PART A :

HEAT TRANSFER

CHAPTER 2 :

STEADY-STATE HEAT CONDUCTION

Before statrting :

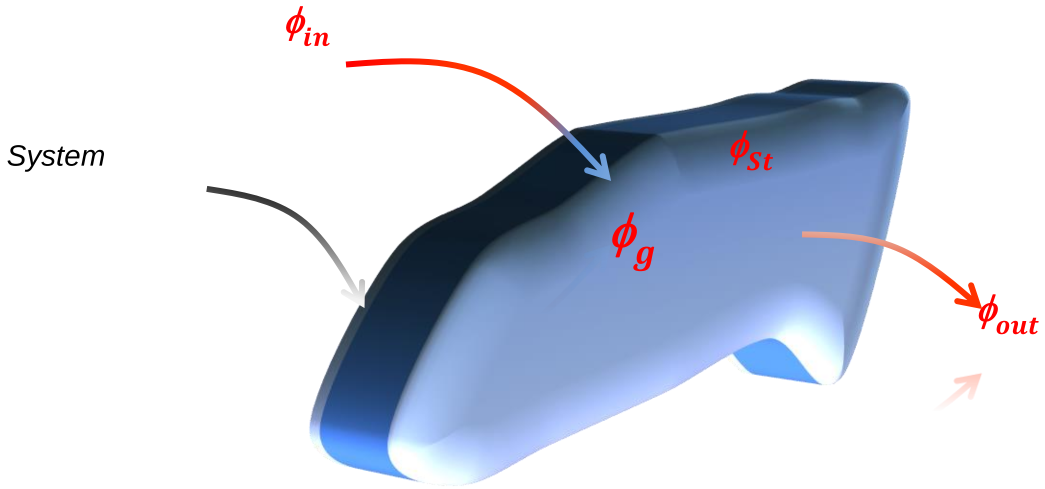
Key Assumptions

1. **Steady-state**: The temperatures at all points in the system remain constant over time.
2. **One-dimensional heat transfer**: Heat flows in a single direction.
3. **Homogeneous material**: The material's thermal properties are uniform.
4. **Constant thermal conductivity**: kkk is assumed to be independent of temperature (though this may vary in real scenarios).

II. 1. Heat equation: energy balance (L’équation de la Chaleur : bilan d’énergie)

Establishing the energy balance of a system comes down to applying the first principle of thermodynamics:

"During of any transformation of a closed system, the variation in its energy is equal to the quantity of energy exchanged with the external environment, by thermal transfer (heat) and mechanical transfer (work)."



[input Energy] + [generated Energy] = [output Energy] + [accumulated (stored) Energy].

Figure 1 Thermal system and energy balance

Applying the first principle of thermodynamics, the energy balance is

Accumulated energy = energy input - energy output + energy generated.

$$\phi_{St} = \phi_{In} - \phi_{Out} + \phi_g$$

$$\dot{q}.V = \phi_E - \phi_S + \rho.V.C.\frac{\partial T}{\partial t}$$

ϕ_{In} : *Input heat flux (W)*;

ϕ_{Out} : *Out heat flux (W);*

ϕ_g : *Generated heat flux (W);*

ϕ_{St} : *Stored heat flux (W) ;*

$\dot{q}$: heat volumetric density $(W.m^{-3})$;

ρ : Density ($kg\ m^{-3}$) ;

C : Heat capacity ($J.K^{-1}.kg^{-1}$) ;

V : Volume (m^{-3}).

Steady state regime :

$$\frac{\partial T}{\partial t} = 0 \Rightarrow \dot{q}.V = \phi_E - \phi_S$$

II. 2. One-Dimensional heat Conduction (Transfert de chaleur unidirectionnel (par conduction))

Let's consider the case of heat diffusion in a single direction. This type of transfer is called unidirectional or one-dimensional. We develop the heat equation by considering a flat plate of area S and thickness e (or a flat wall).

If, for example, we take the direction in which heat propagates to be x, and if we denote by ϕ_x et ϕ_{x+dx} the heat passing through the same surface S per unit time located at x and $x + dx$, then the heat equation becomes (see figure II-2)

$$\boldsymbol{\phi_{St} = \phi_x - \phi_{x+dx} + \phi_g}$$

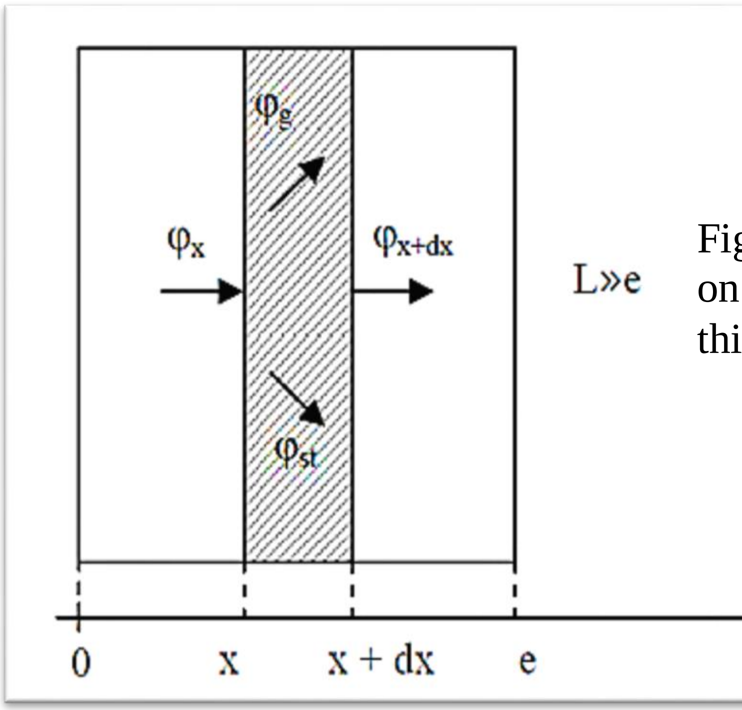


Figure II-2 Heat balance on a flat plate of thickness **e** and area **S**

The amount of heat entering per unit time is given by Fourier's law in a single x direction is :

$$\phi_x = -\left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_x \quad \text{at} \quad x \quad \text{et} \quad \phi_{x+dx} = -\left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_{x+dx} \text{at} \quad x + dx$$

And the heat equation takes the form

$$\rho . V . C . \frac{\partial T}{\partial t} = -\left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_x - \left[-\left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_{x+dx}\right] + \dot{q} . V$$

$$\Rightarrow \rho . S\, .\, dx . C . \frac{\partial T}{\partial t} = \left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_{x+dx} - \left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_x + \dot{q}\, . S . dx$$

Dividing by dx we get

$$\Rightarrow \ \rho . S . C . \frac{\partial T}{\partial t} = \frac{\left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_{x+dx} - \left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_{x}}{dx} + \dot{q}\,.S$$

This gives :

$$\Rightarrow \quad \rho . S . C . \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda\, S\, \frac{\partial T}{\partial x}\right) + \dot{q}\,.S$$

Note that in the general case, conductivity is a function of point coordinates and time: $\lambda = \lambda(x, y, z, t)$

Steady state regime in one dimensionnal direction :

$$\frac{\partial T}{\partial t} = 0 \textit{ and } \frac{\partial}{\partial x}\left(\lambda\, S\, \frac{\partial T}{\partial x}\right) + \dot{q}\,.\,S = 0 \textit{ and } \lambda = \lambda(x, y, z)$$

II. 3. Multidimensional heat conduction

In the previous section, we established the heat equation for one dimentional diffusion, expressing it in the Cartesian coordinate system and considering (Ox) as the direction of diffusion. The equation was then :

$$\rho.S.C.\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda\, S\, \frac{\partial T}{\partial x}\right) + \dot{q}\,.S$$

In the case of diffusion in all three directions, thermal energy will therefore propagate in the other two directions, y and z, in addition to the x direction. The heat equation then takes the most general form:

$$\boldsymbol{\rho . C . \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x \ \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \ \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \ \frac{\partial T}{\partial z}\right) + \dot{q}}$$

Case of no heat generation and steady state regime:

In the case of no heat generation within the system, $\dot{q}$ =0 and the equation will have the form :

$$\rho . C . \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x \ \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \ \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \ \frac{\partial T}{\partial z}\right)$$

Steady stae regime : $\frac{\partial T}{\partial t} = 0$

$$\frac{\partial}{\partial x}\left(\lambda_x \ \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \ \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \ \frac{\partial T}{\partial z}\right) = 0$$

Case of an isotropic medium

In the case where the system medium is isotropic, we have :

$$\lambda_x = \lambda_y = \lambda_z = \lambda$$

And the equation takes the form :

$$\rho . C . \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \ \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \ \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \ \frac{\partial T}{\partial z}\right) + \dot{q}$$

Case of isotropic, homogeneous medium and absence of heat generation

$$(\dot{q} = 0)$$

The heat equation takes the form

$$\rho . C . \frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{d\lambda}{dT} \left[\left(\frac{dT}{dx} \right)^2 + \left(\frac{dT}{dy} \right)^2 + \left(\frac{dT}{dz} \right)^2 \right]$$

Case of isotropic, homogeneous medium, no heat generation and λ independent of T

$$\left(\dot{q} = 0 \ et \ \frac{d\lambda}{dT} = 0\right)$$

$$\rho . C . \frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = \lambda \nabla^2 T = \lambda \Delta T$$

Or again

$$\frac{\rho . C}{\lambda} \frac{\partial T}{\partial t} = \frac{1}{a} \frac{\partial T}{\partial t} = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = \nabla^2 T = \Delta T$$

The ration : $\frac{\lambda}{\rho.C} = a$ **is called the thermal diffusivity, its unit is :**$(\frac{m^2}{s})$**, which quantifies the speed at which heat diffuses inside the medium.**

and the equation takes the form :

$$\rho.C.\frac{\partial T}{\partial t} = \lambda\ \nabla^2 T = \Delta T$$

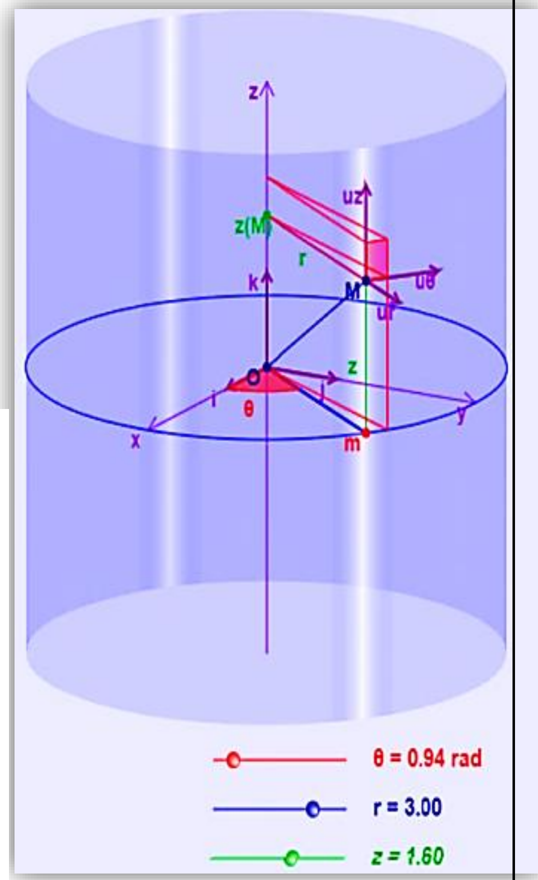
This local equation is valid in the case of an isotropic, homogeneous medium, no heat generation and λ independent of T.

II. 1. Multidimensional heat conduction in cylindrical and spherical coordinates systems

(file:///C:/Users/ASUS/Downloads/25862159.pdf)

• in cylindrical coordinates :

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t}$$

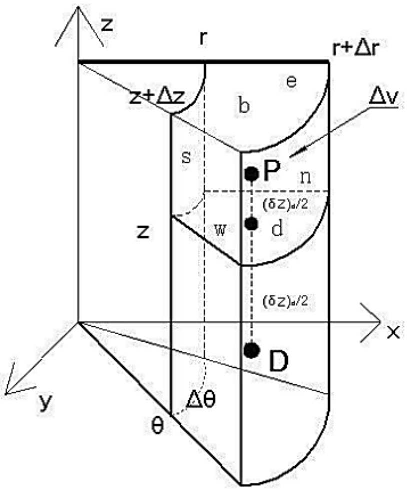


- In the case where T is a function only of r and t; then :

$$\frac{\partial T}{\partial \theta}=\frac{\partial T}{\partial z}=0$$

- And the equation becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{q}}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t}$$



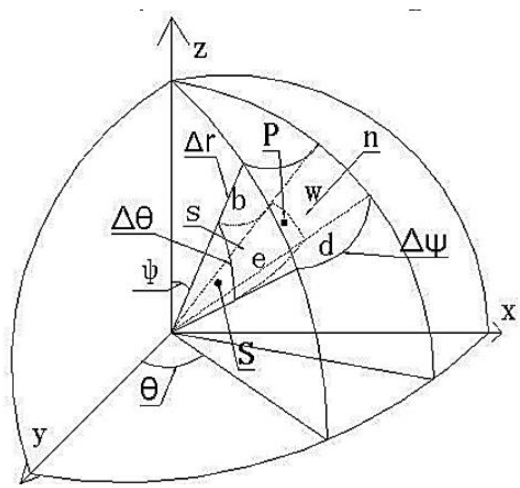
- **in spherical coordinates:**

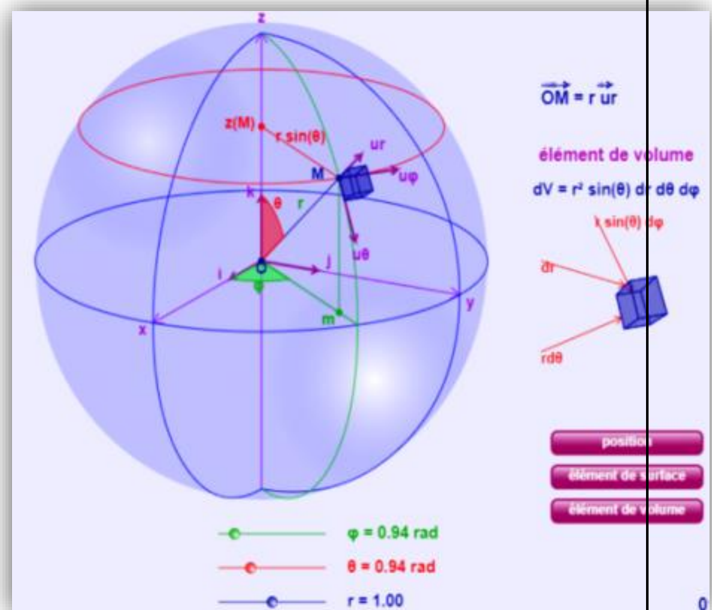
$$\frac{1}{r^2}\frac{\partial}{\partial \mathrm{r}}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2 sin\theta}\ \frac{\partial}{\partial \theta}\left(sin\theta\ \frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2 sin\theta}\ \frac{\partial}{\partial \theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right)+\frac{\dot{q}}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t}$$

Dans le cas où T n’est une fonction que de r et de t ; alors :

$$\frac{\partial T}{\partial \theta}=\frac{\partial T}{\partial \phi}=0$$

$$\boldsymbol{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{q}}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t}}$$





II. 2. Heat Conduction : Steady state (permanent) vs. transient (variable) regime

In the field of heat transfer, problems are often classified as stationary (permanent) or transient (variable).

- The stationary regime indicates that the elements of the thermal study of a system (temperature, heat flow, ...) do not undergo any change with time and thus time independence.

- The unsteady (transient or variable) regime indicates the variation with time and thus the dependence on time..

In the heat equation, the term: $\frac{\partial T}{\partial t}$ is zero

II. 3. Multidimensional heat conduction in Steady-state regime

(Conduction de la chaleur multidimensionnelle en régime permanent)

Under steady-state regime : $\frac{\partial T}{\partial t} = 0$

So, the heat equation then takes the form:

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + \dot{q} = 0$$

In the case of no heat generation (in steady-state regime):

In the case of no heat generation within the system, $\dot{q} = 0$ and the equation will have the form :

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) = 0$$

In the the case of isotropic medium (in steady-state regime):

$$\lambda_x = \lambda_y = \lambda_z = \lambda$$

And the equation takes the form :

$$\frac{\partial}{\partial x}\left(\lambda \ \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \ \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \ \frac{\partial T}{\partial z}\right) + \dot{q} = 0$$

Case of isotropic, homogeneous medium and absence of heat generation (in steady-state regime) :

$$(\dot{q} = 0)$$

The heat equation takes the form

$$\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

$$+ \frac{d\lambda}{dT}\left[\left(\frac{dT}{dx}\right)^2 + \left(\frac{dT}{dy}\right)^2 + \left(\frac{dT}{dz}\right)^2 = 0\right]$$

Case of isotropic, homogeneous medium, no heat generation and λ independent of T (in steady-state regime):

$$\left(\dot{q} = 0 \; et \; \frac{d\lambda}{dT} = 0 \; et \; \frac{\partial T}{\partial t} = 0\right)$$

$$\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \lambda \, \nabla^2 T = \lambda \, \Delta T = 0$$

Or again

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \nabla^2 T = \Delta T = 0$$

II. 4. Temperature from the heat equation

(La température à partir de l'équation de la chaleur)

The local heat equation can be used to determine the temperature T(x,y,z,t) by integration.

To this end, a word of clarification is in order:

i. **An initial condition** $T\left(x,y,z,t=0\right)$ **which defines the initial thermal state of the system.**
ii. **Two boundary conditions. These conditions can be of three types:**

- Dirichlet-type conditions (temperature boundary conditions)
- Neumann-type conditions (heat flux boundary conditions
- Robin Conditions (convective boundary condition)

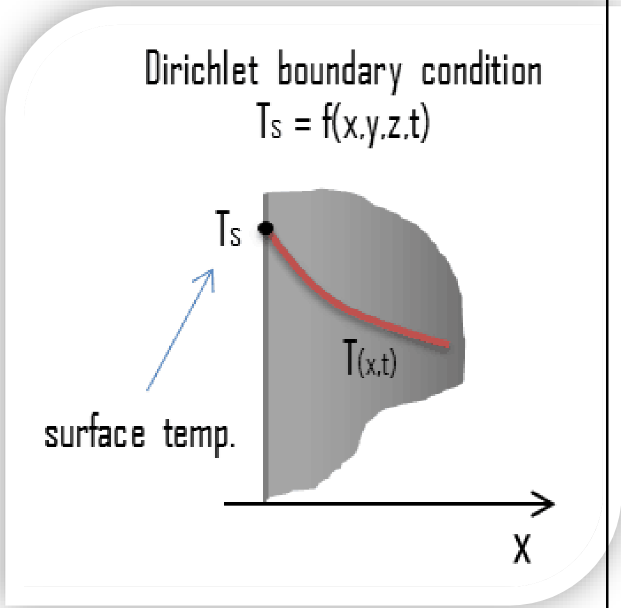
a. Dirichlet-type conditions (temperature boundary conditions or first-type boundary conditions)

Named after a German mathematician Peter Gustav Lejeune Dirichlet (1805–1859).

- Définition

The **Dirichlet-type conditions** refer to boundary conditions where the **temperature** at the boundary is specified as a fixed value. These conditions are also known as **temperature boundary conditions** or **first-type boundary conditions**.

This condition corresponds to a given **fixed surface temperature** in heat transfer problems.



- Characteristics of Dirichlet Conditions:

  - The temperature T at the boundary is constant or prescribed as a function of time or space.

  - Mathematically, it is expressed as:

$$T(x, t) = T_{\text{specified}}(x, y, z, t)$$

  where:
  - $T(x, y, z, t)$ is the temperature at the boundary.
  - $T_{\text{specified}}(x, y, z, t)$ is the known function (which could be a constant).

- Examples in Heat Transfer:

1. **Constant Temperature Wall:** A surface of a body is maintained at a constant temperature, say $T = 100°C$.

2. **Time-Dependent Temperature:** The boundary temperature varies with time, such as $T(t) = T_0 + \alpha t$, where T_0 and α are constants.

3. **Boundary in Thermal Interaction with an Environment:** A boundary exposed to a medium that imposes a fixed temperature (e.g., a body in contact with a large thermal reservoir).

- Example: Heat Transfer in a Rod with Fixed End Temperature

Consider a 1D rod (Fr: tige , baton) of length L, with its ends maintained at constant temperatures. The temperature distribution $T(x,t)$ along the rod is governed by the heat equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

where:

- $T(x,t)$ is the temperature at position x and time t,
- α is the thermal diffusivity of the material.
-

Dirichlet Boundary Conditions:

1. **Fixed temperature at the left end (x = 0):**

$$T(0,t) = T_0,$$

where T_0 is the constant temperature at $x = 0$.

2. **Fixed temperature at the right end** $(x = L)$:

$$T(L,t) = T_L,$$

where T_L is the constant temperature at $x = L$.

- Applications:

Dirichlet boundary conditions are commonly used in problems where the temperature is controlled, such as:

- Surfaces in contact with a heat source or sink (l'évier) at a known temperature.
- Regions where precise thermal conditions are imposed for material processing or experimental setups (dispositifs expérimentaux).

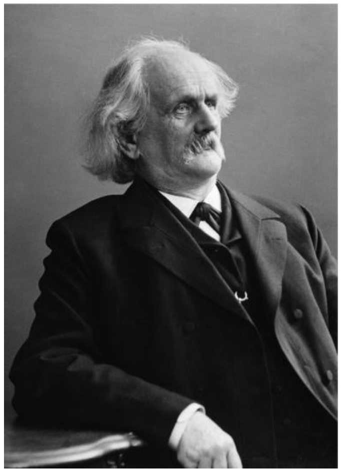
In practice, such conditions might occur in:

- A metal rod with its ends immersed in water baths maintained at specific temperatures.
- A wall of a building exposed to different constant temperatures on its two sides.

b. Neumann-type conditions (heat flux boundary conditions)

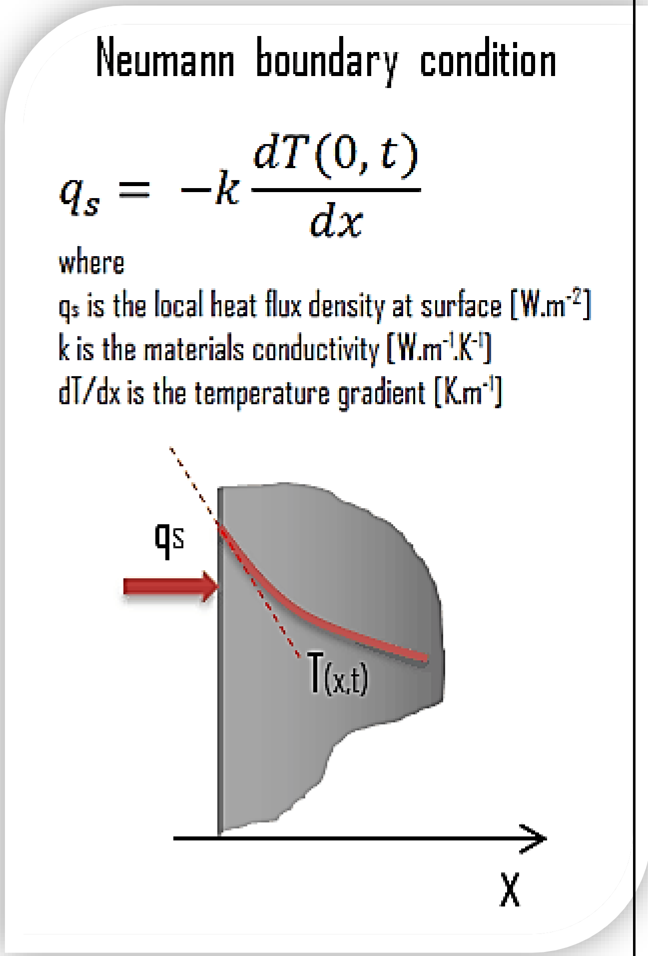
named after a German mathematician Carl Neumann (1832–1925).

- Définition



The Neumann boundary conditions, also known as heat flux boundary conditions or second-type boundary conditions, specify the heat flux across a boundary rather than the temperature.

The Neumann condition corresponds to a **given rate of change of temperature**. In



other words, this condition assumes that the **heat flux** at the material’s surface is known.

Characteristics of Neumann Conditions:

- They describe the rate of heat transfer (or heat flux) at the boundary, either as a constant value, a function, or zero (insulated surface).
- Mathematically, it is expressed as:

$$-\lambda \frac{\partial T}{\partial n} = q_{\text{specified}}(x, t)$$

where:

- ○ λ is the thermal conductivity of the material.
- ○ T is the temperature.
- ○ n is the direction normal to the boundary.
- ○ $q_{\text{specified}}(x, t)$ is the specified heat flux at the boundary (positive for heat leaving the boundary, negative for heat entering).

Common Cases:

1. **Prescribed Heat Flux:** A boundary with a known heat flux, e.g., $q_{\text{specified}} = 500\ \text{W/m}^2$.

2. **Insulated or Adiabatic Boundary**: No heat transfer occurs across the boundary, i.e., $q_{\text{specified}} = 0$. This simplifies to:

$$\frac{\partial T}{\partial n} = 0$$

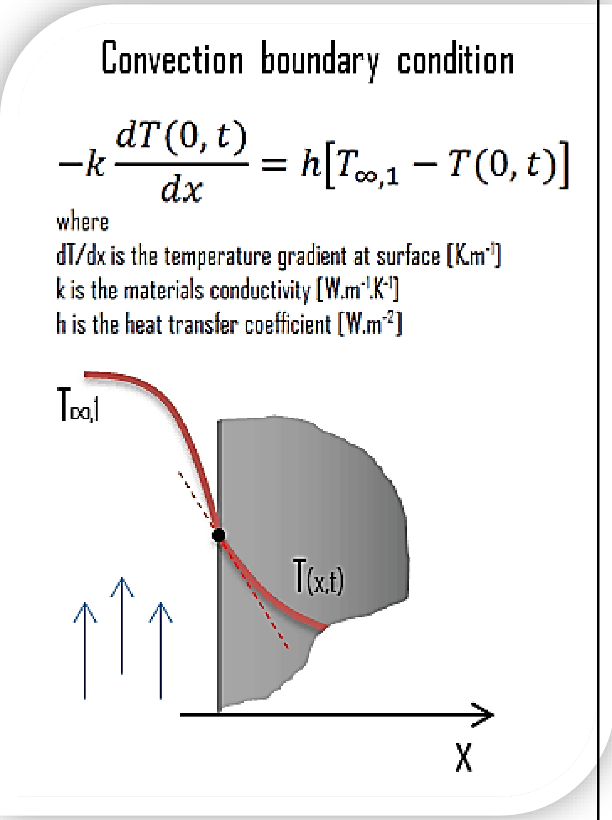
Applications:

Neumann conditions are used in situations where the rate of heat transfer is known or controlled, such as:

- Boundaries with a prescribed heat flux due to a heating element or a cooling process.
- Thermal insulation or adiabatic conditions to prevent heat loss or gain.
- Interfaces between materials where the heat flux must balance due to conservation of energy.

c. The Robin condition (mixed boundary condition)

The **Robin condition** in heat transfer, also known as the **mixed boundary condition** or **convective boundary condition**, is a type of boundary condition that accounts for both conduction and convection at the boundary of a domain.



It is commonly used in problems where a solid surface is in contact with a fluid medium, and heat transfer occurs due to both conduction within the solid and convective exchange with the fluid.

*The **convection boundary condition** is probably the most common boundary condition encountered in practice since most heat transfer surfaces are exposed to a convective environment at specified parameters.*

This condition assumes that the heat conduction at the material’s surface is equal to the heat convection at the surface in the same direction. Since the boundary cannot store energy, the net heat entering the surface from the convective side must leave the surface from the conduction side.

- Mathematical Form

The Robin boundary condition is expressed as:

$$-\lambda \frac{\partial T}{\partial n} = h(T - T_\infty)$$

where:

- λ: Thermal conductivity of the material (W/m·K),
- $\frac{\partial T}{\partial n}$: Temperature gradient normal to the boundary,
- h: Convective heat transfer coefficient ($W/m^2 \cdot K$),
- T: Surface temperature (K),
- T_∞: Temperature of the surrounding fluid (K).

- Key Features:

  - **Conduction Term ($k\frac{\partial T}{\partial n}$):** Represents the heat flux due to conduction at the boundary.

  - **Convection Term ($h(T - T_\infty)$):** Represents the heat flux due to convection at the boundary.

  - The Robin condition is a linear combination of the **Dirichlet boundary condition** ($T = T_{\text{specified}}$) and the **Neumann boundary condition** ($\frac{\partial T}{\partial n} = q/k$).

- Applications:
  - Modeling heat exchange between a solid surface and a surrounding fluid, such as in heat exchangers, cooling fins, and thermal insulation studies.
  - Situations where there is no perfect insulation or constant surface temperature, but rather a combination of heat transfer mechanisms.

The Robin condition is crucial in accurately solving heat transfer problems in real-world applications where boundary heat exchange cannot be simplified to a purely conductive or convective process.