Chapter 02: The field of complex numbers (or the set of complex numbers)

preface: (Introduction): The history of complex numbers goes back to the ancient Greeks who decide (but were perplexed) that no number existed that satisfies: $x^2 = -1$.

which led to constructing another set called the set of complex numbers, denoted by $\mathbb{C}$, where

$\mathbb{C} = \{ z = x+iy \mid x, y \in \mathbb{R} \text{ and } i = \sqrt{-1} \}$

x is called the real part of z and written as: $Re(z) = x$.

y is called the imaginary part of z and written as $Im(z) = y$.

Remark:

1/ Order relations "greater than" and "less than" are not defined for complex numbers.

2/ If the imaginary part of the complex number z is zero, then z is known as purely real number, i.e., $Im(z) = y = 0$ and hence $Re(x+iy) = x$.

3/ If the real part of the complex number z is zero, then z is known as purely imaginary number, i.e., $Re(z) = x = 0$ and hence $Im(x+iy) = y$.

[Example: 2 is a purely real number and $3i$ is a purely imaginary number]

1/ Two complex numbers $z = x+iy$ and $z' = x'+iy'$ are said to be equal if and only if both their real and imaginary parts are equal, i.e.,

$$x+iy = x'+iy' \Leftrightarrow \begin{cases} x = x' \text{ and } (\wedge) \\ y = y' \end{cases} \Leftrightarrow \begin{cases} Re(z) = Re(z') \\ \wedge \\ Im(z) = Im(z'). \end{cases}$$

2/ Operations on complex numbers: (or the algebra of complex numbers)

Let z and z' be any two complex numbers such that: $z = x+iy$ and $z' = x'+iy'$. then:

a/ Addition: $z + z' = (x+iy) + (x'+iy') = (x+x') + i(y+y')$.

b/ Substraction: $z - z' = (x+iy) - (x'+iy') = (x-x') + i(y-y')$.

c/ Multiplication: $zz' = (x+iy)(x'+iy') = xx' - yy' + i(xy' + x'y)$.

d) division: $\frac{z}{z'} = \frac{x+iy}{x'+iy'} = Re\left(\frac{z}{z'}\right) + i\, Im\left(\frac{z}{z'}\right)$

To do this we multiply "top and bottom" by the complex conjugate
(numirator) (denominator)
of the bottom, that is, by $x'-iy'$ (this is called rationalising)

$$\frac{z}{z'} = \frac{x+iy}{x'+iy'} = \frac{x+iy}{x'+iy'} \times \frac{(x'-iy')}{(x'-iy')} = \frac{(xx'+yy') + i\,(yx'-xy')}{x'^2+y'^2}$$

$$= \left(\frac{xx'+yy'}{x'^2+y'^2}\right) + i\left(\frac{yx'-xy'}{x'^2+y'^2}\right) = Re\left(\frac{z}{z'}\right) + i\, Im\left(\frac{z}{z'}\right).$$

3) The conjugate of complex number: (The complex conjugate):

The complex conjugate of a complex number is obtained by changing the sign of the imaginary part. So if $z = x+iy$, its complex conjugate, $\bar{z}$, is defined by: $\bar{z} = x - iy$.

- Any complex number $x+iy$ has a complex conjugate $x-iy$.

Example: Let $z_1 = 2-3i$, then $\bar{z}_1 = 2+3i$ / $z_2 = 5i-1$, then $\bar{z}_2 = -1-5i$

properties: Let $z = x+iy$ and $z' = x'+iy'$ be two complex numbers, then

1) z is said as purely real number $\Leftrightarrow Im(z) = 0 \Leftrightarrow z = \bar{z}$.

2) " " " " purely imaginary number $\Leftrightarrow Re(z) = 0 \Leftrightarrow z + \bar{z} = 0$, where $z \neq 0$

3) $\overline{z+z'} = \bar{z} + \bar{z'}$, 6) $\overline{z^n} = (\bar{z})^n$

4) $\bar{\bar{z}} = z$. , 7) $\overline{\left(\frac{z}{z'}\right)} = \frac{\bar{z}}{\bar{z'}}$, $z' \neq 0$.

5) $\overline{zz'} = \bar{z}\cdot\bar{z'}$,

4) The modulus of a complex number:

The modulus of a complex number $z = x+iy$ is denoted by $|z|$ and is defined by $|z| = \sqrt{x^2+y^2}$.

- The modulus is always a non-negative real number ($|z| = \sqrt{x^2+y^2} \in \mathbb{R}$, $\forall x, y \in \mathbb{R}$

Example: Let $z_1 = 2-3i$, then the modulus or absolute value of z_1 is $|z_1| = \sqrt{13}$.
and Let $z_2 = 5i-1$, then $|z_2| = \sqrt{(-1)^2+(5)^2} = \sqrt{26}$.

Properties: Let $z = x+iy$ and $z' = x'+iy'$ be two complex numbers, then

1) $|zz'| = |z||z'|$, 2) $|z+z'| \leq |z| + |z'|$, 3) $|z| = |\bar{z}| = |-z|$

1') $-|z| \leq Re(z) \leq |z|$, 2') $-|z| \leq Im(z) \leq |z|$, 3') $|z-z'| \geq |z| - |z'|$

4/ $|z^n| = |z|^n$, 5/ $\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}$, $z' \neq 0$, 6/ $|z| = 0 \iff z = 0$

7/ $|z|^2 = z\bar{z}$, 8/ If $|z| = 1$, then $\bar{z} = \frac{1}{z}$.

9/ If $z = x$, then $|z| = |x|$ (absolute value of x) and if $z = iy \Rightarrow |z| = |y|$.

4/ The argument of a complex number:

The argument of a complex number $z = x + iy$, written, for short, $\arg(z)$ is the angle $\theta \in]-\pi, \pi]$ such that: $\cos\theta = \frac{x}{|z|}$ and $\sin\theta = \frac{y}{|z|}$.

Example: Let $z = 2\sqrt{3} + 2i$, then we have $|z| = \sqrt{12+4} = \sqrt{16} = 4$

$\cos\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{2}{4} = \frac{1}{2}$

Therefore : $\theta = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$, then there is an infinite number of possible angles. The one you should normally use is in the interval $-\pi < \theta \leq \theta$, and this is called the principale argument.

Properties: Let $z = x + iy$ and $z' = x' + iy'$ be two complex numbers, where $\arg(z) = \theta$ and $\arg(z') = \theta'$, then :

a/ $\arg(z \times z') = \arg(z) + \arg(z') = \theta + \theta'$.

b/ $\arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z') = \theta - \theta'$.

c/ $\arg(z^n) = n\arg(z) = n\theta$.

d/ $\arg\left(\frac{1}{z}\right) = -\arg(z) = -\theta$

e/ Let $z = x$, then if $x > 0$, then $\arg(z) = 0$ and if $x < 0$, then $\arg(z) = \pi$.

f/ Let $z = iy$, then if $y > 0$, then $\arg(z) = \frac{\pi}{2}$ and if $y < 0$, then $\arg(z) = -\frac{\pi}{2}$.

5/ the forms of a complex number:

a/ Algebraic form: Let z be a complex number.
when we write z as the form $x + iy$, we called it as algebraic form.

b/ Trigonometric form: (polar form) Let P be a point representing a non-zero complex number $z = x + iy$ in the Argand plane.
If OP makes an angle θ with the positive direction of x-axis, then the polar (trigonometric) form of z is: $z = |z|(\cos\theta + i\sin\theta)$

c/ The exponential form: Let z be a complex number such that $|z| = \rho$ and $\arg(z) = \theta$, $\theta \in]-\pi, \pi]$, then the exponential form of z is written as follows: $z = \rho e^{i\theta}$.

Remark: when a complex number z has modulus ρ, which must be non-negative, and argument θ, which is usually taken such that it satisfies $-\pi < \theta \leq \pi$, then z can be represented in the following forms:

① $z = \rho(\cos\theta + i\sin\theta)$, ② $z = [\rho, \theta]$, ③ $z = \rho e^{i\theta}$.

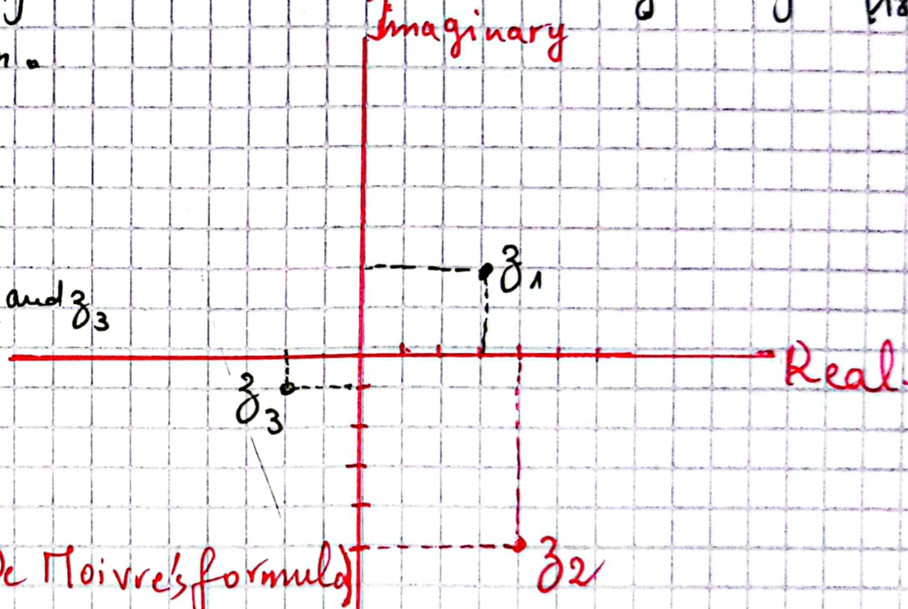
6/ Argand diagram (Geometric representation) of a complex number:

Any complex number $z = x + iy$ can be represented by an ordered pair (x, y) and hence plotted on xy-axes with the real part measured along the x-axis and the imaginary part along the y-axis. This graphical representation of the complex number field is called an **Argand diagram**, named after the Swiss mathematicien Jean Argand (1768 - 1822) or geometric representation.

Example: $z_1 = 3 + 2i$

$z_2 = 4 - 5i$, $z_3 = -2 - i$

Q) plot the complex numbers z_1, z_2 and z_3 on an Argand diagram.



7/ De Moivre's theorem (De Moivre's formula)

Let z be a complex number such that $|z| = 1$ and $\arg(z) = \theta$, that is $z = (\cos\theta + i\sin\theta)$, then $\forall n \in \mathbb{Z}$: $z^n = \cos n\theta + i\sin n\theta$

this relation is called the De Moivre's formula.

Proof: we know that : $\mathbb{Z} = \mathbb{N} \cup \mathbb{N}^-$

a/ for $n \in \mathbb{N}$, we show that : $\forall n \in \mathbb{N}$: $z^n = \cos n\theta + i\sin n\theta$

Let $P(n)$ be the statement: $P(n)$: $z^n = \cos n\theta + i\sin n\theta$, $\forall n \in \mathbb{N}$

$n = 0$ is trivial case.

for $n=1$, we have : $z^1 = z = \cos\theta + i\sin\theta = \cos 1\theta + i\sin 1\theta \Rightarrow p(1)$ is true.

Now, we assume that $p(n)$ is true and we prove that $p(n+1)$ is then true.

Hence, we have : $z^n = \cos n\theta + i\sin n\theta$ --- $(*)$

multiplying $(*)$ by z, we obtain : $z^n z = (\cos n\theta + i\sin n\theta)\, z$

and we know that : $z = \cos\theta + i\sin\theta$, then :

$$z^{n+1} = (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin n\theta)$$
$$= \cos n\theta \cos\theta - \sin n\theta \sin\theta + i(\cos n\theta \sin\theta + \cos\theta \sin n\theta)$$
$$= \cos((n+1)\theta) + i\sin((n+1)\theta)$$, hence $p(n+1)$ is true.

where the last equality holds from : $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

and : $\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$.

It follows that : $\forall n \in \mathbb{N}$: $z^n = \cos n\theta + i\sin n\theta$.

b) Now, it is sufficient to prove that : $\forall n \in \mathbb{N}^-$: $z^n = \cos n\theta + i\sin n\theta$

hence $n \in \mathbb{N}^- \Leftrightarrow n < 0$, we put $n = -n'$ such that $n' \in \mathbb{N}$

$$z^n = (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-n'} = \frac{1}{(\cos\theta + i\sin\theta)^{n'}}$$
$$\overset{\text{(De Moivre) } p(n')}{=} \frac{1}{\cos(n'\theta) + i\sin(n'\theta)}$$
$$\overset{\text{property (8) of Modulus}}{=}$$
$$= \cos(n'\theta) - i\sin(n'\theta)$$
$$= \cos(-n'\theta) + i\sin(-n'\theta)$$
$$= \cos n\theta + i\sin n\theta$$

where the next to last equality follows from : $\begin{cases} \cos(-\alpha) = \cos(+\alpha) \\ \sin(-\alpha) = -\sin\alpha \end{cases}$

So, $\forall n < 0$: $z^n = \cos n\theta + i\sin n\theta$.

From a and b, we deduce that : $z^n = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, the

8/ Euler's formulas: The formulas of Euler are given as :

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \text{ and } \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}),\ |z| = 1$$

where θ is the argument of the complex number $z = (\cos\theta + i\sin\theta)$.

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proof: we have: ① $z = \cos\theta + i\sin\theta$ and $\bar{z} = \cos\theta - i\sin\theta$ — ②

① + ② $\Rightarrow \bar{z} + z = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}(z + \bar{z})$ and from the exponential form of a complex number, we have: $z = \rho e^{i\theta} = e^{i\theta}$ $(\rho = |z| = 1)$

and $\bar{z} = \frac{1}{z} = \frac{1}{e^{i\theta}} = e^{-i\theta}$ (by (8) of the modulus' properties), then:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

in the same way, we get by substracting ② from ①: $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

[i.e., ① − ② $\Rightarrow z - \bar{z} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$].

9/ Applications of De Moivre's formula and Euler's formulas:

a/ linearization of the forms $(\cos\theta)^n$ and $(\sin\theta)^n$:

From Euler's formula, we have:

*/ $$(\cos\theta)^n = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^n = \frac{1}{2^n}\left(e^{i\theta} + e^{-i\theta}\right)^n$$

and from the Newton Binomial, we have:

$$(a+b)^n = \sum_{k=1}^{n} C_n^k a^k b^{n-k}, \text{ where } C_n^k = \frac{n!}{(n-k)!\,k!} \text{ is a combination}$$

Then,

$$(\cos\theta)^n = \frac{1}{2^n}\sum_{k=1}^{n} C_n^k \left(e^{i\theta}\right)^k \left(e^{-i\theta}\right)^{n-k}$$

$$= \frac{1}{2^n}\left(C_n^0 e^{-in\theta} + C_n^1 e^{-i(n-1)\theta} e^{i\theta} + \dots + C_n^{n-1} e^{i(n-1)\theta} e^{-i\theta} + C_n^n e^{in\theta}\right)$$

we know that: $C_n^0 = C_n^n = 1$ and $C_n^1 = C_n^{n-1} = n$ and from the Euler's formula $2\cos A = e^{iA} + e^{-iA}$, we obtain the linearization form of $(\cos\theta)^n$, that is: $(\cos\theta)^n = \frac{1}{2^n}\left(2\cos(n\theta) + 2n\cos(n-2)\theta + \cdots\right)$

In the same way, we get the linearization form of $(\sin\theta)^n$, that is $(\sin\theta)^n = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^n = \frac{1}{(2i)^n}\left(e^{i\theta} - e^{-i\theta}\right)^n$, and then by the Newton Binomial and knowing that, $e^{iA} - e^{-iA} = 2i\sin A$, we complete the proof.

Example: Linearizing the following forms: $(\cos x)^5$, $(\sin x)^4$ and $(\sin x)^3$.

Solution:

1) we have: $\cos x = \frac{1}{2}(e^{ix}+e^{-ix}) \Rightarrow \cos^5 x = \frac{1}{2^5}(e^{ix}+e^{-ix})^5$

we know that: $(a+b)^5 = a^5+b^5+C_5^1 a^4 b + C_5^2 a^3 b^2 + C_5^3 a^2 b^3 + C_5^4 a b^4$,

where $C_5^1 = C_5^4 = 5$, $C_5^2 = C_5^3 = 10$, then:

$\cos^5 x = \frac{1}{2^5}\left(e^{i5x}+e^{-i5x}+5e^{i3x}+5e^{-i3x}+10e^{ix}+10e^{-ix}\right)$

from the relation: $2\cos A = e^{iA}+e^{-iA}$, we obtain:

$\cos^5 x = \frac{1}{2^5}\left(2\cos 5x + 10\cos 3x + 20\cos x\right) = \frac{1}{2^4}\left(\cos 5x + 5\cos 3x + 10\cos x\right)$

2) we have: $\sin x = \frac{1}{2i}(e^{ix}-e^{-ix}) \Rightarrow \sin^4 x = \frac{1}{(2i)^4}(e^{ix}-e^{-ix})^4$

$\Rightarrow \sin^4 x = \frac{1}{2^4}\left(e^{i4x}+e^{-i4x}-4e^{i2x}-4e^{-i2x}+6\right)$

$= \frac{1}{2^4}\left(2\cos 4x - 8\cos 2x + 6\right)$ (by $2\cos A = e^{iA}+e^{-iA}$)

$= \frac{1}{2^3}\left(\cos 4x - 4\cos 2x + 3\right)$, which gives the linearization of $\sin^4 x$.

3) $(\sin x)^3 = \frac{1}{(2i)^3}\left(e^{ix}-e^{-ix}\right)^3$

$= \frac{-1}{2^3 i}\left(e^{i3x}-e^{-i3x}-3e^{ix}+3e^{-ix}\right)$

$= \frac{-1}{2^3 i}\left(2i\sin 3x - 6i\sin x\right)$

$\Rightarrow (\sin x)^3 = -\frac{1}{2^2}\left(\sin 3x - 3\sin x\right) = \frac{1}{2^2}\left(3\sin x - \sin 3x\right)$, which completes the Example.

2/ Developement of $\cos(nx)$ and $\sin(nx)$ in terms of $\sin x$ and $\cos x$.

Example: For finding the developement of $\cos 2x$ and/or $\sin 2x$ expression we use De Moivre's formula and then the Newton's binomial, that is

$\underbrace{(\cos 2x + i\sin 2x)}_{(1)} \overset{\text{DeMoivre}}{=} (\cos x + i\sin x)^2 \overset{\text{Newton}}{=} \sum_{k=0}^{2} C_2^k (\cos x)^k (i\sin x)^{2-k}$

$= \underbrace{\cos^2 x - \sin^2 x + 2i\sin x\cos x}_{(2)}$

comparing (2) with (1), we obtain:

$\cos 2x = \cos^2 x - \sin^2 x$ and $\sin 2x = 2\sin x\cos x$.

Example 2: Find the developement of $\cos 4x$ and $\sin 4x$.

solution: we have:

$$(\cos 4x + i\sin 4x) \overset{\text{De Moivre}}{=} (\cos x + i\sin x)^4 \overset{\text{Newton}}{=} \sum_{k=0}^{4} C_4^k (\cos^k x)(i\sin x)^{4-k}$$

$$= \underset{1}{C_4^0}(i\sin x)^4 + \underset{4}{C_4^1}(\cos x)(i\sin x)^3 + \underset{6}{C_4^2}(\cos x)^2(i\sin x)^2 + \underset{4}{C_4^3}(\cos x)^3(i\sin x)^1 + \underset{1}{C_4^4}(\cos^4 x)(i\sin x)^0$$

$$= \sin^4 x - 4i\cos x\sin^3 x - 6\cos^2 x\sin^2 x + 4i\cos^3 x\sin x + \cos^4 x.$$

$$= \sin^4 x + \cos^4 x - 6\cos^2 x\sin^2 x + i4(\cos^3 x\sin x - \cos x\sin^3 x)$$

hence

$$\boxed{\cos 4x = \sin^4 x + \cos^4 x - 6\cos^2 x\sin^2 x} \text{ and } \boxed{\sin 4x = 4(\cos^3 x\sin x - \cos x\sin^3 x)}$$

10/ Finding n^{th} roots of a complex number: TD: plot the solutions

Let z be a complex number such that $|z| = \rho$ and $\arg(z) = \theta$ and let a be a known complex number with $|a| = r$ and $\arg(a) = \beta$, then for solving the equation: $(*)$ $z^n = a$, we distinguish two cases according to the value of a.

a/ If $a = 0$: then eq $(*)$ has a unique solution of ordre n such that "$z = 0$".

b/ If $a \neq 0$: then we write the eq $[(*)]$ in exponential form, that is,

$\rho^n e^{in\theta} = r e^{i\beta}$, hence we deduce that:

$$\begin{cases} \rho^n = r \\ n\theta = \beta + 2k\pi, \ k = 0,1,2,\dots,n-1 \ (k \in \mathbb{Z}) \end{cases} \Rightarrow \begin{cases} \rho = r^{\frac{1}{n}} = \sqrt[n]{r} \\ \theta = \frac{\beta}{n} + \frac{2k\pi}{n}, \ k = 0,1,2,\dots,n-1 \end{cases}$$

then, the solutions $(*)$ are: $z_k = r^{\frac{1}{n}} e^{i\left(\frac{\beta}{n} + \frac{2k\pi}{n}\right)}$, $k = 0,1,\dots,n-1$

Example: What are the complex solutions to $z^5 = -32$?

Solution: we have: $z = \rho e^{i\theta}$ and $-32 = 32 e^{i\pi}$, hence we obtain

$$\rho^5 e^{i5\theta} = 32 e^{i(\pi + 2k\pi)} \Rightarrow \begin{cases} \rho = \sqrt[5]{32} = 2 \\ 5\theta = \pi + 2k\pi, \ k = 0,1,2,3,4. \end{cases}$$

$$\Rightarrow \begin{cases} \rho = 2 \\ \theta = \frac{\pi}{5} + \frac{2k\pi}{5}, \ k = 0,1,2,3,4 \end{cases} \Rightarrow \text{the solutions to } z^5 = -32 \text{ are } z = 2e^{i\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right)}, \text{ i.e.,}$$

$$S = \left\{2e^{i\frac{\pi}{5}}, 2e^{i\frac{3\pi}{5}}, 2e^{i\left(\frac{5\pi}{5}\right)}, 2e^{i\left(\frac{7\pi}{5}\right)}, 2e^{i\left(\frac{9\pi}{5}\right)}\right\}$$