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University of Jijel

Department of Physics

Functions of a complex variable

TD session two

Exercise 1 : Express in the form $x + iy$, $x, y \in \mathbb{R}$ the value of the given function

$$\sin(2 + i), \quad \cos i, \quad \tan(\pi - 2i).$$

Exercise 2 : Let z, z_1, z_2 be three complex numbers. Prove that

1) $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$.

2) $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$, $\sin(\frac{\pi}{2} - z) = \cos z$.

3) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\sin^2 z + \cos^2 z = 1$.

4) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

5) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$.

6) $\sin(2z) = 2 \sin z \cos z$, $\cos(2z) = \cos^2 z - \sin^2 z$.

7) $\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos^2 z_2 - \cos^2 z_1$.

8) $\cos(z_1 + z_2)\sin(z_1 - z_2) = \frac{1}{2}\sin(2z_1) - \frac{1}{2}\sin(2z_2)$.

9) $\sin z = 0 \Leftrightarrow z = k\pi$, $k \in \mathbb{Z}$.

10) $\cos z = 0 \Leftrightarrow z = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

Exercise 3 :

- Prove that the complex secant and cosecant functions are periodic with period $T = 2\pi$.
- Prove that the complex tangent and cotangent functions are periodic with period $T = \pi$.

Exercise 4 : Prove that for all $z, z_1, z_2 \in \mathbb{C}$, we have

1) $\cosh(iz) = \cos z$, $\cos(iz) = \cosh z$.

2) $\sinh(iz) = i\sin z, \;\; \sin(iz) = i\sinh z.$

3) $\sinh(-z) = -\sinh z, \;\; \cosh(-z) = \cosh z.$

4) $\cosh^2 z - \sinh^2 z = 1.$

5) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2.$

6) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2.$

7) $\sinh(2z) = 2\sinh z \cosh z, \;\; \cosh(2z) = \cosh^2 z + \sinh^2 z.$

8) $\sinh z = 0 \Leftrightarrow z = k\pi i,\ k \in \mathbb{Z}$ and $\cosh z = 0 \Leftrightarrow z = \left(k + \frac{1}{2}\right)\pi i,\ k \in \mathbb{Z}$

Exercise 5 :

1) Express $\sin z$, for all $z = x + iy \in \mathbb{C}$, in the form $u(x, y) + iv(x, y),\ x, y \in \mathbb{R}$.

2) Express $\cos z$, for all $z = x + iy \in \mathbb{C}$, in the form $u(x, y) + iv(x, y),\ x, y \in \mathbb{R}$.

3) Express $\tan z$, for all $z \in \mathbb{C} \setminus \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$, in the form $u(x, y) + iv(x, y),\ x, y \in \mathbb{R}$.

Exercise 6 : Prove that for all $z = x + iy \in \mathbb{C}$, we have

1) $|\sin z|^2 = \sin^2 x + \sinh^2 y.$

2) $|\cos z|^2 = \cos^2 x + \sinh^2 y.$

3) $|\sinh z|^2 = \sinh^2 x + \sin^2 y.$

4) $|\cosh z|^2 = \sinh^2 x + \cos^2 y.$

Exercise 7 : Let $z, z_1, z_2 \in \mathbb{C}^\star$. Prove that

1) If $z = re^{i\theta}$, then $\log z = \ln r + i\theta$.

2) $\log e^z = z + 2k\pi i,\ k \in \mathbb{Z}.$

3) $\log z_1 z_2 = \log z_1 + \log z_2.$

4) $\log \frac{z_1}{z_2} = \log z_1 - \log z_2.$

Exercise 8 : Prove that

1) $\arcsin z = \frac{1}{i} \log\left(iz + (1 - z^2)^{\frac{1}{2}}\right).$

2) $\arccos z = \frac{1}{i} \log\left(z + (z^2 - 1)^{\frac{1}{2}}\right).$

3) $\arctan z = \frac{1}{2i} \log\left(\frac{i-z}{i+z}\right).$

4) $arccotz = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$.

5) $\arctan z + arccotz = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

6) $Arctanz + Arccotz = \begin{cases} \frac{\pi}{2}, & Rez \geq 0 \\ -\frac{\pi}{2}, & Rez < 0. \end{cases}$

<u>Exercise 9</u> :

1) Find all values of $(i)^{-2i}$.

2) Determine the principal value of $(-i)^{i}$.

3) Determine $\arcsin(-i)$.

<u>Exercise 10</u> : Prove that

1) $arcsinhz = \log\left(z + (1+z^2)^{\frac{1}{2}}\right)$.

2) $arccoshz = \log\left(z + (z^2-1)^{\frac{1}{2}}\right)$.