Primitives and integrals

1 Subdivisions and Darboux Sums

Definition 1 *Let $n \in \mathbb{N}^*$. An n-th order subdivision of an interval $[a, b]$ is a finite set $l = \{x_0, x_1, \cdots, x_n\} \subset [a, b]$, such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

We called the step of the subdivision l the real number $h = \max_{1 \leq i \leq n}(x_i - x_{i-1})$.
We say that h is the uniform step of the subdivision when $x_i = x_0 + ih,\ i = 1 \cdots n$ with $h = \dfrac{b-a}{n}\cdot$

Example 1 *In this example, we provide some subdivisions of the interval $[0, 1]$:*

$$l_1 = \{0, \frac{1}{2}, 1\},\ l_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\},\ l_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\},\ l_4 = \{0, \frac{2}{5}, \frac{1}{2}, \frac{5}{6}, 1\}.$$

$l_1,\ l_2,$ and l_3 are uniform with steps of $h_1 = \dfrac{1}{2},\ h_2 = \dfrac{1}{4}$ and $h_3 = \dfrac{1}{3}\cdot$ In contrast, l_4 is not uniform and has a step of $h_4 = \dfrac{2}{5}\cdot$

1.1 Darboux Sums

Let f be a bounded function on $[a, b]$. We consider a subdivision l of $[a, b]$ denoted as:

$$l = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

For $i = 1, \cdots n$, we define

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \qquad \text{and} \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

Definition 2 *The* ***lower Darboux Sum*** *associated with f is given by:*

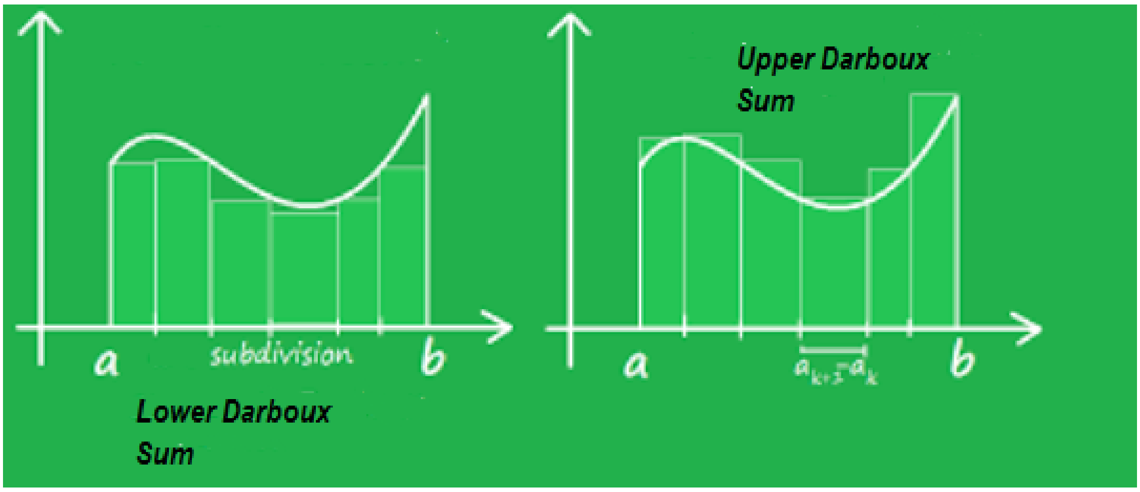
$$s_{[a,b]}(f, l) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

The ***upper Darboux Sum*** *associated with f is given by:*

$$S_{[a,b]}(f, l) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Remark 1 *The two quantities $S(f,l)$ and $s(f,l)$ are finite, because* $\sup_{x\in[x_{i-1},x_i]}$ *and* $\inf_{x\in[x_{i-1},x_i]}$ *are finite real numbers since f is assumed to be bounded. Furthermore, $s(f,l) \leq S(f,l)$ because* $\inf_{x\in[x_{i-1},x_i]} \leq \sup_{x\in[x_{i-1},x_i]}$.

Geometric Interpritation: The upper (respectively, lower) Darboux sum is the sum of the areas of the upper rectangles with base $[x_{i-1} - x_i]$ (respectively, the lower rectangles).



1.2 Riemann-integrable function

Definition 3 *An integral is a mathematical operation that, given a function, finds the area under the curve of that function over a specified interval.*

Definition 4 *Let f be a bounded function on $[a,b]$. We say that f is Riemann-integrable (or integrable in the Riemann sense) on $[a,b]$ if*

$$s(f,l) = S(f,l).$$

This value is denoted by $\int_a^b f(x)dx$, which represents the integral of f over the interval $[a,b]$.

Theorem 1

1. *Every continuous function on a segment $[a,b] \subset \mathbb{R}$ is Riemann-integrable on $[a,b]$.*
2. *If f is Riemann-integrable on $[a,b]$ (with $a < b$), then f is also Riemann-integrable on $[a,b]$ and we have:*

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

This inequality is a consequence of the absolute value inequality for integrals.

1.2.1 Properties of the Riemann Integral

Let f and g be two bounded and integrable functions on an interval $[a, b]$. The following properties hold:

1. **Positivity:** If $f \geq 0$ on $[a, b]$, then
$$\int_a^b f(x)dx \geq 0.$$
2. **Monotonicity:** If $f \leq g$ on $[a, b]$, then
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$
3. **Linearity:**
$$\int_a^b \Big(\alpha f(x) + \beta g(x)\Big)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx, \qquad \forall(\alpha, \beta) \in \mathbb{R}^2.$$

2 Primitive functions

Let's assume that $I \subset \mathbb{R}$ is an interval.

Definition 5 *Let $f : I \to \mathbb{R}$. A primitive of f is a derivative function $F : I \to \mathbb{R}$, such that $F'(x) = f(x), \quad \forall\ x \in I$.*

Theorem 2 *Every continuous function on I admits primitives on I.*

Remark 2 *If a function f admits a primitive, that primitive is not unique.*

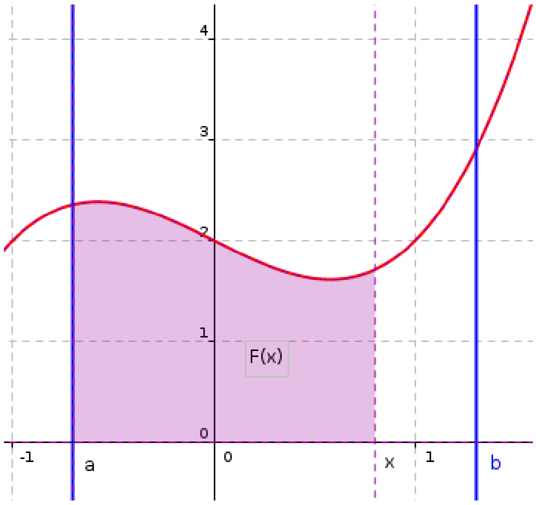
Proposition 1 *If F is a primitive of f on I then, the function $G : I \to \mathbb{R}$ is also a primitive of f if and only if $F - G = c$, where $c \in \mathbb{R}$ is a constant.*

Example 2 *The function $f : I \to \mathbb{R}$ defined by $f(x) = x^3$ has as a primitive the function $F(x) = \frac{1}{4}x^4$, also $G(x) = \frac{1}{4}x^4 + 5$ is a primitive of f because $F(x) - G(x) = c = -5 \in \mathbb{R}$.*

2.1 Primitive function at a point

Theorem 3 *Let f be a continuous function on I, $x_0 \in I$ and $y_0 \in \mathbb{R}$. Then, there exists a primitive F of f, only one, such that $F(x_0) = y_0$.*

Geometric interpretation: Among all the curve presenting the premitives of f on I, there exists one and only one passing through the point (x_0, y_0).



Example 3 *Let f be a function defined on $\mathbb{R}$ by:*

$$f(x) = x^2 - 3x + 2.$$

Determine the primitive F of f on $\mathbb{R}$ which vanishes at the point $x_0 = 1$.
Indeed: *The set of primitive functions of f has the form:*

$$F(x) = \frac{x^3}{3} - \frac{3}{2}x^2 + 2x + c, \qquad c \in \mathbb{R}.$$

The condition $F(1) = 0$, gives

$$F(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = 0 \Longleftrightarrow c = -\frac{5}{6}\cdot$$

Then, the only primitive function of f at the point $x_0 = 1$ is given by:

$$F(x) = \frac{x^3}{3} - \frac{3}{2}x^2 + 2x - \frac{5}{6}\cdot$$

2.2 Primitive functions of elementary functions

In the following table, u is a derivable function on I.

Function f	Primitive F
$k \in \mathbb{R}$	$kx + C$
x^n, $n \in \mathbb{N}$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + C$
$\frac{1}{x^n}$, $n \in \mathbb{N}* - \{1\}$	$\frac{-1}{(n-1)x^{n-1}} + C$
$u'(x)u^n(x)$, $n \in \mathbb{N}*$	$\frac{u^{n+1}(x)}{n+1} + C$
$\frac{u'(x)}{\sqrt{u(x)}}$	$2\sqrt{u(x)} + C$
$\frac{u'(x)}{u^n(x)}$, $n \in \mathbb{N}* - 1\}$	$\frac{-1}{(n-1)u^{n-1}(x)} + C$
$\frac{u'(x)}{u(x)}$	$ln\|u(x)\| + C$
e^x	$e^x + C$
$u'(x)e^{u(x)}$	$e^{u(x)} + C$
$\cos x$	$\sin x + C$
$u'(x)\cos(u(x))$	$\sin(u(x)) + C$
$\sin x$	$-\cos x + C$
$u'(x)\sin(u(x))$	$-\cos(u(x)) + C$

2.3 Operations on primitive functions

Let f and g be two derivable functions on I, with $c \in \mathbb{R}$ is a constant.

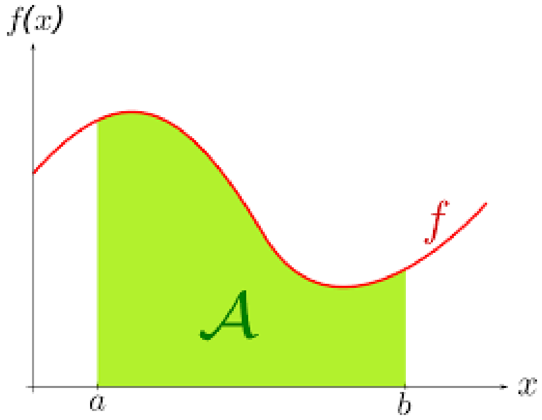
Function	primitive
$f' + g'$	$f + g + c$
kf'	$kf + c$
$f'f^n$ $(n \neq -1, n \in \mathbb{Z})$	$\frac{1}{n+1}f^{n+1} + c$
$\frac{f'}{\sqrt{f}}$	$2\sqrt{f}$
$f'(g' \circ f)$	$f \circ g + c$
$\frac{f'}{\sqrt{1-f^2}}$	$\arcsin f + c$
$\frac{f'}{1+f^2}$	$\arctan f + c$

Example 4 *Determine the primitive of the function:* $f(x) = x(1+x^2)^3$ *on* $\mathbb{R}$.
Indeed: *The function* f *is continuous on* $\mathbb{R}$ *and has the form:* $f(x) = \frac{1}{2}u'(x)u^3(x)$, *where* $u(x) = 1 + x^2$ *and* $u'(x) = 2x$. *Then:* $F(x) = \frac{1}{8}(1+x^2)^4 + c$, *where* $c \in \mathbb{R}$.

3 Integrals

Definition 6 *Let* f *be a function defined on* I *and* F *its primitive and* a, $b \in I$. *The quantity* $F(b) - F(a)$ *(also noted by* $[F(x)]_a^b$*) is the integral of* f *between* a *and* b *and we note:*

$$F(x) = \int_a^b f(x)dx.$$



Attention:

1. The order of a and b in the integral is very important.
2. The number a is the lowest upper bound and b is the highest upper bound of the integral:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

3. The result of calculating the integral dosen't depend on the chosen primitive.
Indeed: If F_1 and F_2 are two primitives of f, then $F_1 = F_2 + c$, where $c \in \mathbb{R}$ is a constant. Then,

$$[F_1(x)]_a^b = [F_2(x) + c]_a^b = F_2(b) + c - F_2(a) - c = F_2(b) - F_2(a) = [F_2(x)]_a^b.$$

Example 5

1. *Calculate the integral:* $\int_{-1}^{2}(2x^2+3)dx$.

   Let $f(x)=(2x^2+3)$ *which is a continuous function on* $[-1,2]$. *Then,* $F(x)=\frac{2}{3}x^3+3x+c$ *is the primitive of* f *on* $[-1,2]$. *Therefore,*

$$\int_{-1}^{2}(2x^2+3)dx = F(2)-F(-1) = \left[\frac{2}{3}x^3+3x\right]_{-1}^{2} = 15.$$

2. $\int(\cos 4x - 3\sin 2x + \cos x)dx = \frac{1}{4}\sin 4x + \frac{3}{2}\cos 2x + \sin x + c, \qquad c \in \mathbb{R}.$

Theorem 4 *Consider the continuous function* $F:[a,b]\to\mathbb{R}$ *which admits a continuous derivative function on* $[a,b]$. *Then,*

$$\int_a^b F'(x)dx = F(b)-F(a).$$

Theorem 5 *Consider the continuous function* $f:[a,b]\to\mathbb{R}$. *Then, for all* $x\in]a,b[$, *we have*

$$\frac{d}{dx}\int_a^x f(x)dt = f(x).$$

Proposition 2 *Let* f *and* g *be two integrable functions on* $[a,b]$. *We have,*

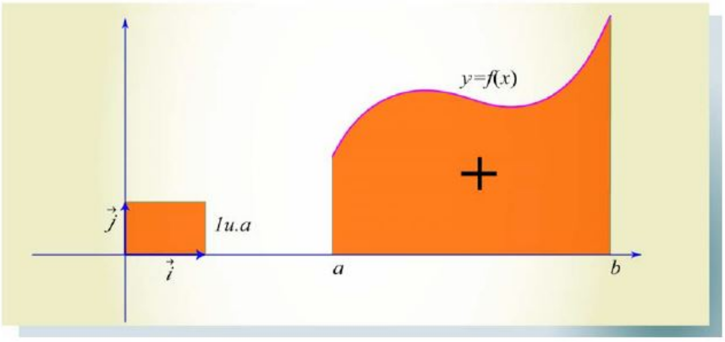
1. $\int_a^b \Big(f(x)+g(x)\Big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$
2. $\int_a^b \lambda f(x)dx = \lambda\int_a^b f(x)dx, \quad$ *where* $\lambda\in\mathbb{R}$.
3. $\int_a^a f(x)dx = 0.$
4. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad \forall c\in[a,b].$

3.1 Integrals and area

Let (C) be the representative curve of the function f in an orthogonal reference form. D is the region bounded by (C), the abscissa axis and the equation lines $x=a$ and $x=b$.
The unit of area is the area of the rectangle framed by the chosen coordinate system.

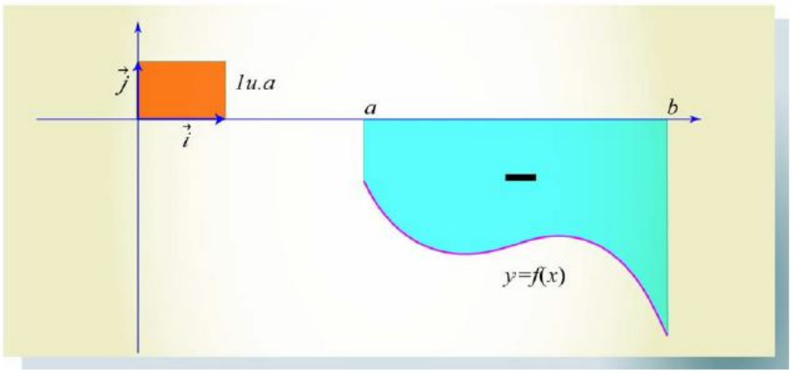
Theorem 6

1. ***Positive function:*** *If f is a positive continuous function on $[a, b]$, then $D = \int_a^b f(x)dx$.*



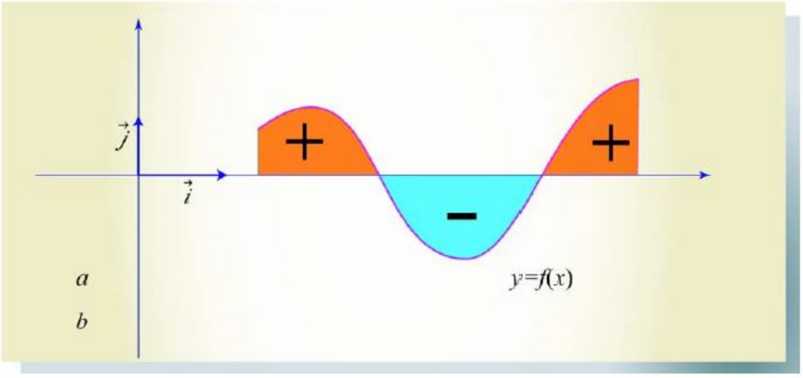
2. ***Negative function:*** *If f is a negative continuous function on $[a, b]$, then*

$$D = -\int_a^b f(x)dx = \int_b^a f(x)dx.$$



3. ***Function for any sign:*** *If f is a continuous function with any sign on $[a, b]$, then*

$$D = \int_a^b f(x)dx = \int_a^c f(x)dx - \int_c^d f(x)dx + \int_d^b f(x)dx.$$



3.2 The integral calculation

In this subsection, we will present 3 methods to calculate the integrals.

(a) Direct Integral: This concerns integrals of functions of the form $u'u^n$, $\dfrac{u'}{u^n}$, $\cdots$ where u is a function whose primitive is known.

Example 6

$$\int_2^3 \frac{xdx}{\sqrt{x^2+1}} = \left[\sqrt{x^2+1}\right]_2^3 = \sqrt{10} - \sqrt{5}.$$

(b) Integration by parts: it is a technique in calculus that allows you to find the integral of the product of two functions by applying a specific formula derived from the product rule for differentiation.
The formula for integration by parts is often written as:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

Or,

$$\int_a^b u(x)v'(x)dx = \left[u(x)v(x)\right]_a^b - \int_a^b u'(x)v(x)dx.$$

Example 7

1. $I = \int xe^x dx$. *Let*

$$u = x \Rightarrow u' = 1\ dx$$
$$v' = e^x dx \Rightarrow v = e^x.$$

Then,

$$I = xe^x - \int e^x dx = xe^x - e^x + c = (x-1)e^x + c.$$

2. $I = \int_1^2 \ln x \; dx$. *Let*

$$u = \ln x \Rightarrow u' = \frac{1}{x}dx$$
$$v' = dx \Rightarrow v = x.$$

Then,

$$I = \Big[x \ln x\Big]_1^2 - \int_1^2 \frac{x}{x}dx = -2\ln(2) - [x]_1^2 = -2\ln(2) + 1.$$

(c) The variable change method: it is a technique used in calculation to simplify and solve integrals by moking a change of variable. This method is particularly useful when dealing with complex integrals or when the integral involves a composite function. The general idea is to replace the variable of integration with a new variable that simplifies the integral.

Theorem 7 *Let $g : [c, d] \to \mathbb{R}$ a continuous differentiable function and strictly monotonic. Suppose that $g([c, d]) = [a, b]$. For all $f : [a, b] \to \mathbb{R}$, we have*

$$\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt.$$

Example 8

1. $I = \int_1^2 \frac{dx}{e^x + e^{-x}}$. *Put:*

$$u(x) = e^x \Rightarrow du = e^x dx.$$

On the other hand,

$$x = 1 \Rightarrow u = e \qquad \text{and} \qquad x = 2 \Rightarrow u = e^2.$$

Thus,

$$I = \int_e^{e^2} \frac{du}{u^2 + 1} = [\arctan]_e^{e^2} = \arctan(e) - \arctan(e^2).$$

2. $I = \int \sqrt{1 - x^2}dx$. *Put:*

$$x = \sin t \Rightarrow dx = \cos t dt.$$

Then,

$$\begin{aligned} I &= \int \sqrt{1-\sin^2 t}\cos t dt = \int \cos^2 t dt = \int (\cos(2t)+1)dt \\ &= \frac{1}{2}\int 2\cos(2t)dt + \int dt \\ &= \frac{1}{2}\sin(2t) + t + c \\ &= \frac{1}{2}\sin t\cos t + t + c \\ &= \frac{1}{2}\sin t\sqrt{1-\sin^2 t} + t + c \\ &= \frac{1}{2}x\sqrt{1-x^2} + \arcsin x + c. \end{aligned}$$

Dr. Kicha Abir