Real functions of one variable

1 The sets of real numbers

1.1 Common sets of numbers

We recall the following notations for subsets of $\mathbb{R}$.

- $\mathbb{N} = \{0,\ 2,\ 3 \cdots\}$ is the set of natural numbers.
- $\mathbb{Z} = \{\cdots, -3,\ 2,\ -1,\ 0,\ 2,\ 3 \cdots\}$ is the set of integers numbers.
- $\mathbb{D} = \left\{\dfrac{a}{10^n};\ a \in \mathbb{Z},\ n \in \mathbb{N}\right\}$ is the set of decimal numbers.
- $\mathbb{Q} = \left\{\dfrac{p}{q};\ p \in \mathbb{Z},\ q \in \mathbb{Z}^*\right\}$ is the set of rational numbers.
- $\mathbb{R}$ represnts the set of real numbers and we have the following inclusions:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R}.$$

- For each of theses sets, the addition of the sign $*$ means that we exclude 0 from the sets: $\mathbb{N}^*,\ \mathbb{Z}^*,\ \mathbb{R}^*, \cdots$

Example 1

1. *Is it* $\sqrt{10} \in \mathbb{Q}$*?. No*
2. *The set of natural numbers* $\mathbb{N}$ *is*
   - *Infinite set.*
   - *Countable set.*
   - *Finite and uncountable set.*
   - *infinite and countable set.* ✓

1.2 Intervals of $\mathbb{R}$

Definition 1 *An interval can be defined as a set of real numbers that contains all real numbers lying within any two specific numbers of the set* $\mathbb{R}$.
We can write the intervals as subsets of a set of real numbers and these intervals can be written as open and closed intervals.
Suppose that a *and* b *are two real numbers* $(a,\ b \in \mathbb{R})$, *such that* $a < b$. *Then,*

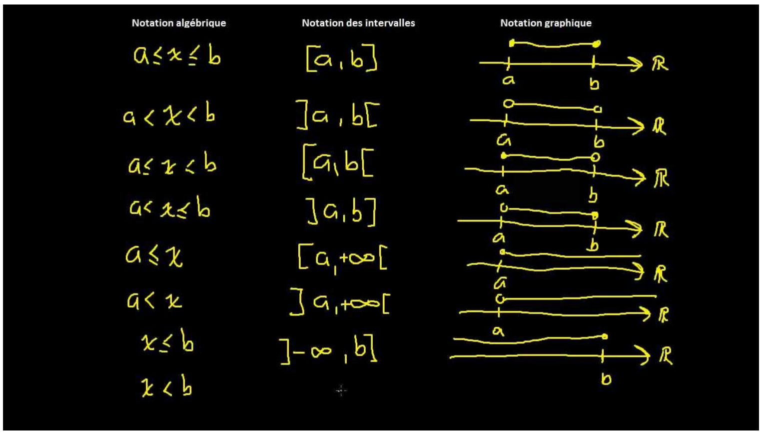


Figure 1: *The intervals of the real line.*

Notations:

- $\mathbb{R}^* =]-\infty, 0[\cup]0,\ +\infty[$.
- $\mathbb{R}_+ = \{x \in \mathbb{R}/\ x \geq 0\} =]0,\ +\infty[$.
- $\mathbb{R}_- = \{x \in \mathbb{R}/\ x \leq 0\} =]-\infty, 0[$.
- $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\ +\infty\} = [-\infty,\ +\infty]$.

1.3 Real functions of one variable

Definition 2 *A function f of a real variable is a rule which assings to each $x \in D \subset \mathbb{R}$ exactly one $y \in \mathbb{R}$. We have:*

$$\forall\ x \in D,\ \exists\ !\ y \in \mathbb{R}:\ \ y = f(x).$$

- *The variable x is called independent variable and y is called dependent.*
- *The set D is called the domain of the function f and denoted $D(f)$ or D_f defined by:*

$$D(f) = D_f = D = \{y = f(x)\ /\ x \in D(f)\}.$$

We write:

$$\begin{array}{rcl} f : D & \rightarrow & f(D) \\ x & \longmapsto & y = f(x), \end{array}$$

where $f(D)$ is called the image of D by the function f.

Example 2 *Determine the set D for each function f :*

1. $f(x) = \dfrac{1}{x}\cdot \qquad D = \mathbb{R}^*.$

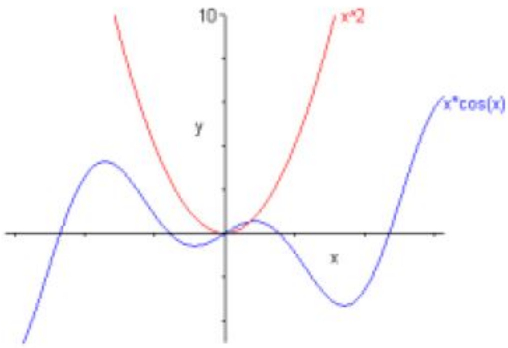
2. $f(x) = \frac{1}{\sqrt{x}}\cdot \qquad D = \mathbb{R}_+^*$.

3. $f(x) = \sqrt{x^2+1}. \qquad D = \mathbb{R}$.

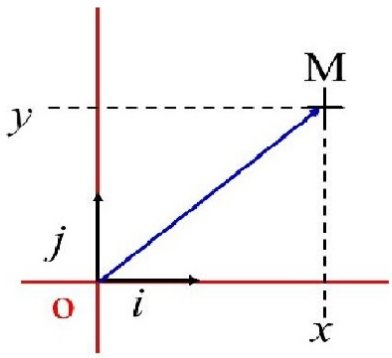
1.4 Graph (Curve) of a function

Definition 3 *The graph of a function f is the set of all points in the plane of the form $(x, f(x))$. We could also define the graph of f to be the graph of the equation $y = f(x)$. Given a function $f : D \to f(D)$. The graph of the function f is the set:*

$$G(f) = \{(x, f(x)) : \ x \in D\}.$$



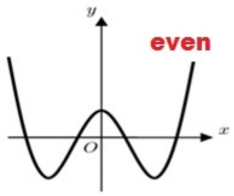
Definition 4 *The graph of f (or representation curve) is the set of points (x, y). On a cartesian plane $(O, \overrightarrow{i}, \overrightarrow{i})$, each couple of points is associated with a unique point M of (x, y), such that: $\overrightarrow{OM} = x\overrightarrow{i} + y\overrightarrow{j}$.*

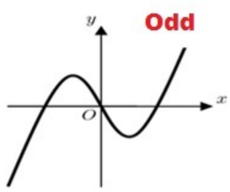


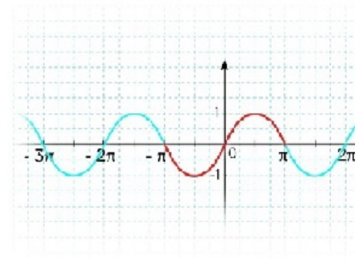
Proposition 1 *Let $f : D \to \mathbb{R}$ be a function and we assume that for all $x \in D$, $-x \in D$. We say that,*

- *f is even on D if $\forall\ x \in D,\ -x \in D:\ \ f(-x) = f(x)$. In this case the curve of f is symmetric with respect to OY.*
- *f is odd on D if $\forall\ x \in D,\ -x \in D:\ \ f(-x) = -f(x)$. In this case the curve of f is symmetric with respect to the origine $O(0, 0)$.*
- *f is periodic on D with a period T if T is the smallest positive real number, such that*

$$\forall\ x \in D,\ f(x+T) = f(x).$$







Definition 5 (Monotonic functions) *Let $f : D \to \mathbb{R}$ be a function. We say that*

- *f is increasing (resp. strictly increasing) if*

$$\forall\ x_1,\ x_2 \in D : x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2) \qquad \big(resp.\ x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)\big).$$

- *f is decreasing (resp. strictly decreasing) if*

$$\forall\ x_1,\ x_2 \in D : x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2) \qquad \big(resp.\ x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)\big).$$

- *f is monotonic on D if it is increasing or decreasing on D.*

1.5 Composition functions

Definition 6 *Let $f : D_f \subseteq \mathbb{R} \to \mathbb{R}$ and $g : D_g \subseteq \mathbb{R} \to \mathbb{R}$, such that $\forall\ x \in D_f:\ \ f(x) \in D_g$ $(f(D_f) \subseteq D_g)$. Then, $g \circ f$ is a function defined on D_f by:*

$$(g \circ f)(x) := g(f(x)), \qquad \forall\ x \in D_f.$$

Noting:

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x)) = (g \circ f)(x).$$

Remark 1 *If the function $g \circ f$ exists, then the composition function $f \circ g$ dosen't always exists, and when $f \circ g$ exists then, in general: $g \circ f \neq f \circ g$.*

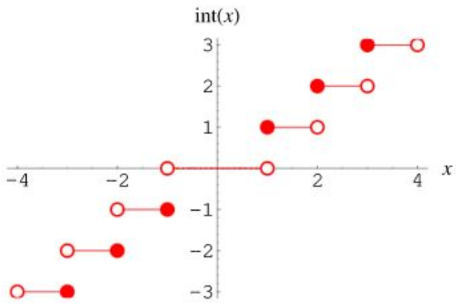
Example 3 *Let $f(x) = \sqrt{x}$, where $D_f = \mathbb{R}_+$ and $g(x) = \dfrac{1+x}{1-x}$, where $D_g = \mathbb{R} - \{1\}$. Then, we have*

$$(g \circ f)(x) = \frac{1+\sqrt{x}}{1+\sqrt{x}} \qquad and \qquad (f \circ g)(x) = \sqrt{\frac{1+x}{1-x}}.$$

1.6 Integer part of a real number

Definition 7 *The integer part of a real number x is the greatest integer less than or equal to x. To be more precise, for $x \in \mathbb{R}$, there exists a unique integer $p \in \mathbb{Z}$, such that $p \leq x < p+1$. The integer p is called the integer part of x and we write $E(x)$ or $[x]$. The function $E(x)$ is defined by:*

$$\begin{aligned} E : \mathbb{R} &\rightarrow \mathbb{Z} \\ x &\mapsto E(x). \end{aligned}$$



Example 4 $E(\pi) = 3, \qquad E(-\frac{3}{2}) = -2, \qquad E(\sqrt{2}) = 1.$

Proposition 2

1. $\forall x \in \mathbb{Z} \Longleftrightarrow E(x) = x.$
2. $\forall x \in \mathbb{R},\ E(x) \leq x < E(x) + 1.$
3. $\forall x \in \mathbb{R},\ E(x+n) = E(x) + n,\ for\ n \in \mathbb{N}.$
4. $\forall x, y \in \mathbb{R},\ E(x) + E(y) \leq E(x+y) \leq E(x) + E(y) + 1.$
5. *The function $E(x)$ is increasing over $\mathbb{R}$, i.e,*

$$\forall x_1, x_2 \in \mathbb{R},\ x_1 \leq x_2 \Longrightarrow E(x_1) \leq E(x_2).$$

Exercise 1 *Demonstrate the following properties:*

(a) *For all $x \in \mathbb{R},\ n \in \mathbb{N}, \quad E(x+n) = E(x) + n.$*
Indeed, we have

$$E(x) \leq x < E(x) + 1 \Longleftrightarrow E(x) + n \leq x + n < E(x) + n + 1.$$

Then, $E(x+n) = E(x) + n.$

(b) *For all $x \in \mathbb{R}, \quad x - 1 < E(x) \leq E(x) + 1.$*
Indeed, We have $E(x) \leq x < E(x) + 1$. Then,

$$E(x) \leq x \qquad and \qquad x < E(x) + 1 \Longrightarrow x - 1 < E(x).$$

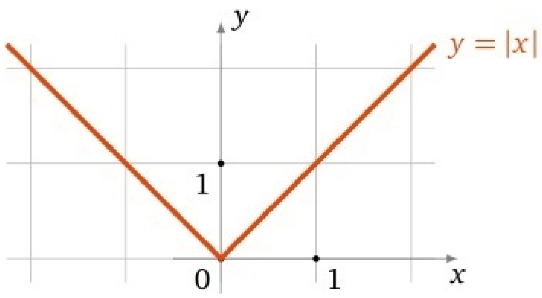
Therefore, $x - 1 < E(x) \leq E(x) + 1.$

1.7 Absolute value function

Definition 8 *Let $x \in \mathbb{R}$, we denote by $|x|$ the absolute value of x defined by*

$$\begin{array}{rcl} |.| : \mathbb{R} & \to & \mathbb{R}_+ \\ x & \mapsto & |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases} \end{array}$$

It is also defined by: $|x| = \max(-x, x)$.



Proposition 3 *Let $x,\ y \in \mathbb{R}$. We have*

1. $|x| \geq 0$.
2. $|x| = 0 \Longleftrightarrow x = 0$.
3. $|x| = |-x|$.
4. $x \leq |x|$ *and* $-x \leq |x|$.
5. $\forall\ \alpha \in \mathbb{R}_+ :\ |x| \leq \alpha \Longleftrightarrow -\alpha \leq x \leq \alpha$.
6. $\forall\ \alpha \in \mathbb{R}_+ :\ |x| \geq \alpha \Longleftrightarrow \alpha \leq x \quad$ *or* $\quad \alpha \geq x$.
7. $\left|\dfrac{x}{y}\right| \leq \dfrac{|x|}{|y|}$, *if* $y \neq 0$.
8. $|xy| = |x||y|$.
9. ***Traingle inequality:*** $\forall x \in \mathbb{R},\ |x + y| \leq |x| + |y|$.
   Indeed, We have

$$\begin{array}{rcl} |x+y|^2 & = & (x+y)^2 \\ & = & x^2 + 2xy + y^2 \\ & = & |x|^2 + 2xy + |y|^2. \end{array}$$

Since that: $2xy \leq 2|xy| = 2|x||y|$*, then*

$$\begin{aligned} |x+y|^2 &\leq |x|^2 + 2|x||y| + |y|^2, \\ &= (|x|+|y|)^2. \end{aligned}$$

Therefore, $|x+y| \leq |x| + |y|$.

10. $\big||x| - |y|\big| \leq |x+y| \leq |x| + |y| \leq |x+y| + |x-y|$.

1.8 Limits at a point

Definition 9 *For a function* $f : D \to \mathbb{R}$*, a real number* L *is said to be a limit of* f *at* x_0 *if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in D,\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

If L *is a limit of* f *at* x_0*, then we also say that* f *converges to* L *at* x_0 *(or* $f(x)$ *approaches* L *as* x *approaches* x_0*). We often write*

$$L = \lim_{x \to x_0} f \qquad or \qquad L = \lim_{x \to x_0} f(x).$$

Proposition 4 *If the function* f *has a limit* L *at a point* x_0*, then this limit is unique.*

1.8.1 Right-hand limit and left-hand limit

Definition 10 *Let* $f : D \to \mathbb{R}$ *and* $x_0 \in \mathbb{R}$*. We say that*

(i) f *has a right-hand limit at* x_0 *if*

$$\lim_{x \to x_0^+} f(x) = \lim_{x \overset{>}{\to} x_0} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in D,\ x_0 < x < \delta + x_0 \Rightarrow |f(x) - L| < \varepsilon.$$

(ii) f *has a left-hand limit at* x_0 *if*

$$\lim_{x \to x_0^-} f(x) = \lim_{x \overset{<}{\to} x_0} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in D,\ x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon.$$

Definition 11 *Let* $f : D \to \mathbb{R}$ *and* $x_0 \in \mathbb{R}$*. We have*

* $\lim_{x \to x_0} f(x) = +\infty \iff \forall M > 0,\ \exists \delta > 0 :\ 0 < |x - x_0| < \delta \Longrightarrow f(x) > M.$
* $\lim_{x \to x_0} f(x) = -\infty \iff \forall M > 0,\ \exists \delta > 0 :\ 0 < |x - x_0| < \delta \Longrightarrow f(x) < -M.$
* $\lim_{x \to +\infty} f(x) = +\infty \iff \forall M > 0,\ \exists K > 0 :\ x > K \Longrightarrow f(x) > M.$
* $\lim_{x \to -\infty} f(x) = +\infty \iff \forall M > 0,\ \exists K > 0 :\ x < -K \Longrightarrow f(x) > M.$
* $\lim_{x \to +\infty} f(x) = -\infty \iff \forall M > 0,\ \exists K > 0 :\ x > K \Longrightarrow f(x) < -M.$
* $\lim_{x \to -\infty} f(x) = -\infty \iff \forall M > 0,\ \exists K > 0 :\ x < -K \Longrightarrow f(x) < -M.$

1.8.2 Operations on limits

Proposition 5 *Let $f,\ g : D \subseteq \mathbb{R} \to \mathbb{R}$ be two functions and $x_0 \in \mathbb{R}$, such that*

$$\lim_{x\to x_0} f(x) = L \qquad \text{and} \qquad \lim_{x\to x_0} g(x) = L'.$$

Thus,

1. $\lim_{x\to x_0} (f+g)(x) = L + L'$.
2. $\lim_{x\to x_0} (f\times g)(x) = L \times L'$.
3. $\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{L}{L'}\cdot,\ L' \neq 0$.
4. $\lim_{x\to x_0} \alpha f(x) = \alpha L, \quad \alpha \in \mathbb{R}$.
5. $f(x) \leq g(x) \Longrightarrow L \leq L'$.

Remark 2 *The indeterminate forms of limits are:*

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0,\ -\infty+\infty,\ 1^{\infty},\ 0^0,\ 0^{\infty},\ \infty^{\infty}.$$

2 Continuous functions

Definition 12 *The function f is continuous at some point $x_0 \in D_f$ if the limit of $f(x)$, as x approach x_0 through the domain of f, exists and is equal to $f(x_0)$. In mathematical notation:*

$$\forall x \in D_f : \qquad \lim_{x\to x_0} f(x) = f(x_0).$$

Example 5 *The function $f(x) = \dfrac{1}{x-1}$ is not continuous on $\mathbb{R}$, but it is continuous on $]1,+\infty[$ or $]-\infty,1[$.*

Proposition 6 *The function f is continuous at $x_0 \in D_f \Longleftrightarrow \lim_{x \overset{>}{\to} x_0} f(x) = f(x_0) = \lim_{x \overset{<}{\to} x_0} f(x)$.*

Definition 13 *A function f is continuous on D_f if it is continuous at every point $x_0 \in D_f$, i.e. $\lim_{x \overset{<}{\to} x_0} f(x) = \lim_{x \overset{>}{\to} x_0} f(x) = f(x_0)$.*

Proposition 7 *Let $I \subset \mathbb{R}$ and $f, g : I \to \mathbb{R}$ are continuous functions at $x_0 \in I$, then the functions λf $(\lambda \in \mathbb{R})$, $f+g$, fg, $\dfrac{f}{g}$ $(g \neq 0)$ and $|f|$ are continuous at x_0.*

3 Derivable functions

Definition 14 *Let $x_0 \in D$, assume that there exists $r > 0$, such that $]x_0 - r,\ x_0 + r[\subset D$. We say that $f : D \to \mathbb{R}$ is derivable at x_0 if the limit $\lim\limits_{x\to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists and finite and we have:*

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Example 6 *Prove that the function $f(x) = \sqrt{x}$ is derivable at every point $x_0 > 0$ and $f'(x_0) = \dfrac{1}{2\sqrt{x_0}}$.*
We have,

$$\begin{aligned} \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} \\ &= \lim_{x\to x_0} \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} \\ &= \frac{1}{2\sqrt{x_0}}. \end{aligned}$$

Then, the limit exists and it is unique, thus f is differentiable at $x_0 > 0$ and we have $f'(x_0) = \dfrac{1}{2\sqrt{x_0}}$.

Proposition 8 *Let $f : I \to \mathbb{R}$ be a function, such that I is an open interval of $\mathbb{R}$ and $x_0 \in I$.*

- *If f is derivable at x_0 (or on I), then f is continuous at x_0 (or on I). The receprocal is false.*

  Example 7 *We know that the function $f(x) = |x|$ is continuous at $x_0 = 0$ ($\lim\limits_{x\to 0} f(x) = 0 = f(0)$.)*
  On the other hand,

  $$\lim_{x\to x_0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to x_0} \frac{|x|}{x} = \begin{cases} 1, & x \leq 0 \\ -1, & x < 0. \end{cases}$$

  Then,

  $$\lim_{x \overset{<}{\to} x_0} \frac{f(x) - f(0)}{x} \neq \lim_{x \overset{>}{\to} x_0} \frac{f(x) - f(0)}{x}.$$

  Therefore, f is not derivable at x_0.

- *We say that f is right-derivable (resp. left-derivable) at x_0 if $\lim\limits_{x \overset{>}{\to} x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ $\Big($resp. $\lim\limits_{x \overset{<}{\to} x_0} \dfrac{f(x) - f(x_0)}{x - x_0}\Big)$ exists and finite.*

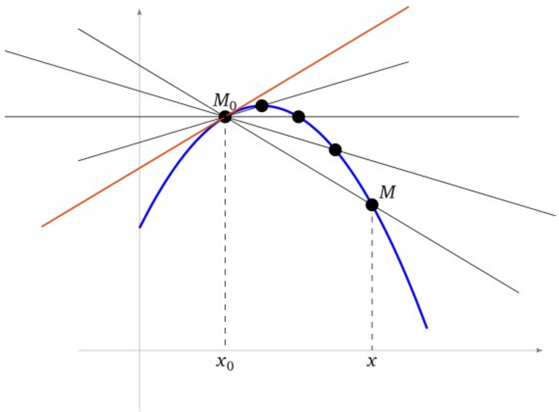
- *f is derivable at x_0 if and only if* $\lim_{x\to h} \frac{f(x+h)-f(x_0)}{h}$ *exists and finite.*

Definition 15 *We say that f is derivable on D if f is derivable at every point $x \in D$. The derivative function of f is denoted by: f' or $\frac{df}{dx}$.*

3.1 Tangent to a curve

The line passing through the distinct points $(x_0, f(x_0))$ and $(x, f(x))$ has the slope $\frac{f(x)-f(x_0)}{x-x_0}$. Passing to the limits, we find that the slope is $f'(x_0)$. The tangent equation at a point $(x_0, f(x_0))$ is given by:

$$y = (x - x_0)f'(x_0) + f(x_0).$$



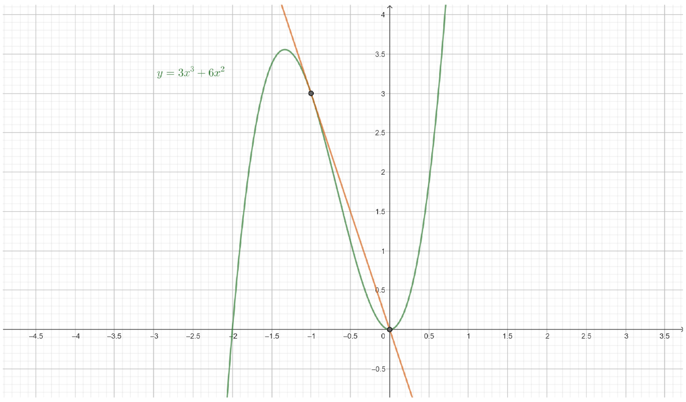
Definition 16 *A tangent is a line that touches a curve at a single point and does not cross through it. The point where the curve and the tangent meet is called the point of tangency. We know that for a line $y = ax + b$ its slope is a.*

Method: To trace the tangent, we need to determine two points of the tangent line using its equation and plot the line passing through those points.

Example 8 *Let $f(x) = 3x^3 + 6x^2$, we will prove how to trace the tangent of this function at $x = -1$.*
To determine the point where the curve and the tangent touch, we can read that $f(-1) = 3$. Then, the curve and the tangent touch at the point $(-1, 3)$.
Now, we must determine the equation of the tangent. For that, we must determine the derivative function of f. We have: $f'(x) = 9x^2 + 12x$. Also, we determine the equation of the tangent as follows:

$$y = f'(-1)(x - x_0) + f(-1) = -3(x + 1) + 3 = -3x.$$

We can then determine another tangent point. When $x = 0$ and $y = 0$. so we can use the two points to draw the tangent line.



3.2 Derivative calculation

Proposition 9 *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be two derivative functions on I. Then,, for all $x \in I$, we have:*

- $(f + g)'(x) = f'(x) + g'(x)$.
- $(\lambda f)'(x) = \lambda f'(x)$, *where λ is a fixed real number.*
- $(f \times g)'(x) = f'(x)g(x) + f(x)g'(x)$.
- $\left(\frac{1}{f}\right)'(x) = -\left(\frac{f'(x)}{f^2(x)}\right), \qquad (f(x) \neq 0)$.
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}, \qquad (g(x) \neq 0)$.

Remark 3 *If you have to derivate a function with an exponent dependent on x, it is absolutely neccessary to convert it back to the exponential form. For example: if $f(x) = 2^x$, then firstable we write $f(x) = e^{x \ln 2}$, after that we calculate the derivative $f'(x) = \ln 2 e^{x \ln 2} = 2^x \ln 2$.*

3.3 Derivative of elementary functions

The first table as a summary of the main formulas to know, where x is a variable. In the second table, u represent a function $x \mapsto u(x)$.

Function	Derivative
x^n	nx^{n-1} $(n \in \mathbb{Z})$
$\frac{1}{x}$	$-\frac{1}{x^2}$ $(x \neq 0)$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$, $(x > 0)$
x^α	$\alpha x^{\alpha-1}$ $(\alpha < 1 \text{ et } x \neq 0) \vee (\alpha \geq 1)$
e^x	e^x
$\ln x$	$\frac{1}{x}$ $(x \neq 0)$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$

Function	Derivative
u^n	$nu'u^{n-1}$ $(n \in \mathbb{Z})$
$\frac{1}{u}$	$\frac{-u'}{u^2}$, $(u \neq 0)$
$\sqrt{u}$	$\frac{u'}{2\sqrt{u}}$ $(u \neq 0)$
u^α	$\alpha u'u^{\alpha-1}$, $(\alpha < 1 \text{ et } u \neq 0) \vee (\alpha \geq 1)$
e^u	$u'e^u$
$\ln u$	$\frac{u'}{u}$
$\cos u$	$-u' \sin u$
$\sin u$	$u' \cos u$
$\tan u$	$u'(1 + \tan^2 u) = \frac{u'}{\cos^2 u}$

3.4 Derivative of a composite function

Theorem 1 *Let f be a function defined on $I \subseteq \mathbb{R}$, g be a function defined on $J \subseteq \mathbb{R}$, with $f(I) \subseteq J$ and $x_0 \in I$. If f is derivative at x_0 and g is derivative at $f(x_0)$, then $g \circ f$ is derivative at x_0 and we have:*

$$(g \circ f)'(x_0) = g'(f(x_0)) = f'(x_0)g'(f(x_0)).$$

Example 9 *Calculate the derivative of the function $H(x) = \ln(1 + x^2)$.*
Put: $g(x) = \ln(x)$ with $g'(x) = \frac{1}{x}$ and $f(x) = 1 + x^2$ with $f'(x) = 2x$.
Then, $H(x) = (g \circ f)(x)$. Thus,

$$H'(x) = (g(f(x)))' = f'(x)g'(f(x)) = 2xg'(1 + x^2) = \frac{2x}{1 + x^2}.$$

3.5 Successive derivatives

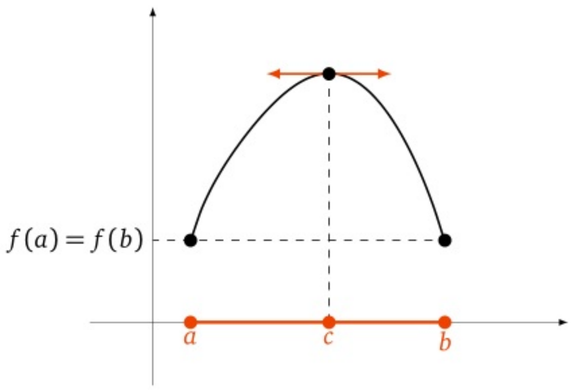
Definition 17 *Let $f : I \to \mathbb{R}$. We note $f^{(0)} = f$. Suppose that $f^{(n-1)}$ exists and it is derivable on I. So, we define the function $f^{(n)} = (f^{(n-1)})'$.*
If the function $f^{(n)} : I \to \mathbb{R}$ exists, then we say that f is n times derivable on I.

Proposition 10 *Let $\lambda \in \mathbb{R}$, $m \in \mathbb{N}$, $f, g : I \to \mathbb{R}$ be two functions n times derivable on I. Then, $\forall\ n \leq m$:*

- *$f + g$ is n times derivable on I and $(f+g)^m = f^m + g^m$.*
- *λf is n times derivable on I and $(\lambda f)^m = \lambda f^m$.*
- *fg is n times derivable on I and $(fg)^m = \sum_{k=0}^{m} C_m^k f^{(k)} g^{(m-k)}$, where $C_m^k = \dfrac{m!}{k!(m-k)!}$. (Leibniz formulat).*
- *If $\forall\ x \in I,\ g(x) \neq 0$, then $\dfrac{f}{g}$ is n times derivable on I.*

3.6 Rolle's theorem

Theorem 2 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$, derivable on $]a,b[$, such that $f(a) = f(b)$. Then, there exists $c \in]a,b[$, such that $f'(c) = 0$.*



Geometric Interpretation: geometrically, the theorem states that there is at least one point $c \in]a.b[$, distinct from a and b, for which the tangent to the curve at this point is horizental.

Example 10 *Let f be a function defined by:*

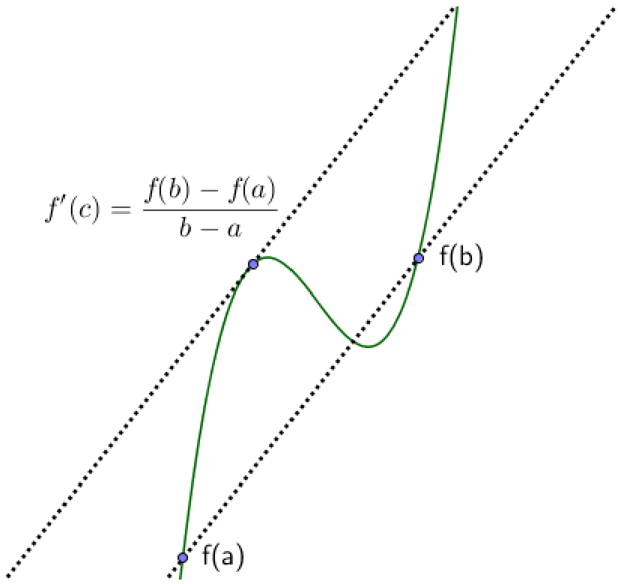
$$f(x) = 3x^4 - 11x^3 + 12x^2 - 4x + 2.$$

Prove that $f'(x) = 0$ at least once on $]0,1[$. We have the function f is continuous on $[0,1]$ and differentiable on $]0,1[$, such that $f(0) = f(1) = 2$. Then, from Rolle's theorem there exists $x \in]0,1[$, such that $f'(x) = 0$.

Theorem 3 (Mean value Theorem) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and derivable on $]a, b[$. Then, there exists $c \in]a, b[$, such that*

$$f(b) - f(a) = (b - a)f'(c) \Longleftrightarrow f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note that $\frac{f(b) - f(a)}{b - a}$ is the direction coefficient of the line connecting $(a, f(a))$ to $(b, f(b))$ (which we call a chord between two of the curve of f) while $f'(c)$ is the direction coefficient of the tangent at c.



4 The Hospital Rule

Theorem 4 *Let $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be two derivables functions and let $x_0 \in I$. Suppose that $f(x_0) = g(x_0) = 0$ and $g'(x) \neq 0$ for all $x \in I - \{x_0\}$.*

$$If \quad \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l \qquad then \qquad \lim_{x \to x_0} \frac{f(x)}{g(x)} = l \in \mathbb{R}.$$

Remark 4

- *The Hopital's rule removes indeterminations of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$.*
- *On the other words, it can be used to determine $\lim_{x \to x_0} \frac{f(x)}{g(x)}$, if when x tends to x_0, $f(x)$ and $g(x)$ tends to 0 or ∞.*
- *Acording to this rule, if $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$.*
  Example:

- $\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$

- $\lim_{x \to 0} \frac{\ln(x+1)}{x} = \frac{0}{0} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1.$

- *It is possible to apply the Hospital rule several times successively when calculating the limit to remove the case of* $\frac{0}{0}$.
  Example:

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos x} = 2.$$

Note: The Hospital rule is used to confirm the calculation of limits.

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