2024/2025

University of Jijel
Department of Physics
Functions of a complex variable

TD session one

Exercise 1 : Express in the form $x + iy$:

$$z_1 = 1 + i + i(2-3i).\ \ z_2 = (3+2i)(2+i).\ \ z_3 = \frac{1}{1-2i}.$$

$$z_4 = \frac{1+3i}{2-i}.\ \ z_5 = \frac{2i}{1+i} + \frac{1-i}{2i}.$$

Exercise 2 : Express in trigonometric form :

$$z_1 = -8 + \frac{4}{i} + \frac{25}{3-4i}.\quad z_2 = \frac{(1-i)(\sqrt{3}+i)}{(1+i)(\sqrt{3}-i)}.$$

Exercise 3 : Solve the following equations

$$z^2 = 2i.\quad z^4 = -16.\quad z^n = 1.$$

Exercise 4 : Let z, z_1, z_2 be complex numbers and let $\overline{z}$ be the conjugate of z. Prove that

1) $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$.

2) $z = \overline{z} \Longleftrightarrow z \in \mathbb{R}$.

3) $|z| = \sqrt{z.\overline{z}}$.

4) $|z| = |\overline{z}|$.

5) $|z_1.z_2| = |z_1||z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$.

6) $|z_1 + z_2| \leq |z_1| + |z_2|$.

7) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

8) $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$, $\forall z \in \mathbb{C}^\star$.

Exercise 5 : By using the triangle inequality, prove that

1) $\left|z^3 + 3z - 1\right| \leq 15$, $\forall z \in \mathbb{C}$, $|z| = 2$.

2) $\left|z^2 + iz + 2\right| \geq 6,\ \forall z \in \mathbb{C},\ |z| = 4.$

3) $0 \leq \left|\frac{z-1}{z^2+2}\right| \leq 2,\ \forall z \in \mathbb{C},\ |z| = 1.$

Exercise 6 : Shade the following region and determine whether they are domain.

1) $D_1 = \left\{z \in \mathbb{C},\ 1 < |z-1| < \sqrt{3}\right\}.$

2) $D_2 = \left\{z \in \mathbb{C},\ \frac{1}{2} < \left|z - \frac{1}{2}\right| < 1\right\} \cup \left\{z \in \mathbb{C},\ \frac{1}{2} < \left|z + \frac{1}{2}\right| < 1\right\}.$

3) $D_3 = \{z \in \mathbb{C},\ |z| < |z-1|\}.$

4) $D_4 = \{z \in \mathbb{C},\ |z-1| < 1\} \cup \{z \in \mathbb{C},\ |z-3| < 1\}.$

Exercise 7 : Describe the domain of definition for each of the following functions

$$1)\ f(z) = z^2 - iz + 5 + 4i. \quad 2)\ f(z) = \frac{z}{z-i}. \quad 3)\ f(z) = \frac{z}{z+\overline{z}}. \quad 4)\ f(z) = \frac{1}{4-|z|^2}.$$

Exercise 8 :

1) Find the real part and the imaginary part of the function defined by $f(z) = z^3 + 2z + 1$.

2) Express in term of z the function defined by

$$f(z) = x^2 - y^2 + i(2x - 2xy),\ \forall z = x + iy \in \mathbb{C}.$$

Exercise 9 : Prove that if a function has a limit, then this limit is unique.

Exercise 10 : Find each of the following limits

$$1)\ \lim_{z\to 2i} \frac{z^3+5}{iz}, \quad 2)\ \lim_{z\to i} \frac{z^2+1}{z-i}, \quad 3)\ \lim_{z\to\infty} \frac{4z^2+1}{z^2+z+3+i}, \quad 4)\ \lim_{z\to\infty} \frac{z^4-i}{z^3+2z+4}.$$

Exercise 11 : Prove that $\lim\limits_{z\to 0} \frac{\overline{z}^2}{z} = 0.$

Exercise 12 : Let $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ be two complex functions and let $z_0 \in \mathbb{C}$. Prove that if f is continuous at z_0 and g is continuous at $w_0 = f(z_0)$, then the function $g \circ f$ is continuous at z_0.

Exercise 13 : Discuss the continuity of the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(z) = \begin{cases} \frac{z^3-1}{z-1} & \text{if } |z| \neq 1, \\ 3 & \text{if } |z| = 1 \end{cases}$$

at $z_0 = 1$, $z_1 = -1$, $z_2 = -i$ and $z_3 = i$.

Exercise 14 : For each of the following functions, determine $f'(z)$.

1) $f(z) = \frac{z+2}{3z-1}, \ \forall z \in \mathbb{C} \setminus \left\{\frac{1}{3}\right\}.$

2) $f(z) = (1+z^2)^3, \ \forall z \in \mathbb{C}.$

Exercise 15 : Determine the coefficients a, b and c so that the Cauchy-Riemann equations are satisfied for the function f defined by

$$f(z) = ay^3 + ix^3 + xy(bx + icy), \ \forall z = x + iy \in \mathbb{C}.$$

Exercise 16 : Prove that the Cauchy-Riemann equations can be written, in polar coordinates as

$$\frac{\partial u}{\partial r}(r,\theta) = \frac{1}{r}\frac{\partial v}{\partial \theta}(r,\theta), \ \ \frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta) = -\frac{\partial v}{\partial r}(r,\theta).$$

Where $x = r\cos\theta$ and $y = r\sin\theta$.

Exercise 17 : Let $u : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a harmonic function in D. Prove that

1) If $v : D \longrightarrow \mathbb{R}$ is a conjugate of u, then $-u$ is also a conjugate of v.

2) If $v_1 : D \longrightarrow \mathbb{R}$ and $v_2 : D \longrightarrow \mathbb{R}$ are conjugates of u, then v_1 and v_2 differ by a real constant.

3) If $v : D \longrightarrow \mathbb{R}$ is a conjugate of u, then v is also a conjugate of $u + c$, where c is a real constant.