be a set. We say that A is an open set if every point of A is an interior point of A.*

Example 1.1.2 *$\mathbb{C}$ and $\emptyset$ are open.*

Definition 1.1.11 *Let $A \subset \mathbb{C}$ be a set and $z_0 \in \mathbb{C}$.*

- *We say that z_0 is an exterior point of A if there is some neighborhood of z_0 that does not contain any points of A.*
- *We say that z_0 is a boundary point of A if every neighborhood of z_0 contains at least one point of A and at least one point not in A.*
- *We denote by ∂A the set of all boundary points of A and it is called the boundary (or frontier) of A.*

Definition 1.1.12 *Let $A \subset \mathbb{C}$ be a set. We say that A is closed if it contains all of its boundary points, that is $\partial A \subseteq A$.*
The set $A \cup \partial A = \overline{A}$ is called the closure of A.

Example 1.1.3 **1)** *$\mathbb{C}$ and $\emptyset$ are closed.*

2) *The closed disk $\overline{B}(z_0, r)$ is closed and it is the closure of the open disk $B(z_0, r)$.*

3) *The circular annulus $A = \{z \in \mathbb{C},\ r_1 < |z| \leq r_2\}$ where $r_1, r_2 \in \mathbb{R}_+^\star$ is neither open nor closed and its boundary is given by $\partial A = \{z \in \mathbb{C},\ |z| = r_2\} \cup \{z \in \mathbb{C},\ |z| = r_1\}$.*

Remark 1.1.6 *Let A be a subset of $\mathbb{C}$. Then, A is open if and only if its complement $A^c_{\mathbb{C}} = \{z,\ z \in \mathbb{C}$ and $z \notin A\}$ is closed.*

Definition 1.1.13 *Let $A \subset \mathbb{C}$ be a set and $z_0 \in \mathbb{C}$. We say that z_0 is an accumulation point (or limit point) of A if every neighborhood of z_0 contains infinitely many points of A.*

Remark 1.1.7 *Let A be a subset of $\mathbb{C}$. Then, A is closed if it contains all of its accumulation points.*

Definition 1.1.14 *Let A be a subset of $\mathbb{C}$. We say that A is bounded if there exists a positive real number K such that $|z| \leq K$, $\forall z \in A$.*
If A is not bounded, we say that A is unbounded.

Definition 1.1.15 *Let A be a subset of $\mathbb{C}$. We define the diameter of A by*

$$diam\, A = \sup_{z,w \in A} |z - w| \,.$$

Remark 1.1.8 *It is clear that a set A is bounded if and only if $diam\, A < +\infty$.*

Theorem 1.1.1 (Bolzano-Weierstrass) *Let A be an infinite bounded set of complex numbers, then A has at least one accumulation point.*

Definition 1.1.16 *Let A be a subset of $\mathbb{C}$. we say that A is compact if it is closed and bounded.*

Example 1.1.4 *The closed disk $\overline{B}(z_0, r)$ is compact but the open disk $B(z_0, r)$ is not compact.*

Definition 1.1.17 **1)** *Let $z_1, z_2 \in \mathbb{C}$. The set $[z_1, z_2] = \{z \in \mathbb{C},\ z = (1-t)z_1 + tz_2,\ t \in [0,1]\}$ is called the line segment joining z_1 and z_2.*

2) *Let $z_1, z_2, \ldots, z_{n+1} \in \mathbb{C}$ and let l_k be the line segment joining z_k and z_{k+1}, $k \in \{1, 2, \ldots, n\}$. The successive line segments $l_1, l_2, \ldots, l_n$ from a continuous chain is called a polygonal path joining z_1 to z_{n+1}.*

Definition 1.1.18 *Let A be an open subset of $\mathbb{C}$. we say that A is connected if every pair of points $z_1, z_2 \in A$ can be joined by a polygonal path that lies entirely in A.*
A domain is an open connected set.

Example 1.1.5 *The open disk $B(z_0, r)$ is a domain.*

Definition 1.1.19 *A region is a domain together with some, none, or all of its boundary points.*

Definition 1.1.20 *Let A be a subset of $\mathbb{C}$. We say that A is convex if each pair of points $z_1, z_2 \in A$ can be joined by a line segment $[z_1, z_2]$ such that every point in $[z_1, z_2]$ lies in A.*

Example 1.1.6 *Open disks and closed disks are convex.*

Remark 1.1.9 **1)** *If A and B are two convex sets, then $A \cap B$ is also convex.*

2) *If A is a convex set, then it is connected.*

1.2 Complex-valued functions

Definition 1.2.1 *Let E be a subset of $\mathbb{C}$. A complex function (or complex-valued of a complex variable) f from E to $\mathbb{C}$ is a rule that assigns to each $z \in E$ a complex number $w \in \mathbb{C}$ and this w is called the value of f at z.*
The set E is called the domain of f.
The set $\{w \in \mathbb{C}, \exists z \in E, w = f(z)\}$ is called the rang of f.

Definition 1.2.2 *Let E be a subset of $\mathbb{C}$ and let $f : E \longrightarrow \mathbb{C}$ be a complex function.*

1) *We say that f is single-valued if for each element $z \in E$, f assigns one and only one value $w = f(z)$.*

2) *We say that f is multi-valued if for each element $z \in E$, f assigns a finite or infinite non-empty subset of $\mathbb{C}$.*

3) *We say that f is periodic in E if there exists a constant $T \in \mathbb{C}$ such that $f(z+T) = f(z)$, $\forall z \in E$ and T is a period of f.*

Example 1.2.1 **a)** *The function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by* $f(z) = iz$, $\forall z \in \mathbb{C}$ *is single-valued.*

b) *The function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by* $f(z) = \arg z$, $\forall z \in \mathbb{C}$ *is multi-valued.*

Remark 1.2.1 *A branch of a multi-valued function is a single-valued function.*

Definition 1.2.3 *Let* $E \subset \mathbb{C}$, $f : E \longrightarrow \mathbb{C}$ *be a complex function and* $z = x + iy \in E$, *then the value of* f *at* z *can be written as* $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$ *and* u *is called the real part of* f *and* v *is called the imaginary part of* f.

Example 1.2.2 **(i)** *The domain of the function* f *defined by* $w = f(z) = z^2 + 5z$ *is* $\mathbb{C}$ *and for all* $z = x+iy$, $x, y \in \mathbb{R}$ *we have* $w = f(x+iy) = x^2-y^2+5x+i(2xy+5y)$, *then the real part of* f *is the function* $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $u(x, y) = x^2-y^2+5x$, $\forall(x, y) \in \mathbb{R}^2$. *The imaginary part of* f *is the function* $v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $v(x, y) = 2xy + 5y$, $\forall(x, y) \in \mathbb{R}^2$.

(ii) *The domain of the function* f *defined by* $w = f(z) = |z|$ *is* $\mathbb{C}$ *and it is clear that* f *is a real-valued function, then the real part of* f *is the function* $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $u(x, y) = \sqrt{x^2 + y^2}$, $\forall(x, y) \in \mathbb{R}^2$. *The imaginary part of* f *is* $v \equiv 0$.

Definition 1.2.4 *Let* $z_0 \in \mathbb{C}$ *and let* f *be a complex function defined in some neighborhood of* z_0. *We say that* f *has a limit* w_0 *as* z *approaches* z_0 *if we have*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ |z - z_0| \leq \delta \Longrightarrow \left|f(z) - w_0\right| < \varepsilon.$$

In this case we write $\lim\limits_{z\to z_0} f(z) = w_0$.

Example 1.2.3 **1)** *Let* f *be the function defined by* $f(z) = 8z + 1$, $\forall z \in \mathbb{C}$. *We will prove that* $\lim\limits_{z\to i-1} f(z) = -7 + 8i$. *Let* $z_0 = -1 + i$ *and* $w_0 = -7 + 8i$. *We have*

$$\begin{aligned} \left|f(z) - w_0\right| &= |8z + 1 - (-7 + 8i)| \\ &= 8\,|z - (-1 + i)|, \end{aligned}$$

hence,

$$\forall \varepsilon > 0,\ \exists \delta = \frac{\varepsilon}{8} > 0,\ |z - z_0| \leq \delta \Longrightarrow \left|f(z) - w_0\right| < \varepsilon.$$

2) *Let f be the function defined by $f(z) = \frac{z^2+4}{z-2i}$, $\forall z \in \mathbb{C} \setminus \{2i\}$. We will prove that $\lim\limits_{z\to 2i} f(z) = 4i$.*

Let $z_0 = 2i$ and $w_0 = 4i$. We have

$$\begin{aligned} \left|f(z) - w_0\right| &= \left|\frac{z^2+4}{z-2i} - 4i\right| \\ &= |z - 2i|, \end{aligned}$$

hence,

$$\forall \varepsilon > 0,\ \exists \delta = \varepsilon > 0,\ |z - z_0| \leq \delta \Longrightarrow \left|f(z) - w_0\right| < \varepsilon.$$

Theorem 1.2.1 *Let $f : E \longrightarrow \mathbb{C}$ be a function defined by*

$$f(z) = f(x+iy) = u(x,y) + iv(x,y),\ \forall z = x+iy \in E$$

and let $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$, $x_0, y_0, u_0, v_0 \in \mathbb{R}$. Then

$$\lim_{z\to z_0} f(z) = w_0 \Longleftrightarrow \begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \\ \text{and} \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0, \end{cases}$$

Proposition 1.2.1 *Let $E \subset \mathbb{C}$, $z_0 \in E$ and let $f, g : E \longrightarrow \mathbb{C}$ be two functions. Suppose that each function has a limit as z approaches z_0. Then*

1) $\lim\limits_{z\to z_0}(f(z) + g(z)) = \lim\limits_{z\to z_0} f(z) + \lim\limits_{z\to z_0} g(z)$.

2) $\lim\limits_{z\to z_0}(f(z)g(z)) = (\lim\limits_{z\to z_0} f(z))(\lim\limits_{z\to z_0} g(z))$.

2) *If $\lim\limits_{z\to z_0} g(z) \neq 0$, then $\lim\limits_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim\limits_{z\to z_0} f(z)}{\lim\limits_{z\to z_0} g(z)}$.*

Remark 1.2.2 *We use the notation $z \to \infty$ if we have $|z| \to +\infty$ and we use the notation $f(z) \to \infty$ if we have $|f(z)| \to +\infty$.*

Definition 1.2.5 *Let $z_0, w_0 \in \mathbb{C}$ and let f be a complex function defined in some neighborhood of z_0.*

1) *We say that* $\lim\limits_{z\to z_0} f(z) = \infty$ *if we have*

$$\forall\alpha > 0, \exists\delta > 0, |z - z_0| \leq \delta \Longrightarrow |f(z)| > \alpha.$$

2) *We say that* $\lim\limits_{z\to\infty} f(z) = w_0$ *if we have*

$$\forall\varepsilon > 0, \exists\beta > 0, |z| > \beta \Longrightarrow |f(z) - w_0| < \varepsilon.$$

3) *We say that* $\lim\limits_{z\to\infty} f(z) = \infty$ *if we have*

$$\forall\alpha > 0, \exists\beta > 0, |z| > \beta \Longrightarrow |f(z)| > \alpha.$$

Example 1.2.4 **1)** *Let* $f(z) = \frac{z+1}{5z-2}$, *then* $\lim\limits_{z\to\infty} f(z) = \frac{1}{5}$.

2) *Let* $f(z) = \frac{z-i}{2z^2+1}$, *then* $\lim\limits_{z\to\infty} f(z) = 0$.

Definition 1.2.6 *Let* $E \subset \mathbb{C}$, $z_0 \in E$ *and* $f : E \longrightarrow \mathbb{C}$ *be a complex function.*

- *We say that* f *is continuous at* z_0 *if* $\lim\limits_{z\to\infty} f(z) = f(z_0)$.
- *We say that* f *is continuous on* E *if it is continuous at each point of* E.

Example 1.2.5 *The following complex functions are continuous on* $\mathbb{C}$.

♣ $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = Rez$.

♣ $g : \mathbb{C} \longrightarrow \mathbb{C}$, $g(z) = z^2 + z - 3$.

Remark 1.2.3 *Let* $E \subset \mathbb{C}$, $z_0 = x_0 + iy_0 \in E$ *and let* $f : E \longrightarrow C$ *be a complex function such that* $f(z) = u(x, y) + iv(x, y)$, $\forall z \in E$. *Then* f *is continuous at* z_0 *if and only if* u *and* v *are continuous at* (x_0, y_0).

Proposition 1.2.2 *Let* $E \subset \mathbb{C}$, $z_0 \in E$ *and let* $f, g : E \longrightarrow C$ *be two complex functions. If* f *and* g *are continuous at* z_0, *then* $f \pm g$, fg *and* $\frac{f}{g}$, $g(z_0) \neq 0$ *are continuous at* z_0.

Remark 1.2.4 **1)** *Polynomial functions defined by* $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $\forall z \in \mathbb{C}$, $a_j \in \mathbb{R}$, $j \in \{0, 1, \ldots, n\}$ *are continuous on the whole plane* $\mathbb{C}$.

2) *Rational functions defined by* $R(z) = \frac{P(z)}{Q(z)}$ *where* P *and* Q *are two polynomial functions, are continuous at each point* z *such that* $Q(z) \neq 0$.

Example 1.2.6 *Let us consider the functions* f, g, h *defined respectively by* $f(z) = z^3 - 2z^2 + 6$, $g(z) = \frac{z+2i}{z}$ *and* $h(z) = \frac{z^2+4}{z(z-3i)}$. *Then* f *is continuous on* $\mathbb{C}$, g *is continuous on* $\mathbb{C}^\star$ *and* h *is continuous on* $\mathbb{C} \setminus \{0, 3i\}$.

1.3 Holomorphic and harmonic functions

Definition 1.3.1 *Let* $E \subset \mathbb{C}$, $z_0 \in E$ *and let* $f : E \longrightarrow \mathbb{C}$ *be a complex single-valued function. We say that* f *is differentiable at* z_0 *if the following limit exists and belongs to* $\mathbb{C}$.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{1.1}$$

In this case, we denote by $f'(z_0)$ *(or* $\frac{df}{dz}(z_0)$*) the derivative of* f *at* z_0 *and*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If f *is differentiable at every point* $z \in E$, *we say that* f *is differentiable in* E.

Remark 1.3.1 *If we put* $\Delta z = z - z_0$, *then the limit given in* (2.2.1) *becomes*

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{1.2}$$

Example 1.3.1 **1)** *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be the function defined by* $f(z) = z^2$, $\forall z \in \mathbb{C}$ *and let* $z_0 \in \mathbb{C}$. *Then*

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \frac{z^2 - z_0^2}{z - z_0} \\ &= \frac{(z - z_0)(z + z_0)}{z - z_0} \\ &= z + z_0. \end{aligned}$$

Hence

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0,$$

then f is differentiable at z_0 and $f'(z_0) = 2z_0$.

Similarly, we have

$$\frac{d}{dz}(z^n) = nz^{n-1}, \ \forall n \in \mathbb{N}^\star.$$

2) *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \overline{z}$, $\forall z \in \mathbb{C}$ and let $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{\overline{z_0 + \Delta z} - \overline{z_0}}{\Delta z} \\ &= \frac{\overline{z_0} + \overline{\Delta z} - \overline{z_0}}{\Delta z} \\ &= \frac{\overline{\Delta z}}{\Delta z}. \end{aligned}$$

Let $z = x + iy$, $x, y \in \mathbb{R}$, then $\Delta z = \Delta x + i\Delta y$ and

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \begin{cases} 1, & \text{if } \Delta y = 0 \\ -1, & \text{if } \Delta x = 0, \end{cases}$$

therefore, the limit given by (1.2) does not exist, and f is not differentiable at any $z_0 \in \mathbb{C}$.

Remark 1.3.2 *All rules known for real functions are also satisfied for complex functions.*

Proposition 1.3.1 *Let $E \subset \mathbb{C}$, $z_0 \in E$, c be a real number and let $f, g : E \longrightarrow \mathbb{C}$ be complex single-valued functions. If f and g are differentiable at z_0, then*

1) *$f \mp g$ is differentiable at z_0 and $(f \mp g)'(z_0) = f'(z_0) \mp g'(z_0)$.*

2) *cf is differentiable at z_0 and $(cf)'(z_0) = cf'(z_0)$.*

3) *fg is differentiable at z_0 and $(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0)$.*

4) *$\frac{f}{g}$ is differentiable at z_0 and $\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$, if $g(z_0) \neq 0$.*

5) *If f is differentiable at $g(z_0)$, then $f \circ g$ is differentiable at z_0 and $(f \circ g)'(z_0) = f'\left(g(z_0)\right) g'(z_0)$.*

Theorem 1.3.1 *Let $E \subset \mathbb{C}$, $z_0 \in E$ and $f : E \longrightarrow \mathbb{C}$ be a complex single-valued function. If f is differentiable at z_0, then it is continuous at z_0.*
The converse is not true.

Example 1.3.2 *The function f defined by $f(z) = |z|^2$, $\forall z \in \mathbb{C}$ is continuous in $\mathbb{C}$, but it is differentiable only at $z = 0$.*

Theorem 1.3.2 (Cauchy-Riemann equations) *Let $E \subset \mathbb{C}$, $z_0 = x_0 + iy_0 \in E$ and let $f : E \longrightarrow \mathbb{C}$ be a complex single-valued function such that $f(z) = u(x, y)+iv(x, y)$, $\forall z = x+iy \in E$. If f is differentiable at z_0, then the first order partial derivatives of u and v satisfy, at (x_0, y_0), the following equations*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{1.3}$$

The equations given in (1.3) are called the Cauchy-Riemann equations.

Example 1.3.3 *Let $f(z) = x^2 + y + i(y^2 - x) = u(x, y) + iv(x, y)$, $\forall z = x + iy \in \mathbb{C}$. Then*

$$u(x, y) = x^2 + y \text{ and } v(x, y) = y^2 - x,$$

therefore

$$\frac{\partial u}{\partial x}(x, y) = 2x,\ \frac{\partial v}{\partial y}(x, y) = 2y \text{ and } \frac{\partial u}{\partial y}(x, y) = 1,\ \frac{\partial v}{\partial x}(x, y) = -1.$$

Hence f is not differentiable at any point $z \in \mathbb{C} \setminus \{z \in \mathbb{C},\ z = x + ix,\ x \in \mathbb{R}\}$.

Theorem 1.3.3 *Let $E \subset \mathbb{C}$, $z_0 = x_0 + iy_0 \in E$ and let $f : E \longrightarrow \mathbb{C}$ be a complex single-valued function such that $f(z) = u(x, y) + iv(x, y)$, $\forall z = x + iy \in E$. If u and v have continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in a neighborhood of z_0 and if they satisfied, at (x_0, y_0), the Cauchy-Riemann equations given in (1.3), then f is differentiable at z_0.*
Furthermore,

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \end{aligned}$$

Example 1.3.4 *Let* $f(z) = z^3$, $\forall z \in \mathbb{C}$, *then*

$$f(z) = f(x+iy) = x^3 - 3xy^2 + i(3x^2y - y^3) = u(x,y) + iv(x,y)$$

where

$$u(x,y) = x^3 - 3xy^2 \text{ and } v(x,y) = 3x^2y - y^3.$$

Then for all $(x,y) \in \mathbb{R}^2$ *we have*

$$\frac{\partial u}{\partial x}(x,y) = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}(x,y) \text{ and } \frac{\partial u}{\partial y}(x,y) = -6xy = -\frac{\partial v}{\partial x}(x,y).$$

Therefore, u and v have continuous partial derivatives and they satisfy (1.3), *hence f is differentiable in* $\mathbb{C}$ *and for all* $z \in \mathbb{C}$ *we have*

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \\ &= 3x^2 - 3y^2 + i6xy \\ &= 3(x^2 - y^2 + 2ixy) = 3z^2. \end{aligned}$$

Definition 1.3.2 *Let* $E \subset \mathbb{C}$, $z_0 \in E$ *and let* $f : E \longrightarrow \mathbb{C}$ *be a complex single-valued function. We say that f is holomorphic at* z_0 *if it is differentiable at* z_0 *and in a neighborhood of* z_0.

Remark 1.3.3 *A function can be differentiable at a point but not holomorphic at the same point.*

Example 1.3.5 *The function* $z \longmapsto |z|^2$ *is differentiable at* $z = 0$ *but it is not holomorphic at* $z = 0$.

Definition 1.3.3 *Let* $E \subset \mathbb{C}$ *be a domain and let* $f : E \longrightarrow \mathbb{C}$ *be a complex single-valued function. We say that f is holomorphic in E if it is holomorphic at each point* $z_0 \in E$.

Definition 1.3.4 *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be a complex single-valued function. We say that f is entire if it is holomorphic in whole* $\mathbb{C}$.

Definition 1.3.5 *We say that a a complex single-valued function f is holomorphic at* $z = \infty$ *if* $z \mapsto f\left(\frac{1}{z}\right)$ *is holomorphic at* $z = 0$.

Example 1.3.6 *Polynomial complex functions are entire.*

Proposition 1.3.2 **(L'Hôpital)** *Let $E \subset \mathbb{C}$, $z_0 \in E$ and let $f, g : E \longrightarrow \mathbb{C}$ be two complex single-valued functions such that f and g are holomorphic at z_0 with $f(z_0) = g(z_0) = 0$, but $g'(z_0) \neq 0$, then*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Example 1.3.7 *Let $z_0 = i$ and*

$$h(z) = \frac{z^3 + i}{z^2 + 1} = \frac{f(z)}{g(z)}$$

where $f(z) = z^3 + i$ and $g(z) = z^2 + 1$, then f and g are holomorphic at z_0. In addition, $f(i) = g(i) = 0$ and $g'(z_0) = 2i \neq 0$, hence by applying the Proposition 1.3.2 we get

$$\lim_{z \to z_0} h(z) = \lim_{z \to i} \frac{z^3 + i}{z^2 + 1} = \lim_{z \to i} \frac{f'(z)}{g'(z)} = \lim_{z \to i} \frac{3z^2}{2z} = \frac{3}{2} i.$$

Theorem 1.3.4 *Let $E \subset \mathbb{C}$ be a domain and let $f : E \longrightarrow \mathbb{C}$ be a complex single-valued function. If f is holomorphic in E and $f'(z) = 0$, $\forall z \in E$, then $f(z) = k \in \mathbb{C}$, $\forall z \in E$. That is f is constant in E.*

Example 1.3.8 *Let $E = \{z \in \mathbb{C}, |z| < 1\} \cup \{z \in \mathbb{C}, |z| > 2\}$ and let $f : E \longrightarrow \mathbb{C}$ be the function defined by*

$$f(z) = \begin{cases} 1, & \text{if } |z| < 1 \\ 2, & \text{if } |z| > 2. \end{cases}$$

It is clear that E is not a domain, in addition f is holomorphic in E and $f'(z) = 0$ for all $z \in E$, but f is not constant in E.

Theorem 1.3.5 *Let $E \subset \mathbb{C}$ be a domain and let $f : E \longrightarrow \mathbb{C}$ be a complex single-valued function. If f is holomorphic in E and if $|f|$ is constant in E, then f is constant in E.*

Definition 1.3.6 *Let $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ and let $h : D \longrightarrow \mathbb{R}$ be a real function. We say that h is harmonic at (x_0, y_0) if it has continuous partial derivatives of the first order and of the second order and these partial derivatives satisfy, at (x_0, y_0), the following Laplace equation*

$$\triangle h(x_0, y_0) = \frac{\partial^2 h}{\partial x^2}(x_0, y_0) + \frac{\partial^2 h}{\partial y^2}(x_0, y_0) = 0.$$

We say that h is harmonic in D if it is harmonic at each point (x, y) *of D.*

Theorem 1.3.6 *Let* $E \subset \mathbb{C}$ *be a domain and let* $f : E \longrightarrow \mathbb{C}$ *be a complex single-valued function such that* $f(z) = u + iv$, $\forall z \in E$. *If f is holomorphic in E, then u and v are harmonic in E.*

Definition 1.3.7 *Let* $D \subset \mathbb{R}^2$ *and let* $u, v : D \longrightarrow \mathbb{R}$ *be two real functions. If u and v are harmonic in D and their first order partial derivatives satisfy Cauchy-Riemann equations* (1.3), *then we say that v is a harmonic conjugate of u.*

Example 1.3.9 *Let* $u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be two real functions defined by*

$$u(x, y) = x^2 - y^2,\ v(x, y) = 2xy,\ \forall (x, y) \in \mathbb{R}^2,$$

then

$$\frac{\partial u}{\partial x}(x, y) = 2x,\ \frac{\partial u}{\partial y}(x, y) = -2y,\ \forall (x, y) \in \mathbb{R}^2,$$

hence

$$\frac{\partial^2 u}{\partial x^2}(x, y) = 2,\ \frac{\partial^2 u}{\partial y^2}(x, y) = -2,\ \forall (x, y) \in \mathbb{R}^2$$

and

$$\triangle u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0,$$

then u is harmonic in $\mathbb{R}^2$
Similarly, we can prove that v is harmonic in $\mathbb{R}^2$. *Furthermore, u and v satisfy Cauchy-Riemann equations* (1.3), *then v is a conjugate of u.*

Example 1.3.10 *Find a holomorphic function f whose real part is given by*

$$u(x, y) = e^{-x} \sin y,\ \forall (x, y) \in \mathbb{R}^2.$$

We have

$$\frac{\partial u}{\partial x}(x, y) = -e^{-x} \sin y,\ \frac{\partial u}{\partial y}(x, y) = e^{-x} \cos y,\ \forall (x, y) \in \mathbb{R}^2,$$

hence

$$\frac{\partial^2 u}{\partial x^2}(x, y) = e^{-x} \sin y,\ \frac{\partial^2 u}{\partial y^2}(x, y) = -e^{-x} \sin y,\ \forall (x, y) \in \mathbb{R}^2$$

and

$$\triangle u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0,$$

then u is harmonic in $\mathbb{R}^2$.

Let $v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a conjugate of u, then u and v satisfy Cauchy-Riemann equations (1.3)*, that is*

$$\frac{\partial v}{\partial y}(x, y) = \frac{\partial u}{\partial x}(x, y) = -e^{-x} \sin y, \ \forall (x, y) \in \mathbb{R}^2 \tag{1.4}$$

and

$$\frac{\partial v}{\partial x}(x, y) = -\frac{\partial u}{\partial y}(x, y) = -e^{-x} \cos y, \ \forall (x, y) \in \mathbb{R}^2. \tag{1.5}$$

By integrating (1.4) *with respect to y, we get*

$$v(x, y) = e^{-x} \cos y + \varphi(x). \tag{1.6}$$

Substituting (1.6) *in* (1.5) *we obtain*

$$-e^{-x} \cos y + \varphi'(x) = -e^{-x} \cos y,$$

then $\varphi'(x) = 0$, $\forall x \in \mathbb{R}$, hence $\varphi(x) = k \in \mathbb{R}$. Therefore,

$$f(z) = e^{-x} \sin y + ie^{-x} \cos y + ik = ie^{-z} + ik, \ k \in \mathbb{R}.$$

CHAPTER 2

ELEMENTARY COMPLEX-VALUED FUNCTIONS

2.1 Complex exponential function

Definition 2.1.1 *The exponential function is the function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by*

$$w = f(z) = e^z = e^x \cos y + ie^x \sin y, \ \forall z = x + iy \in \mathbb{C}, \ x, y \in \mathbb{R}.$$

By applying the Theorem 1.3.3 we can easily obtain the following result.

Proposition 2.1.1 *Let* $w = f(z) = e^z$, $\forall z \in \mathbb{C}$, *then* f *is entire and for all* $z \in \mathbb{C}$ *we have* $f'(z) = e^z$.

Proposition 2.1.2 *Let* z, z_1 *and* z_2 *be three complex number. Then*

1) $e^z = 1 \Leftrightarrow z = 2k\pi i$, $k \in \mathbb{Z}$.

2) $e^{z_1} = e^{z_2} \Leftrightarrow z_1 = z_2 + 2k\pi i,\ k \in \mathbb{Z}$.

3) $\overline{e^z} = e^{\overline{z}}$.

Proof.

1) $\Leftarrow$? Let $z = 2k\pi i,\ k \in \mathbb{Z}$, then $e^z = e^{2k\pi i} = \cos(2k\pi) + i\sin(2k\pi) = 1$.

$\Rightarrow$? Let $z = x + iy,\ x, y \in \mathbb{R}$, then

$$\begin{aligned} e^z = 1 \quad &\Rightarrow \quad e^x e^{iy} = 1 \\ &\Rightarrow \quad e^x |e^{iy}| = 1 \\ &\Rightarrow \quad e^x = 1 \\ &\Rightarrow \quad x = 0, \end{aligned}$$

hence

$$e^z = e^{iy} = \cos y + i \sin y = 1,$$

then

$$\cos y = 1 \text{ and } \sin y = 0.$$

Therefore, $y = 2k\pi,\ k \in \mathbb{Z}$, that is $z = 2k\pi i,\ k \in \mathbb{Z}$.

2) $\Leftarrow$? Let $z_1 = z_2 + 2k\pi i,\ k \in \mathbb{Z}$, then

$$e^{z_1} = e^{z_2 + 2k\pi i} = e^{z_2} e^{2k\pi i} = e^{z_2}.$$

$\Rightarrow$? Suppose that $e^{z_1} = e^{z_2}$, then $e^{z_1 - z_2} = 1$. By using 1) we obtain $z_1 - z_2 = 2k\pi i,\ k \in \mathbb{Z}$, that is $z_1 = z_2 + 2k\pi i,\ k \in \mathbb{Z}$.

3) Let $z = x + iy,\ x, y \in \mathbb{R}$, then

$$e^z = e^{x+iy} = e^x \cos y + i e^x \sin y,$$

hence

$$\overline{e^z} = e^x \cos y - i e^x \sin y = e^{x - iy} = e^{\overline{z}}.$$

■

Corollary 2.1.1 *The function $z \longmapsto e^z$ is periodic with period $T = 2\pi i$.*

2.2 Complex trigonometric Functions

Definition 2.2.1 *We define the complex sine and the complex cosine respectively by*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \ \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

From the Proposition 2.1.1 and the Definition 2.2.1, we have the following result.

Proposition 2.2.1 *The complex functions* $z \longmapsto \cos z$ *and* $z \longmapsto \sin z$ *are entire and for all* $z \in \mathbb{C}$, *we have*

$$\frac{d}{dz}\sin z = \cos z \text{ and } \frac{d}{dz}\cos z = -\sin z.$$

Proposition 2.2.2 *For all* $z \in \mathbb{C}$, *we have*

$$\overline{\sin z} = \sin \overline{z} \text{ and } \overline{\cos z} = \cos \overline{z}.$$

Proof. By using the Definition 2.2.1, the Proposition 2.1.2 and that $\overline{iz} = -i\overline{z}$, $\forall z \in \mathbb{C}$, we have

$$\begin{aligned} \overline{\sin z} &= \frac{\overline{e^{iz}} - \overline{e^{-iz}}}{\overline{2i}} \\ &= \frac{e^{\overline{iz}} - e^{\overline{-iz}}}{-2i} \\ &= \frac{e^{i\overline{z}} - e^{-i\overline{z}}}{2i} \\ &= \sin \overline{z}. \end{aligned}$$

Similarly, we can prove that $\overline{\cos z} = \cos \overline{z}$. ■

Proposition 2.2.3 *Let* z, z_1, z_2 *be three complex numbers, then*

1) $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$.

2) $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$, $\sin(\frac{\pi}{2} - z) = \cos z$.

3) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\sin^2 z + \cos^2 z = 1$.

4) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

5) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$.

6) $\sin(2z) = 2\sin z \cos z, \quad \cos(2z) = \cos^2 z - \sin^2 z$.

7) $\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos(2z_2) - \cos(2z_1)$.

8) $\cos(z_1 + z_2)\sin(z_1 - z_2) = \sin(2z_1) - \sin(2z_2)$.

9) $\sin z = 0 \Leftrightarrow z = k\pi, \ k \in \mathbb{Z}$.

10) $\cos z = 0 \Leftrightarrow z = \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$.

Proof. Exercise. ■

Corollary 2.2.1 *The complex functions $z \longmapsto \cos z$ and $z \longmapsto \sin z$ are periodic with period $T = 2\pi$.*

Definition 2.2.2 *We define the complex tangent, cotangent, secant and cosecant functions by*

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

From the Definition 2.2.2, the Proposition 2.2.1 and the Proposition 2.2.3 we get the following result.

Proposition 2.2.4 **1)** *The complex functions $z \longmapsto \cot z$ and $z \longmapsto \csc z$ are holomorphic in $\mathbb{C} \setminus \{k\pi, k \in \mathbb{Z}\}$.*

2) *The complex functions $z \longmapsto \tan z$ and $z \longmapsto \sec z$ are holomorphic in $\mathbb{C} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$. Furthermore,*

$$\frac{d}{dz}\tan z = \sec^2 z, \quad \frac{d}{dz}\sec z = \sec(\tan z), \quad \frac{d}{dz}\cot z = -\csc^2 z, \quad \frac{d}{dz}\csc z = -\csc(\cot z).$$

Remark 2.2.1 • *The complex secant and cosecant are periodic functions with period $T = 2\pi$.*

• *The complex tangent and cotangent are periodic functions with period $T = \pi$.*

- *The inequalities* $|\sin x| \leq 1$ *and* $|\cos x| \leq 1$ *are satisfied for all real numbers* x*, but they, in general, are not satisfied for complex numbers. Indeed* $|\cos i| \simeq 1.5431 > 1$ *and* $|\sin(2+i)| \simeq 1.4859 > 1$.

2.3 Complex hyperbolic Functions

Definition 2.3.1 *We define the complex hyperbolic functions* cosh, sinh, tanh *and* coth *by*

$$\sinh z = \frac{e^z - e^{-z}}{2}, \ \cosh z = \frac{e^z + e^{-z}}{2}, \ \tanh z = \frac{\sinh z}{\cosh z}, \ \coth z = \frac{\cosh z}{\sinh z}.$$

From the Proposition 2.1.1 and the Definition 2.3.1, we have the following result.

Proposition 2.3.1 *The complex functions* $z \longmapsto \cosh z$ *and* $z \longmapsto \sinh z$ *are entire and for all* $z \in \mathbb{C}$*, we have*

$$\frac{d}{dz}\sinh z = \cosh z \text{ and } \frac{d}{dz}\cosh z = \sinh z.$$

Proposition 2.3.2 *Let* $z, z_1, z_2 \in \mathbb{C}$*. Then*

1) $\cosh(iz) = \cos z, \ \ \cos(iz) = \cosh z.$

2) $\sinh(iz) = i\sin z, \ \ \sin(iz) = i\sinh z.$

3) $\sinh(-z) = -\sinh z, \ \ \cosh(-z) = \cosh z.$

4) $\cosh^2 z - \sinh^2 z = 1.$

5) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2.$

6) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2.$

7) $\sinh(2z) = 2\sinh z\cosh z, \ \ \cosh(2z) = \cosh^2 z + \sinh^2 z.$

8) $\sinh z = 0 \Leftrightarrow z = k\pi i, \ k \in \mathbb{Z}$ *and* $\cosh z = 0 \Leftrightarrow z = \left(k + \frac{1}{2}\right)\pi i, \ k \in \mathbb{Z}$

Proof. Exercise. ■

Corollary 2.3.1 **1)** *The complex functions* $z \longmapsto \cosh z$ *and* $z \longmapsto \sinh z$ *are periodic with period* $T = 2\pi i$.

2) *The complex function* $z \longmapsto \coth z$ *is holomorphic in* $\mathbb{C} \setminus \{k\pi i,\ k \in \mathbb{Z}\}$.

3) *The complex function* $z \longmapsto \tanh z$ *is holomorphic in* $\mathbb{C} \setminus \left\{\left(k + \frac{1}{2}\right)\pi i,\ k \in \mathbb{Z}\right\}$.

4) $\frac{d}{dz} \tanh z = \frac{1}{\cosh^2 z},\ \forall z \in \mathbb{C} \setminus \left\{\left(k + \frac{1}{2}\right)\pi i,\ k \in \mathbb{Z}\right\}$.

5) $\frac{d}{dz} \coth z = -\frac{1}{\sinh^2 z},\ \forall z \in \mathbb{C} \setminus \{k\pi i,\ k \in \mathbb{Z}\}$.

2.4 Complex logarithmic Function

Definition 2.4.1 *We define the complex logarithm function as an inverse to the complex exponential function. That is a logarithm of non zero complex number z is any complex number w that* $e^w = z$ *and we write* $w = \log z = \ln|z| + i \arg z$.

Remark 2.4.1 $z \longmapsto \log z$ *is a multi-valued function.*

Example 2.4.1 **1)** $\log 3 = \ln|3| + i \arg 3 = \ln 3 + 2k\pi i,\ k \in \mathbb{Z}$.

2) $\log(-1) = \ln|-1| + i \arg(-1) = (2k+1)\pi i,\ k \in \mathbb{Z}$.

3) $\log(1+i) = \ln|1+i| + i \arg(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right),\ k \in \mathbb{Z}$.

Proposition 2.4.1 *Let* $z, z_1, z_2 \in \mathbb{C}^\star$*, then*

1) *If* $z = re^{i\theta}$*, then* $\log z = \ln r + i\theta$.

2) $\log e^z = z + 2k\pi i,\ k \in \mathbb{Z}$.

3) $\log z_1 z_2 = \log z_1 + \log z_2$.

4) $\log \frac{z_1}{z_2} = \log z_1 - \log z_2$.

Proof. Exercise. ■

Definition 2.4.2 *We define the principal branch of the complex logarithm function* log *by*

$$Logz = \ln|z| + iArgz.$$

Theorem 2.4.1 *The function $z \longmapsto Logz$ is holomorphic in the domain $E = \mathbb{C}\backslash]-\infty, 0[$*

$$\forall z \in E,\ \frac{d}{dz}Logz = \frac{1}{z}.$$

Proof. Let $w = Logz$, $z_0 \in E$ and $w_0 = Logz_0$, then

$$\begin{aligned} \lim_{z\to z_0}\frac{Logz - Logz_0}{z - z_0} &= \lim_{w\to w_0}\frac{w - w_0}{e^w - e^{w_0}} \\ &= \lim_{w\to w_0}\frac{1}{\frac{e^w - e^{w_0}}{w - w_0}} \\ &= \frac{1}{e^{w_0}} \\ &= \frac{1}{e^{Logz_0}} \\ &= \frac{1}{z_0}. \end{aligned}$$

■

2.5 Complex power Function

Definition 2.5.1 *Let $z \in \mathbb{C}^\star$ and $\alpha \in \mathbb{C}$. We define the $\alpha - th$ power z^α of z by*

$$z^\alpha = e^{\alpha \log z} = e^{\alpha \ln|z| + i \arg z}.$$

Remark 2.5.1 **1)** *The complex function $z \longmapsto z^\alpha$ is a multi-valued function.*

2) *Let $z \in \mathbb{C}^\star$ and $\alpha \in \mathbb{C}$. Then*

$$z^\alpha = e^{\alpha\left(\ln|z| + iArgz\right)} e^{2\alpha k\pi i},\ k \in \mathbb{Z}.$$

Example 2.5.1 • $1^i = e^{i\log 1} = e^{-2k\pi}$, $k \in \mathbb{Z}$.

• $(-2)^i = e^{i\log(-2)} = e^{i\ln 2}e^{-(\pi+2k\pi)}$, $k \in \mathbb{Z}$.

Remark 2.5.2 *Let $z \in \mathbb{C}^\star$ and $\alpha \in \mathbb{C}$. Then*

- *$z \longmapsto z^\alpha$ is single-valued function if $\alpha \in \mathbb{Z}$.*
- *$z \longmapsto z^\alpha$ takes finitely many values if α is real rational number.*
- *$z \longmapsto z^\alpha$ takes infinitely many values in all other cases.*

Definition 2.5.2 *Let $z \in \mathbb{C}^\star$ and $\alpha \in \mathbb{C}$ be a constant. The principal branch of the function $z \longmapsto z^\alpha$ is the single-valued function defined by $f(z) = z^\alpha = e^{\alpha Logz}$.*

Remark 2.5.3 *Since $z \longmapsto e^z$ is entire and $z \longmapsto Logz$ is holomorphic in $E = \mathbb{C}\backslash]-\infty, 0]$, then the principal branch of $z \longmapsto z^\alpha$ is holomorphic in E. Furthermore,*

$$\forall z \in E,\ \frac{d}{dz}\left(z^\alpha\right) = \frac{d}{dz}\left(e^{\alpha Logz}\right) = e^{\alpha Logz}\frac{d}{dz}\left(\alpha Logz\right) = \frac{\alpha}{z}e^{\alpha Logz}.$$

2.6 Inverse trigonometric and hyperbolic functions

Definition 2.6.1 **1)** *The complex arcsine function is the solution to the equation*

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

2) *The complex arccosine function is the solution to the equation*

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}.$$

Proposition 2.6.1 **1)**

$$\arcsin z = \frac{1}{i}\log\left(iz + \left(1 - z^2\right)^{\frac{1}{2}}\right).$$

2)

$$\arccos z = \frac{1}{i} \log\left(z + i\left(1 - z^2\right)^{\frac{1}{2}}\right).$$

Proof. Exercise. ■

Definition 2.6.2 • *The principal value of the complex arcsine function is defined by*

$$Arcsinz = \frac{1}{i} Log\left(iz + \left|1 - z^2\right|^{\frac{1}{2}} e^{\frac{i}{2} Arg\left(1-z^2\right)}\right).$$

• *The principal value of the complex arccosine function is defined by*

$$Arccosinz = \frac{1}{i} Log\left(z + i\left|1 - z^2\right|^{\frac{1}{2}} e^{\frac{i}{2} Arg\left(1-z^2\right)}\right).$$

Proposition 2.6.2 **1)** $\arcsin z + \arccos z = \frac{1}{i} \log i$.

2) $Arcsinz + Arccosz = \frac{\pi}{2}$.

Proof.

1) Let $z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$, then by letting $v = e^{iw}$ we obtain

$$v^2 - 2izv - 1 = 0,$$

hence

$$v = iz + \sqrt{1 - z^2},$$

then

$$\frac{i}{v} = \frac{i}{iz + \sqrt{1 - z^2}} = \frac{i\left(-iz + \sqrt{1 - z^2}\right)}{\left(iz + \sqrt{1 - z^2}\right)\left(-iz + \sqrt{1 - z^2}\right)} = z + i\sqrt{1 - z^2}.$$

Therefore,

$$\arcsin z + \arccos z = \frac{1}{i}\left(\log v + \log\left(\frac{i}{v}\right)\right) = \frac{1}{i}\left(\log\left(v \times \frac{i}{v}\right)\right) = \frac{1}{i} \log i.$$

2) From the Definition 2.6.8 we have

$$\begin{aligned} Arcsinz + Arccosz &= \frac{1}{i} Log\left(\frac{i}{v}\right) + \frac{1}{i} Logv \\ &= \frac{1}{i}\left(Log|v| + Log\frac{1}{|v|} + iArgv + iArg\left(\frac{i}{v}\right)\right) \\ &= Argv + Arg\left(\frac{1}{v}\right). \end{aligned}$$

On the other hand we have

$$v = iz + \sqrt{|1-z^2|}e^{\frac{i}{2}Arg\left(1-z^2\right)} \text{ and } \frac{i}{v} = z + i\sqrt{|1-z^2|}e^{\frac{i}{2}Arg\left(1-z^2\right)},$$

then

$$Re(\pm iz) = \mp Imz,\ Rev = -Imz + \sqrt{|1-z^2|}\cos\left(\frac{1}{2}Arg\left(1-z^2\right)\right),$$
$$Re\left(\frac{1}{v}\right) = Imz + \sqrt{|1-z^2|}\cos\left(\frac{1}{2}Arg\left(1-z^2\right)\right).$$

We know that the sign of the real part of any complex number z is the same as the sign of the real part of $\frac{1}{z}$, then we can easily prove that $Rev > 0$, hence

$$Argv + Arg\left(\frac{i}{v}\right) = Argi = \frac{\pi}{2}.$$

■

Definition 2.6.3 **1)** *The complex arctangent function is the solution to the equation*

$$z = \tan w = \frac{\sin w}{\cos w}.$$

2) *The complex arccotangent function is the solution to the equation*

$$z = \cot w = \frac{\cos w}{\sin w}.$$

Proposition 2.6.3 **1)**

$$\arctan z = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right).$$

2)

$$arccotz = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right).$$

Proof. Exercise. ■

Definition 2.6.4 • *The principal value of the complex arctangent function is defined by*

$$Arctanz = \frac{1}{2i}Log\left(\frac{i-z}{i+z}\right).$$

• *The principal value of the complex arccotangent function is defined by*

$$Arccotz = \frac{1}{2i}Log\left(\frac{z+i}{z-i}\right).$$

Proposition 2.6.4 **1)** $\arctan z + arccotz = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

2) $Arctanz + Arccotz = \begin{cases} \frac{\pi}{2}, & Rez \geq 0 \\ -\frac{\pi}{2}, & Rez < 0. \end{cases}$

Proof. Exercice. ■

Definition 2.6.5 **1)** *The complex inverse hyperbolic sine function is the solution to the equation*

$$z = \sinh w = \frac{e^{w} - e^{-w}}{2}.$$

2) *The complex inverse hyperbolic cosine function is the solution to the equation*

$$z = \cosh w = \frac{e^{w} + e^{-w}}{2}.$$

Proposition 2.6.5 **1)**

$$arcsinhz = \log\left(z + \left(1+z^2\right)^{\frac{1}{2}}\right).$$

2)

$$arccoshz = \log\left(z + \left(z^2-1\right)^{\frac{1}{2}}\right).$$

Proof. Exercise. ■

Definition 2.6.6 • *The principal value of the complex inverse hyperbolic sine function is defined by*

$$Arcsinhz = Log\left(z + \left|1+z^2\right|^{\frac{1}{2}} e^{\frac{i}{2}Arg\left(1+z^2\right)}\right).$$

- *The principal value of the complex inverse hyperbolic cosine function is defined by*

$$Arccoshz = Log\left(z + \left|z^2 - 1\right|^{\frac{1}{2}} e^{\frac{i}{2}Arg\left(z^2-1\right)}\right).$$

Definition 2.6.7 **1)** *The complex inverse hyperbolic tangent function is the solution to the equation*

$$z = \tanh w = \frac{\sinh w}{\cosh w}.$$

2) *The complex inverse hyperbolic cotangent function is the solution to the equation*

$$z = \coth w = \frac{\cosh w}{\sinh w}.$$

Proposition 2.6.6 **1)**

$$arctanhz = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

2)

$$arccothz = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right).$$

Proof. Exercise. ■

Definition 2.6.8 • *The principal value of the complex inverse hyperbolic tangent function is defined by*

$$Arctanhz = \frac{1}{2}Log\left(\frac{1+z}{1-z}\right).$$

- *The principal value of the complex inverse hyperbolic cosine function is defined by*

$$Arccothz = \frac{1}{2}Log\left(\frac{z+1}{z-1}\right).$$