Some mathematical symbols:

$\forall$: for all, for any, for each, for every.

$\exists$: there exists, there is, there are.

$\exists!$: there exists exactly one (uniqueness quantification)

$\in$: in, is an element of, belongs to.

$\notin$: not in, is not element of, not belongs to

$\subset$: subset (is a subset of ...)

$\supset$: superset (is a superset of ...)

$\cup$: the Union. (the union of A and B = $A \cup B$)

$\cap$: the intersection (A is intersected with B = $A \cap B$)

$\vee$: or, logical disjunction or join in a lattice.

$\wedge$: and, logical conjunction or meet in a lattice.

$\emptyset$; $\{\}$: empty set, the set with no elements.

$\rightarrow$: from ... to ... . ($f: E \rightarrow F$: f is defined from E to F)

$\leftarrow$: converse implication ($a \leftarrow b$: b implies a, then a is the converse implication of : this reads as "a if b" or " not b without a".

$\Rightarrow$: implication (implies, if ... then ...).

$\leqslant$: is less than.

$\geqslant$: " greater " .

$\ll$: is much less than.

$\gg$: " " greater than.

$\Leftrightarrow$: equivalent.

$=$: equality, is equal to, equals,

$\neq$: is not equal to, inequality

∞: infinity

$+$: plus

$-$: minus

$\times$: multiplication, cross product, Kartesian product, times.

$\cdot$: dot product, multiplication.

$\div$: division ($6 \div 3$ means 6 is devided by 3 or division of 6 by 3).

Chapter one: The field of real numbers

1/ Introduction (preface): The set of natural numbers, denoted by $\mathbb{N}$ and consisting of elements $\mathbb{N} = \{1, 2, 3, \dots\}$ is considered a primitive notion.

- The absence of elements from the set $\mathbb{N}$, where their sum with 1, or 2 or any other non-zero natural number gives the number "0", led to the creation of the set of integers $\mathbb{Z}$, where

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$

- Then after that, the absence of elements from the set $\mathbb{Z}$, whose multiplication with the numbers 2 or 3 gives the numbers 1 led to the creation of the set of rational numbers $\mathbb{Q}$, where

$\mathbb{Q} = \{\frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}\}$.

- Now, based on the Vithagora's theorem (VT) for a square whose side length is 1, the diagonale length x must satisfy the equation $x^2 = 1^2 + 1^2 = 2$.

However, there is no rational number which satisfies this equation. This leads to the creation of the set of real numbers $\mathbb{R}$, and we have:

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$\mathbb{N}$: The set of natural numbers.

$\mathbb{Z}$: The set of integers.

$\mathbb{Q}$: The set of rational numbers.

$\mathbb{R}$: The set of real numbers.

2/ the set of real numbers:

2-1/ the existence and unity of the set $\mathbb{R}$:

Definition: we accept the existence and unity of the set of real numbers $\mathbb{R}$ with the internal addition "+" and multiplication "." operations defined as follows:

$\forall x, y \in \mathbb{R}: x * y = x + y \wedge x \cdot y = xy$.
and they satisfy the following theorem

Theorem 1: $(\mathbb{R}, +, \cdot)$ is a commutative field. that is satisfies the following statement:

a) for addition: for any $x, y, z \in \mathbb{R}$, we have:

1/ $x+y = y+x$ ("+" is a commutative or a permutative) $\boxed{\text{commutativity}}$

2/ $x+(y+z) = (x+y)+z$ ("+" is a compilation or an associative process) $\boxed{\text{associativity}}$

3/ $x+0 = 0+x = x$ ("0" is a neutral element with respect to "+")

4/ $\forall x \in \mathbb{R}, \exists (-x) \in \mathbb{R}: x+(-x) = 0$ ($(-x)$ is the inverse element of x with the respect to "+")

b) for multiplication: for any $x, y, z \in \mathbb{R}$, we have:

1/ $xy = yx$ ("$\cdot$" is a commutative)

2/ $x(yz) = (xy)z$ ("$\cdot$" is an associative)

3/ $x \cdot 1 = 1 \cdot x = x$ ("1" is a neutral element with respect to "$\cdot$")

4/ $\forall x \in \mathbb{R}, \exists \left(\frac{1}{x}\right) \in \mathbb{R}^*: x\left(\frac{1}{x}\right) = 1$ ($\frac{1}{x}$ is the inverse element of x with respect to "$\cdot$")

c) Distributivity (distributativity)

1/ $x(y+z) = xy + xz$ (multiplication is distributive over addition from the left)

2/ $(y+z)x = yx + zx$ (multiplication is distributive over addition from the right)

Remark: from the above, we can deduce that:

1/ $0 \cdot x = 0$, 2/ $(-1)x = -x$, 3/ $-xy = -(xy)$, 4/ $(-x)(-y) = xy$

• in addition to the algebraic structure of the set $\mathbb{R}$, we assume that it is associated with the order relationship "$\leq$".

Theorem 2: The relationship "$\leq$" is a total ordering relationship consistent with the algebraic structure of $\mathbb{R}$, that is, $\mathbb{R}$ is a well-ordered field, i.e.,

1/ $\forall x \in \mathbb{R}: x \leq x$ (reflexivity)

2/ $\forall x, y \in \mathbb{R}: (x \leq y) \wedge (y \leq x) \Rightarrow x = y$. (antisymmetric relationship)

3/ $\forall x,y,z\in\mathbb{R}\ (x\leq y \wedge y\leq z) \Rightarrow x\leq z$ (transitivity)

4/ $\forall x,y\in\mathbb{R},\ x\leq y \vee y\leq x$. (total order)

5/ $\forall x,y\in\mathbb{R}:\ x\leq y \Leftrightarrow y-x\in\mathbb{R}_+$ (Rule 1)

6/ $\forall x,y,z\in\mathbb{R}:\ x\leq y \Leftrightarrow x+z\leq y+z$ (Rule 2)

7/ $\left.\begin{array}{l}\forall x,y\in\mathbb{R}, \forall z>0:\ x\leq y \Leftrightarrow xz\leq yz \\ \quad ,\forall z<0:\ x\leq y \Leftrightarrow xz\geq yz\end{array}\right\}$ (Rule 3)

8/ $\forall x,y\in\mathbb{R}_+^*:\ x\leq y \Leftrightarrow \frac{1}{x}\geq\frac{1}{y}$ (Rule 4, called Reciprocal Rule)

9/ $\forall x,y\in\mathbb{R}^*:\ \forall p>0:\ x\leq y \Leftrightarrow x^p\leq y^p$ (Rule 5, called Power Rule)

Remark: Rules 1-5 are used for rearranging inequalities.

Example: solve the inequality $\frac{1}{2x^2+2}<\frac{1}{4}$.

solution: since $2x^2+2>0$, we have

$$\frac{1}{2x^2+2}<\frac{1}{4} \Leftrightarrow 2x^2+2>4 \quad \text{(by Rule 4)}$$
$$\Leftrightarrow x^2+1>2 \quad \text{(by Rule 3)}$$
$$\Leftrightarrow x^2-1>0 \quad \text{(by Rule 1)}$$
$$\Leftrightarrow (x-1)(x+1)>0$$

This last inequality holds precisely when $x<-1$ or $x>1$.

It follows that the solution of the given (original) inequality is:

$$\left\{x:\ \frac{1}{2x^2+2}<\frac{1}{4}\right\} =]-\infty,-1[\ \cup\ ]1,+\infty[.$$

2.2: Integer part function: دالة الجزء الصحيح:

Definition: Let x be a real number, the integer part of x, denoted by $[x]$, is the (greater / largest) integer number less than or equal to the number x, that is,

$\forall x\in\mathbb{R}:\ [x]\leq x$, where: $[\cdot]:\mathbb{R}\longrightarrow\mathbb{Z}$, $x\longmapsto[x]$

Remarks: From the definition, we conclude that:

$\forall x\in\mathbb{R}:\quad [x]\leq x<[x]+1$

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Example: Find the integer part of the following number:

$-1{,}1$; $2{,}8$; π ; 5 ; $6{,}3$; -3 ; $\frac{1}{3}$; $-\frac{1}{3}$.

solution:

$[-1{,}1] = -2$; $[2{,}8] = 2$; $[\pi] = 3$; $[5] = 5$;

$[6{,}3] = 6$; $[-3] = -3$; $[\frac{1}{3}] = 0$; $[-\frac{1}{3}] = -1$.

Remark 2: Let x be a real number, then we have

$$[x+k] = [x] + k, \ \forall k \in \mathbb{Z}.$$

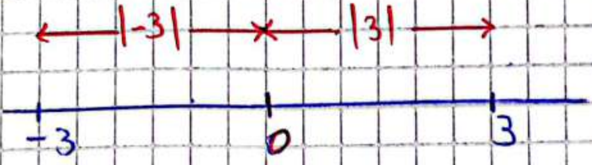
2.3. Absolute value function: دالة القيمة المطلقة

Definition: Let x be a real number, the modulus or absolute value of x, denoted by $|x|$ is defined by:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Remark ① It is often useful to think of $|x|$ as the distance along the real line from 0 to x.

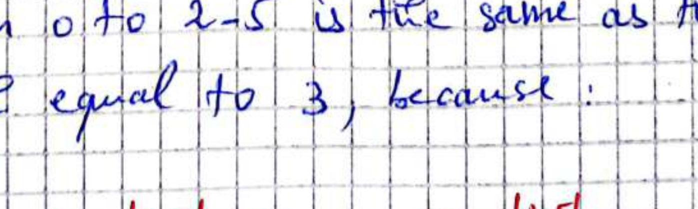
e.g: $|-3| = |3| = 3$



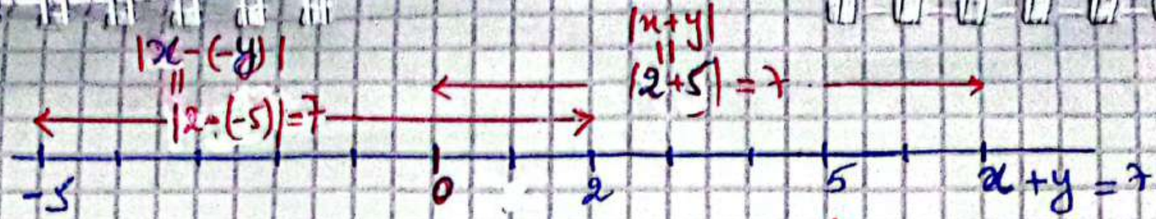
② In the same way, $|x-y|$ is the distance along the real line from 0 to $x-y$, which is the same as the distance from x to y.

e.g: the distance from 0 to $2-5$ is the same as the distance from 2 to 5 which equal to 3, because:

$|2-5| = |-3| = 3$



③ Also, we note that: $|x+y| = |x-(-y)|$ is the distance from x to $(-y)$ (e.g: $|2+3| = |2-(-3)| = |5| = 5$)



properties of the absolute value (or modulus): for any real number x and y, we have the following properties:

1/ $|x| \geq 0$, 2/ $|x| = 0$ iff ($\Leftrightarrow$) $x = 0$, 3/ $-x \leq x \leq |x|$

4/ $|x-y| = |y-x|$, 5/ $|xy| = |x| \times |y|$

6/ $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}_+ : |x| < y \Leftrightarrow -y < x < y$ (Rule 6)

example: write the inequality $|x-2| < 1$ without absolute values (or solve the inequality $|x-2| < 1$)

solution: using Rule 6, we obtain

$$|x-2| < 1 \Leftrightarrow -1 < x-2 < 1$$
$$\Leftrightarrow 1 < x < 3$$

So, the solution set of the given inequality is

$$\{x : |x-2| < 1\} =]1, 3[.$$

Remark: We can use Rule 5 with $p = 2$, we get

$$|x-2| < 1 \Leftrightarrow (x-2)^2 < 1$$
$$\Leftrightarrow x^2 - 4x + 3 < 0$$
$$\Leftrightarrow (x-1)(x-3) < 0$$

this proves that the solution set of the original inequality is

$$\{x : |x-2| < 1\} =]1, 3[.$$

7/ $\forall x, y, z \in \mathbb{R} : x < y \wedge y < z \Rightarrow x < z$ (transitive Rule)

8/ If $x < y$ and $z < t$ then

ⓐ $x + z < y + t$ (sum Rule)

ⓑ $xz < yt$ (product Rule, with $x, z \geq 0$)

9/ $\forall x, y \in \mathbb{R} : |x+y| \leq |x| + |y|$ (tri Inequality) (TI)

; $||x| - |y|| \leq |x-y|$ (the "reverse form" of the tri inequality)

10/ For any real numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$, we have:

$$(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$$

[called **Cauchy-Schwarz Inequality**]

Proof: (Cauchy-Schwarz inequality)

If $a_i = 0, \forall i = \overline{1,n}$ and b_i, $\forall i = \overline{1,n}$, then the result is obvious.

So we need only examine the case when not all the as are zero.

If we denote the sum $\sum_{i=1}^{n} a_i^2 = a_1^2 + a_2^2 + \dots + a_n^2$ by A, then $\boxed{A > 0}$.

Also, we denote $\sum_{i=1}^{n} b_i^2 = b_1^2 + b_2^2 + \dots + b_n^2$ by B and

$\sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$ by C.

Now, for any real number λ, we have $(\lambda a_i + b_i)^2 \geq 0$, so that:

$$\sum_{i=1}^{n} (\lambda^2 a_i^2 + 2\lambda a_i b_i + b_i^2) = \sum_{i=1}^{n} (\lambda a_i + b_i)^2 \geq 0,$$

which we may rewrite in the form:

$\lambda^2 A + 2\lambda C + B \overset{[\times(A)>0]}{\geq} 0$. But this inequality is equivalent to the following inequality: $(\lambda A + C)^2 + AB \geq C^2$, for any real number.

Since A is non-zero, we may choose $\lambda = -\frac{C}{A}$. It follows from the last inequality that $AB \geq C^2$, which is exactly what we had to pro[ve]

Properties of $\mathbb{R}$:

1/ Archimedes' axiom in $\mathbb{R}$: First recall the Archimedes' axiom in $\mathbb{Q}$

$\forall r \in \mathbb{Q}, \exists n \in \mathbb{N} : n > r$

This axiom can be generalized to $\mathbb{R}$, which is given as follow:

/ theorem ①

(Archimedes' axiom in $\mathbb{R}$) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$

Proof: We assume the opposite, that is,

$\exists x \in \mathbb{R}, \forall n \in \mathbb{N} : n \leq x$ --- (*)

Let $n_0 = [x] + 1$, from definition of integer part, we have

$x < n_0$ $\quad ([x] \leq x < [x] + 1 = n_0)$

Hence, we found $n_0 \in \mathbb{N}$ such that $x < n_0$ which contradicts th[e] assumption (*). And thus:

$\boxed{\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x}$

b/ Generalization of Archimedes' axiom in $\mathbb{R}$:

theorem ② If $x \in \mathbb{R}, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$.

[$\forall x \in \mathbb{R}_+^*, \forall y \in \mathbb{R}, \exists n \in \mathbb{N} : nx > y$]

proof/ the same way as the previous proof, only we replace x by $\frac{y}{x} \in \mathbb{R}$, we get the desired inequality.

[that is: $\exists x \in \mathbb{R}_+^*, \exists y \in \mathbb{R}, \forall n \in \mathbb{N} : nx \leq y \Leftrightarrow n \leq \frac{y}{x}$

let $n_0 = [\frac{y}{x}] + 1$ then, we have $\frac{y}{x} < n_0 \Leftrightarrow y \leq n_0 x$ contradiction]

2/ $\mathbb{Q}$ is dense in $\mathbb{R}$:

theorem ③ If $x \in \mathbb{R}, y \in \mathbb{R}$ and $x < y$, then there exists a number $r \in \mathbb{Q}$ such that $x < r < y$.

[$\forall x, y \in \mathbb{R}, \exists r = \frac{p}{q} \in \mathbb{Q} : x < r < y$, i.e., Between any two real numbers there is a rational one]

Proof: Since $x < y$, we have $y - x > 0$, and theorem ② furnishes (تعطي - تقدم) a positive integer n such that: $n(y-x) > 1$. —①

Apply theorem ② again, to obtain positive integers m_1 and m_2 such that $m_1 > nx$, $m_2 > -nx$. then

$$-m_2 < nx < m_1.$$

Hence there is an integer m (with $-m_2 \leq m \leq m_1$) such that

$$m - 1 \leq nx < m. \quad —(2)$$

If we combine these inequalities (1) and (2), we obtain

$$nx \overset{(2)}{<} m \overset{(2)}{\leq} nx + 1 \overset{(1)}{<} ny \Rightarrow nx < m < ny$$

Since $n > 0$, it follows that: $x < \frac{m}{n} < y$

this proves theorem (3), with $r = \frac{m}{n}$.

2nd mtd: there is three cases, according to the sign of x and y.

case 1: if $x < 0 < y$, it is enough to take $\boxed{r = 0}$.

case 2: if $y > x \geq 0$, then we have $y - x > 0$, from theorem 2,

$\exists q \in \mathbb{N}^*,\ q(y-x) > 1 \Leftrightarrow \exists q \in \mathbb{N}^* : y - x > \frac{1}{q}$ —(1)

put $p = [qy]$, then we have:

$p = [qy] \leq qy < p + 1 = [qy] + 1$ —(2)

it produces that: $x < \frac{p}{q} \Rightarrow x < \frac{p}{q} \leq y$

because, if we assume that $x \geq \frac{p}{q}$ we obtain from (1) that

$y - x > \frac{1}{q} \Leftrightarrow y > \frac{1}{q} + x > \frac{1}{q} + \frac{p}{q} = \frac{p+1}{q} \Leftrightarrow qy > p + 1$ (contradiction with (2))

then, we deduce that: $x < \frac{p}{q}$ and hence:

$\exists \frac{p}{q} = r \in \mathbb{Q} : \boxed{x < \frac{p}{q} \leq y}$

- if $y \neq \frac{p}{q}$, then it satisfies to take $\boxed{r = \frac{p}{q}}$, hence theorem (3) holds
- if $y = \frac{p}{q}$, then it is enough to take $\boxed{r = \frac{p-1}{q}}$ hence $x < \boxed{\frac{p-1}{q} < y = \frac{p}{q}}$

it's clear that $\frac{p-1}{q} < y = \frac{p}{q}$.

Now, we prove that $x < \frac{p-1}{q}$?

from (1), we have: $y - x > \frac{1}{q} \Leftrightarrow \frac{p}{q} - x > \frac{1}{q} \Leftrightarrow x < \frac{p-1}{q}$

hence: $\exists r = \frac{p-1}{q} \in \mathbb{Q} : \boxed{x < \frac{p-1}{q} < y}$

case 3: if $x < y \leq 0 \Leftrightarrow -x > -y \geq 0$, which is the case (2) himself, only, we replace x by $(-x)$ and y by $(-y)$.

The Intervals

Definition 1: we call an interval of $\mathbb{R}$ any subset I of $\mathbb{R}$ which satisfies the following property:

$$\forall a, b \in I,\ \forall x \in \mathbb{R} : (a \leq x \leq b \Rightarrow x \in I)$$

Definition 2: The open interval is a subset of $\mathbb{R}$ written as follows:

$$]a, b[= \{x \in \mathbb{R} ;\ a < x < b\} \text{ with } a, b \in \mathbb{R} \cup \{\pm\infty\}$$

Forms of the intervals:

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ the closed interval

$]a,b[= \{x \in \mathbb{R} : a < x < b\}$ the open interval

$[a,b[= \{x \in \mathbb{R} : a \leq x < b\} \wedge]a,b] = \{x \in \mathbb{R} : a < x \leq b\}$ The semi-open interval

$]-\infty, b] = \{x \in \mathbb{R} : x \leq b\} \wedge [a, +\infty[= \{x \in \mathbb{R} : x \geq a\}$ the closed interval

$]-\infty, b[= \{x \in \mathbb{R} : x < b\} \wedge]a, +\infty[= \{x \in \mathbb{R} : x > a\}$ the open interval.

$$\mathbb{R} =]-\infty, +\infty[$$

Definition 3: Let a be a point of $\mathbb{R}$ ($a \in \mathbb{R}$), we call the part V of $\mathbb{R}$ ($V \subset \mathbb{R}$) "a" neighborhood of a if it contains an open interval $\underline{I}$ its center is "a", and let it be of the form $]a-\alpha, a+\alpha[$ with $\alpha > 0$ such that

$$a \in I \subset V \text{ , that is (i.e.,)}$$

$$a \in V \Leftrightarrow \exists I \text{ open} : a \in I \subset V$$

Example: $V = [0,1]$ is a neighborhood of the point $\frac{1}{2}$ because there exists an open interval $I =]0,1[$ such that:

$$\frac{1}{2} \in]0,1[\subset [0,1] = V$$

whereas V isn't a neighborhood neither of point 1 nor of point 0, because there is no open interval I such that: $1 \in I \subset V$ (or $0 \in I \subset V$)

Remark: 1/ Empty sets and one-element sets (single sets) are intervals (Domains) but not neighbors of any point, because:

$$\emptyset =]a,a[\text{ and } \{a\} = [a,a].$$

2/ We often use the following property:

$$(*) \quad \boxed{x \in [a,b] \Leftrightarrow \exists t \in [0,1] : x = at + (1-t)b}$$

((*) is the relation that connects a point "x" to a line segment $[a,b]$)

(المستقيم الحقيقي الكامل) The completed real line

Definition: The obtained set by adding $\pm\infty$ to $\mathbb{R}$ is called the completed real line and we denote it by $\overline{\mathbb{R}}$, meaning that

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

Operations in $\overline{\mathbb{R}}$: Let x be a qualitative point of $\mathbb{R}$ then

1/ $x + (\pm\infty) = \pm\infty$

2/ $+\infty + (+\infty) = +\infty$

3/ $-\infty + (-\infty) = -\infty$

4/ $x(\pm\infty) = \begin{cases} \pm\infty & \text{if } x > 0 \\ \mp\infty & \text{if } x < 0 \end{cases}$

5/ $(-\infty)(-\infty) = +\infty$

6/ $(-\infty)(+\infty) = -\infty$

Remark ①: the two operations $+\infty + (-\infty)$ and $0(\pm\infty)$ are undefined in $\mathbb{R}$, that is, they are indeterminacy forms.

Remark ②:

a/ the difference between $\mathbb{R}$ and $\overline{\mathbb{R}}$ is that:

$\min \overline{\mathbb{R}} = -\infty$ and $\max \overline{\mathbb{R}} = +\infty$ whereas $\min \mathbb{R}$ and $\max \mathbb{R}$ are not exist in $\mathbb{R}$.

b/ The operations of addition and multipliation in $\overline{\mathbb{R}}$ aren't defined in all cases. e.g.: $0(\pm\infty)$ and $+\infty + (-\infty)$ aren't defined in $\overline{\mathbb{R}}$ but they are defined in $\mathbb{R}$ (because $\pm\infty \notin \mathbb{R}$).

Bonded Sets:

Introductions (preface) Any finite set $\{x_1, x_2, \ldots, x_n\}$ of real numbers obviously has a greatest element and a least element, but this property doesn't necessarily hold for infinite sets.

Example ① The interval $]0,2]$ has greatest element 2, but neither of the sets $\mathbb{N} = \{1, 2, 3 \ldots\}$ nor $[0,2[$ has a greatest element. However the set $[0,2[$ is bounded above by, since all points of $[0,2[$ are less than or equal to 2.

Definition ① 1/ A set $A \subseteq \mathbb{R}$ is "bounded above" if there is a real number $M \in \mathbb{R}$, called an "upper bound" of A, such that:

$\forall x \in A : \; x \leq M$

- If the upper bound M belongs to A, then M is called the "maximum element" of A, denoted by $\max A$.

2/ A set $A \subseteq \mathbb{R}$ is "bounded below" if there is a real number $m \in \mathbb{R}$, called a "lower bound" of A, such that:

$\forall x \in A : \; x \geq m$

- If the lower bound m belongs to A, then m is called the "minimum element" of A, denoted by $\min A$.

Example ② Determine which of the following sets are bounded above, and/or bounded below, and which have a maximum element and/or a minimum element.

$E_1 = [0, 2[$, $E_2 =]-\infty, 1]$, $E_3 = \mathbb{N}$, $E_4 = \{1 - \frac{1}{n}, n = 1, 2 \dots\}$, $E_5 = \{\frac{1}{n}, n = 1, 2 \dots\}$

$E_6 = \{n^2, n = 1, 2 \dots\}$

solution:

The set E_1 is bounded above and below, because $\forall x \in E_1 : \; 0 \leq x < 2$

However, E_1 has no maximum element, but E_1 has a minimum element $\min E_1 = 0$

•) for E_2 we have: $\forall x \in E_2 : \; x \leq 1$, hence

the set E_2 is bounded below but it's not bounded above.

hence E_2 cannot has a maximum element, but E_2 has a minimum element

$\min E_2 = 1$

•) the same way for all sets, where

$\min E_3 = 0$, $\min E_4 = 0$ / $\max E_5 = 1$, $\min E_6 = 1$

E_3 hasn't a max / E_4 has no max / E_5 cannot has a min / E_6 hasn't a minimum element

E_3 hasn't an upper bound / E_4 has an upper bound $= 1$ / E_5 has a lower bound $= 0$ / E_6 hasn't a lower bound

E_3 isn't bounded above / E_4 is bounded / E_5 is bounded / E_6 not bounded below

Definition ② A set $A \subseteq \mathbb{R}$ is bounded if it is bounded above and bounded below.

Example ⑧ The set $E_1 = \{1 - \frac{1}{n}, n = 1, 2, \dots\}$ is bounded, but the sets $E_2 =]-\infty, 1]$ and $E_3 = \{n^2, n = 1, 2, \dots\}$ aren't bounded.

Least upper bounds and greatest lower bounds:

Definition ③ A real number M is the least upper bound, or supremum, of a set $A \subseteq \mathbb{R}$ if:

1/ M is an upper bound of A.

2/ if $M' < M$, then M' is not an upper bound of A. ($M' < M$, $\exists y \in \mathbb{R}$: $M' < y < M$ by the density property of $\mathbb{R}$)

In this case, we write $M = \sup A$.

Definition ④. A real number m is the greatest lower bound, or infimum of a set $A \subseteq \mathbb{R}$ if:

1/ m is a lower bound of A.

2/ if $m' > m$, then m' is not a lower bound of A.

In this case, we write $m = \inf A$.

Remark: ① The least upper bounds and the greatest lower bounds aren't necessarily elements of the set A, that is, they can be $\sup A \in A \vee \sup A \notin A$ and also $\inf A \in A \vee \inf A \notin A$.

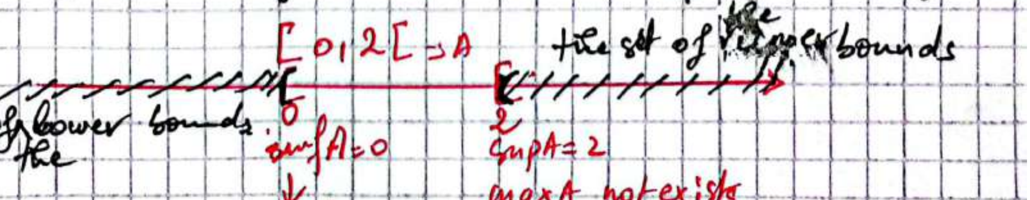
Bolzano axiom (مسلمة بولزانو):

every non-empty subset of $\mathbb{R}$ bounded above (respectively below) admitted a least upper bounds (respectively a greatest lower bounds)

Theorem ①: (Uniqueness of the least upper bounds and the greatest lower bounds)

Let A be a non-empty subset of $\mathbb{R}$.

the least upper bounds (resp. the greatest lower bounds) of set B, if it exists, it's unique.



proof: Let A be bounded above

we assume that there is two different upper bound M_1 and M_2 of A.

Hence:

M_1 is the least upper bounds $\Rightarrow$ M_1 is an upper bound of A.

M_2 " " " " $\Rightarrow$ M_2 " " " " A.

M_1 is the least upper bounds and M_2 is an upper bound of A

implies that: $M_1 \leq M_2$ — ①

on the other hand:

M_2 is the least upper bounds and M_1 is an upper bound of A $\Rightarrow$

$M_2 \leq M_1$ — ②

from ① and ②, we deduce that $M_1 = M_2$.

which proves the uniqueness of the least upper bounds)

in the same way, we prove the uniqueness of the greatest lower bou

Theorem②: (Analytical definition of least upper bounds and greatest lower bounds)

1/ Let A be a non-empty subset of $\mathbb{R}$, bounded above and let M its leas upper bound, then we have the following equivalent

$$M = \sup A \iff \begin{cases} \forall x \in A: \quad x \leq M & \text{— (a)} \\ \forall \varepsilon > 0, \exists x \in A: \ M - \varepsilon < x & \text{— (b)} \end{cases}$$

2/ Let A be a non-empty subset of $\mathbb{R}$, bounded below and let m its greatest lower bound, then we have the following equivalent

$$m = \inf A \iff \begin{cases} \forall x \in A: \quad x \geq m & \text{— (a')} \\ \forall \varepsilon > 0, \exists x \in A: \quad x < m + \varepsilon & \text{— (b')} \end{cases}$$

proof: for (1), we first prove the necessity of the conditions i.e

$\Rightarrow$) that is, we assume that $M = \sup A$ and we prove (a) and (b)

we have $M = \sup A \Rightarrow$ Since M is a least upper bounds, then M is and we have m is a greatest bounds, then $m \leq m+\varepsilon$ contradiction.

Hence (b'). Thus, the necessity of the condition.

Now, we prove the sufficiency of the condition ($\overset{?}{\Leftarrow}$), that is, we assume that m is a lower bound of A and that it satisfies the relations (a') and (b'). we prove that m is the greatest lower bound

From (a'), we have: $\forall x \in A: x \geq m \Rightarrow m$ is a lower bound of A.

we now prove that m is the greatest lower bounds by the opposite:

$\exists m' \in \mathbb{R}: m < m'$ (m isn't the greatest lower bounds)

m' is a greatest lower bounds $\Leftrightarrow \forall x \in A: x \geq m'$ —(c')

we put: $m' = m+\varepsilon$ ($m' > m$), hence $m' - m = \varepsilon > 0$

but from (b'), we have: $\forall x \in A: m+\varepsilon > x \Leftrightarrow \forall x \in A: m' > x$

contradiction with (c'). Hence (b'). Thus,

m is a greatest lower bounds of A, i.e, $m = \inf A$.

Example: Let A be a set defined as follows: $A = \{\frac{1}{n+1}, n \in \mathbb{N}^*\}$

1/ prove that: A is bounded and that $\inf A = 0$ and $\sup A = \frac{1}{2}$.

2) Find the maximum element and the minimum element if it exists.

solution: Let $a_n = \frac{1}{n+1}$, $\forall n \in \mathbb{N}^*$

1/ we have $n \in \mathbb{N}^* \Rightarrow n \geq 1 \Rightarrow n+1 \geq 2 \Rightarrow 0 < \frac{1}{n+1} \leq \frac{1}{2}$, hence:

$\forall n \in \mathbb{N}^*: 0 < a_n \leq \frac{1}{2} \Rightarrow A$ is a bounded set ($\forall x \in A: 0 < x \leq \frac{1}{2}$)

- Now, we have $\forall n \in \mathbb{N}^*: a_n \geq 0 \Rightarrow m=0$ is a lower bound of A. (by Definition 3)

and we prove that 0 is the greatest lower bounds, i.e., if $m' > 0$, then m' isn't a lower bound

we assume that there exists $m' > m = 0$ and by the density property of $\mathbb{R}$

$\exists y \in A: m' > y > 0$, hence $y \in A$ ($y > 0$) and so m' can not be a lower bound of $A \Rightarrow (y < m'$ not $y \geq m') \Rightarrow m = 0$ is the greatest lower bounds $\Rightarrow \inf A = 0$

- the same, we have $\forall n \in \mathbb{N}^*: a_n \leq \frac{1}{2} \Rightarrow M = \frac{1}{2}$ is an upper bound (by definition 3) and we prove that $\frac{1}{2}$ is the least upper bounds of A.

we assume that there exists $M' < M = \frac{1}{2}$ and by the density property

of $\mathbb{R}$, $\exists y \in \mathbb{R}$: $M' < y < \frac{1}{2} \Rightarrow M'$ isn't an upper bound of A.
Hence $\sup A = \frac{1}{2}$, i.e, $M = \frac{1}{2}$ is the least upper bounds of A.

2) Finding minimum and maximum of A if it exists:

for $n = 1$, we have: $a_n = \frac{1}{2} \in A \Rightarrow \sup A \in A \Rightarrow \max A = \frac{1}{2}$
since $\inf A = 0 \notin A$, then $\min A$ cannot exists.

2nd mtd to prove that: $\inf A = 0$: From theorem ② we have:

$$\inf A = 0 \Leftrightarrow \begin{cases} \forall x \in A: \ x \geq 0 \\ \forall \varepsilon > 0, \exists x \in A: \ x < 0 + \varepsilon \end{cases}$$

From (P_1) we have. $\forall x \in A$: $0 < x \leq \frac{1}{2} \Rightarrow x \geq 0 \Rightarrow 0$ is an upper bound.
Now, it satisfies to prove that there exist $x \in A$ such that: $x < \varepsilon, \forall \varepsilon > 0$.

$\overset{?}{\exists} x \in A: x < \varepsilon + 0 \Leftrightarrow \overset{?}{\exists} n \in \mathbb{N}^*: \ \frac{1}{1+n} < \varepsilon \ , \ \forall \varepsilon > 0$

$$\Leftrightarrow \forall \varepsilon > 0, \overset{?}{\exists} n \in \mathbb{N}^*: \ n + 1 > \frac{1}{\varepsilon}$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists n \in \mathbb{N}^*: \ n > \frac{1}{\varepsilon} - 1 = \frac{1-\varepsilon}{\varepsilon}$$

Hence it satisfies to take $n = \left[\frac{1-\varepsilon}{\varepsilon}\right] + 1 \in \mathbb{N}^* \Rightarrow$ "0" is the greatest lower bounds.

Remark ① Let A be a non-empty subset of $\mathbb{R}$. To show that M (resp. m) is the least upper bounds (resp. the greatest lower bounds) or supremum (resp. infimum) of A, is equivalent to check that:

1/ $x \leq M$, (resp. $x \geq m$) for all $x \in A$

2/ if $M' < M$ (resp $m' > m$), then there is some $x \in A$ such that $x > M'$ (resp $x < m'$).

Remark ② (supplementary Remarks)

- If the set A is not defined by expression, it's prefer to use the first method to prove the least upper bonds and the greatest upper bonds of it.
- If the set A has an expression like the previous example, it's prefer to use the second mtd to prove it least upper bonds or greatest lower bonds.

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