# Chapter 03: Numerical sequences

## 1/ Definitions:

**Definition 1:** We call a numerical sequence each function defined from $\mathbb{N}$ to $\mathbb{R}$ as follows: $U : \mathbb{N} \longrightarrow \mathbb{R}$

$n \longmapsto U(n) := U_n$.

Then, we denote by $U_n$ the general term of the sequence and by $(U_n)_{n\in\mathbb{N}}$ (or $(U_n)_n$) for the numerical sequence itself.

**Remark 1:** A numerical sequence can be defined explicitly or implicitly (or regressive or progressive), i.e., $\begin{cases} U_n = f(n) \longrightarrow \text{explicit form} \\ \vee \\ U_n = f(U_{n-1}) \longrightarrow \text{implicit form} \end{cases}$

**Example**

1. The numerical sequence $(U_n)_n = (n)_{n\in\mathbb{N}^*}$ such that $U_n = n$ is its general term is defined explicitly, that is: $U_n = f(n)$, $U_1 = 1, U_2 = 2, \ldots, U_n = n, \ldots$

2. the numerical sequence defined by its general term $\begin{cases} U_n = \dfrac{U_{n-1}}{n^2} + 1, & n\in\mathbb{N}^* \\ U_0 = 1, \end{cases}$

is defined implicitly, that is: $U_n = f(U_{n-1})$ and we have:

$U_1 = 2, \ U_2 = \dfrac{3}{2}, \ldots$

3. Arithmetic and Geometric sequences with a base $r$ and the first term $U_0$ for each them are sequences defined

a/ explicitly when we write it by: $U_n = U_0 + nr$ (resp.) $U_n = U_0 r^n$.

b/ implicitly when we write it by: $U_n = U_{n-1} + r$ (resp.) $U_n = U_{n-1} \cdot r$.

4. The sequence defined by $(U_n) = (-1)^n, n\in\mathbb{N}$ is called an alternating sequence and we have:

$U_0 = 1, \ U_1 = -1, \ U_2 = 1, \ U_3 = -1, \ldots$

**Remark 2:** ⓐ Operations with sequences: Let $(U_n)_n$ and $(V_n)_n$ be two numerical sequences, then

- we define their sum as: $(U_n)_n + (V_n)_n = (U_n + V_n)_n$ (addition)
- " " " difference as: $(U_n)_n - (V_n)_n = (U_n - V_n)_n$ (subtraction)
- " " " scalar multiple as: $\lambda (U_n)_n = (\lambda U_n)_n$ (multiplication by a scalar)
- " " " product as: $(U_n)_n \cdot (V_n)_n = (U_n \cdot V_n)_n$ (multiplication)

(b) equality of two sequences:

two sequences $(U_n)_n$ and $(V_n)_n$ are said to be equal or identical if and only if $n^{th}$ terms of both sequences are equal, that is,
$U_n = V_n$, $\forall n \in \mathbb{N}$.

(c) The order of terms is of importance in a sequence: Thus the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \quad, \dots\}$ is different from the sequence $\{\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \dots\}$ even though both contain the same terms.

Definition (2): A sequence $(U_n)_n$ is

a/ bounded above, if there exists a real number $M$ such that $U_n \leq M$ for all positive integers $n$, that is $\exists M \in \mathbb{R} : U_n \leq M, \forall n \in \mathbb{N}$

b) bounded below, if $\exists m \in \mathbb{R} : U_n \geqslant m, \forall n \in \mathbb{N}$

c/ bounded sequence if it is bounded above and bounded below, i.e.,
$\exists A > 0 : |U_n| \leq A, \forall n \in \mathbb{N}$

d/ if a sequence is not bounded, it is unbounded sequence.

Example: the sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}^*}$ is bounded above because $\frac{1}{n} \leq 1, \forall n \in \mathbb{N}^*$
it is also bounded below because $\frac{1}{n} > 0, \forall n \in \mathbb{N}^*$. Therefore, $\left(\frac{1}{n}\right)_n$ is a bounded sequence.

On the other hand, consider the sequence $(2^n)_{n \in \mathbb{N}^*}$. This sequence $(2^n)_n$ is bounded below because $2^n \geqslant 2, \forall n \geqslant 1$. However, the sequence isn't bounded above. Therefore, $(2^n)_n$ is an unbounded sequence.

Definition (3):

a/ A sequence $(U_n)_n$ is called increasing if: $U_{n+1} \geqslant U_n$ for all $n \in \mathbb{N}$ ($\forall n \in \mathbb{N}$)

b/ " " " " " decreasing if: $U_{n+1} \leq U_n$, $\forall n \in \mathbb{N}$.

c/ if $(U_n)_n$ is increasing or decreasing, then it is called a monotone sequence.

• The sequence is called strictly increasing (resp. strictly decreasing) if $U_{n+1} > U_n, \forall n \in \mathbb{N}$ (resp. $U_n > U_{n+1}, \forall n \in \mathbb{N}$)

**Remark (3):** It is easy to show by induction that if $(U_n)_n$ is an increasing sequence, then $U_n \le U_m$ whenever $n \le m$.

- To study the monotonicity of the sequence $(U_n)_n$:

1/ we study the sign of the difference $U_{n+1} - U_n$

2/ or we compare the ratio $\frac{U_{n+1}}{U_n}$ with the number 1 if all terms of the sequence $(U_n)_n$ are positive.

**Example:**

1/ the sequence $(U_n)_n$ that its general term equal to $U_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}, \forall n \in \mathbb{N}$ is increasing, because:

$U_{n+1} - U_n = \frac{1}{(n+1)!} > 0, \forall n \in \mathbb{N} \Rightarrow U_{n+1} > U_n$

2/ the sequence $(U_n)_n$ that its general term is defined as: $U_n = \frac{n!}{n^n}, \forall n \in \mathbb{N}^*$ is decreasing because:

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{(n+1)^{n+1}} \cdot n^n = \left(\frac{n}{n+1}\right)^n < 1, \forall n \in \mathbb{N} \quad (n < n+1)$$

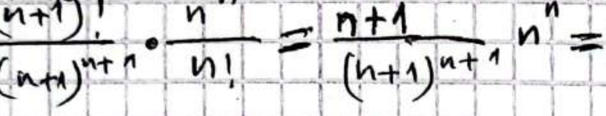
$\Rightarrow U_{n+1} < U_n, \forall n \in \mathbb{N}$.

## 2/ Limit of a sequence (the convergent sequences)

**Definition (1)** Let $(U_n)_n$ be a numerical sequence. The sequence $(U_n)_n$ **converges** to a real number $l$ if, for every positive number $\varepsilon$, there exists an $N \in \mathbb{N}$ such that for all $n \ge N$: $|U_n - l| < \varepsilon$, that is,

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \ge N : |U_n - l| < \varepsilon$.

$l$ is called the **limit** of the sequence $(U_n)_n$ and we write $\lim\limits_{n \to +\infty} U_n = l$.




$U_n \in ]l-\varepsilon, l+\varepsilon[$

**Remark (4):** If $\lim\limits_{n \to +\infty} U_n = \pm\infty$ or if $\lim\limits_{n \to +\infty} U_n$ takes two different values then the sequence $(U_n)_n$ does not converge, it is said to **diverge**.

**Example:** the sequences $(U_n)_n = (-1)^n$ and $(V_n)_n = (n)_n$ are not convergent because: $\lim\limits_{n \to +\infty} U_n = \begin{cases} 1 & \text{if } n = 2p \\ -1 & \text{if } n = 2p+1 \end{cases}$ and $\lim\limits_{n} V_n = +\infty \Rightarrow (U_n)$ and $(V_n)$ don't converge.

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Example: Show by definition that the general term sequence $U_n = \frac{n}{3n+1}, \forall n \in \mathbb{N}$ converges to $\frac{1}{3}$.

Solution:

$$\lim_n U_n \overset{?}{=} \frac{1}{3} \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq N : |U_n - l| < \varepsilon$$

$$\left|U_n - \frac{1}{3}\right| = \left|\frac{n}{3n+1} - \frac{1}{3}\right| = \left|\frac{-1}{9n+3}\right| = \frac{1}{9n+3} < \varepsilon \Rightarrow 9n+3 > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{9\varepsilon} - \frac{1}{3}$$

$$\Rightarrow n > \frac{1-3\varepsilon}{9\varepsilon}$$

So, it is enough to take $N = \left[\frac{1-3\varepsilon}{9\varepsilon}\right] + 1$ to obtain that $|U_n - \frac{1}{3}| < \varepsilon$.

Remark(5): Studying the nature of a sequence depends on the study of its convergence or its divergence

Properties of Sequences:

Remark(6): (properties sequences)

1) A sequence $(U_n)_n$ is identical to a sequence $(V_n)_n$ if they both attain the same values in exactly the same order.

2) If $(U_n)_n$ and $(V_n)_n$ differ from each other in only a finite number of terms, then both sequences converge to the same value or they both diverge

3) Suppose that $(U_n)_n$ is defined for all $n \in \mathbb{N}$. Then $(U_n)_n$ converges to $l$ if and only if for any $k \in \mathbb{N}$ the sequence $\{U_n\}_{n=k}^{\infty} = \{U_n, U_{n+1}, \dots\}$ converges to $l$. That is, $\lim_n U_n = l$ iff $\lim_{n\to\infty} U_{n+k} = l$, for $k \in \mathbb{N}$.

Equivalently, if $V_n = U_{n+k}$, then $(V_n)_n$ must also converge to $l$.

(Thus, no matter where we start a converging sequence, the new sequence will converge to the same value as the original sequence did. This is like taking an infinite sequence and removing a finite number of terms from the beginning).

4) If the sequence $(U_n)_n$ converges and $(U_n) = l$ for infinitely many values of $n$, where $l$ is a constant, then $(U_n)_n$ converges to $l$,

In particular, $\lim_{n\to+\infty} l = l$.

5) If $(U_n) = \ell$ for infinitely many values of $n$, where $\ell$ is a constant, then $(U_n)_n$ does not necessarily have to converge to $\ell$, for example when $(U_n)_n$ is defined by its general term $U_n = (-1)^n$, and we take $\ell = 1$ or $\ell = -1$ (How is this statement different from the previous one?)

6) $\lim\limits_n U_n = \ell$ iff for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have: $|U_n - \ell| < k\varepsilon$ for any fixed $k > 0$.

7/ $\lim\limits_n U_n = \ell$ implies that for any given constant $K > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|U_n - \ell| < K$.
(That is, $\varepsilon$ in the definition, can be replaced by any constant positive. The definition just requires $|U_n - \ell|$ to be able to be made arbitrarily small for $n$ sufficiently large)

8/ If $\ell \neq \ell'$ are real constants and $U_n = \ell$ for infinitely many values of $n$, and also $U_n = \ell'$ for infinitely many values of $n$, then $(U_n)_n$ diverges.

<u>Theorem①(uniqueness of the limit)</u>

Any two limits of a convergent sequence are the same.

<u>proof</u>: Assume that $U_n \to \ell$ and $U_n \to \ell'$ (both as $n \to \infty$). Then we will show that $\ell = \ell'$ must follow.

$\lim\limits_n U_n = \ell \Leftrightarrow \forall \varepsilon > 0, \exists n_1 \in \mathbb{N} \mid \forall n \in \mathbb{N}, n \geq n_1 : |U_n - \ell| < \varepsilon/2$ — ①

$\lim\limits_{n \to \infty} U_n = \ell' \Leftrightarrow \forall \varepsilon > 0, \exists n_2 \in \mathbb{N} ; \forall n \in \mathbb{N}, n \geq n_2 : |U_n - \ell'| < \varepsilon/2$ — ②

Now, let $N > \max\{n_1, n_2\}$ so that $N > n_1$ and $N > n_2$. Then for all $n \geq N$:

$$|\ell - \ell'| = |\ell - U_n + U_n - \ell'| \overset{TI}{\leq} |\ell - U_n| + |U_n - \ell'| \overset{prop}{\leq} |U_n - \ell| + |U_n - \ell'| \overset{①②}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus for any $\varepsilon > 0$, we have shown that $|\ell - \ell'| < \varepsilon$. This implies that $|\ell - \ell'| = 0$, i.e., that $\ell = \ell'$.

<u>Theorem②</u>: Any <u>convergent</u> sequence is <u>bounded</u>

<u>proof</u>: suppose $(U_n)$ converges to $\ell$. This means that for any $\varepsilon > 0$, there is an $N$ such that for any $n \geq N$; $|U_n - \ell| < \varepsilon$. Then we get

$$|u_n| = |u_n - \ell + \ell| \underbrace{\leq}_{\text{triangle inequality}} |u_n - \ell| + |\ell| < \varepsilon + |\ell| \quad (\text{since } |u_n - \ell| < \varepsilon)$$

So the "tail" ذيل of the sequence is difinitely bounded, that is:

$\forall n \geq N$: $|u_n| < \varepsilon + |\ell|$ $\left(-\varepsilon - |\ell| < u_n < \varepsilon + |\ell|\right) \Leftrightarrow \left(-\varepsilon - |\ell| < u_N, u_{N+1}, \dots < |\ell| + \varepsilon\right)$

Now, $\{u_0, u_1, u_2, \dots u_{N-1}\}$ is a finite set of real numbers.

Let $K = \max_{n<N} \{|u_n|\}$. Then $|u_n| \leq K$ for all $n < N$. Note that we are taking the maximum of the absolute value of the sequence terms. this important.

Now, let $M = \max\{|\ell| + \varepsilon, K\}$, and this gives us that $|u_n| \leq M$ for all $n \in \mathbb{N}$. Thus the sequence $(u_n)_n$ is bounded.

**Remark ④:** The converse of the previous theorem is false, that is,

$(u_n)_n$ is bounded $\not\Rightarrow$ $(u_n)$ is convergent. for example $u_n = (-1)^n$.

**Theorem ③ (Monotone convergence theorem)**

Let $(u_n)_n$ be a sequence of real numbers. The following hold:

a/ If $(u_n)_n$ is increasing and bounded above, then it is convergent

b/ If $(u_n)_n$ is decreasing and bounded below, then it is convergent.

**proof:**

a/ let $(u_n)$ be an increasing sequence that is bounded above. Define $A = \{u_n : n \in \mathbb{N}\}$. Then $A$ is a subset of $\mathbb{R}$ that is nonempty and bounded above and, hence, $\sup A$ exists.

Let $\ell = \sup A$ and let $\varepsilon > 0$. By Theorem ② [of bounded set]:

$$\ell = \sup A \Leftrightarrow \begin{cases} u_n \leq \ell, \ \forall n \in \mathbb{N} \\ \forall \varepsilon > 0, \exists N \in \mathbb{N} : \ell - \varepsilon < u_N \end{cases} \Rightarrow \begin{cases} \forall n \in \mathbb{N} : u_n \leq \ell \\ \forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N : \underbrace{u_n \geq u_N}_{(u_n)\nearrow} \geq \ell \end{cases}$$

there exists $N \in \mathbb{N}$ such that: $\ell - \varepsilon < u_N \leq \ell$.

Since $(u_n)_n$ is increasing, $\ell - \varepsilon < u_N \leq u_n$ for all $n \geq N$.

On the other hand, since $\ell$ is an upper bound for $A$, we have $u_n \leq \ell$ for all $n$. Thus, $\ell - \varepsilon < u_n < \ell + \varepsilon$, $\forall n \geq N$. Therefore, $\lim_{n\to\infty} u_n = \ell$

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b/ Let $(U_n)_n$ be a decreasing sequence that is bounded below.
Define $V_n = -U_n$, then $(V_n)_n$ is increasing and bounded above (if $M$ is a lower bound for $(U_n)_n$, then $-M$ is an upper bound for $(V_n)_n$). Let $\ell = \lim_n V_n = \lim_n (-U_n) = -\lim_n (U_n)$
Then $(U_n)_n$ converges to $(-\ell)$. livre b1_Full

Remark ⑧: It follows from the proof of theorem ③ that if $(U_n)_n$ is increasing and bounded above, then $\lim_n U_n = \sup\{U_n : n \in \mathbb{N}\}$; similarly, if $(U_n)$ is decreasing and bounded below, then
$\lim_n U_n = \inf\{U_n ; n \in \mathbb{N}\}$.

Example: Consider the sequence $(U_n)_n$ defined as follows:

$$\begin{cases} U_1 = 2 \\ U_{n+1} = \dfrac{U_n+5}{3} \quad \text{for } n \geq 1 . \end{cases} \quad \text{—} (*) \begin{cases} (U_n) \nearrow \\ \forall n \in \mathbb{N} : U_n \leq 3 \end{cases}$$

Solution: First we will show that the sequence is increasing.
we prove by induction that $\forall n \in \mathbb{N}$: $U_n < U_{n+1}$.
Since $U_2 = \frac{U_1+5}{3} = \frac{7}{3} > 2$, the statement is true for $n=1$. Next, suppose that $U_k < U_{k+1}$ for some $k \in \mathbb{N}$. Then $U_k+5 < U_{k+1}+5$ and $(U_k+5)/3 < \frac{U_{k+1}+5}{3}$. Therefore, $U_{k+1} < U_{k+2}$.
It follows by induction that the sequence is increasing.
Next, we prove that the sequence is bounded by 3. Again, we proceed by induction. The statement is clearly true for $n=1$. Suppose that $U_k \leq 3$ for some $k \in \mathbb{N}$. then $U_{k+1} = \frac{U_k+5}{3} \leq \frac{3+5}{3} = \frac{8}{3} < 3$.
It follows that $U_n \leq 3 \ \forall n \in \mathbb{N}$.
From the monotone convergence theorem (theorem ③), we deduce that there is $\ell \in \mathbb{R}$ such that $\lim_n U_n = \ell$.
Since the subsequence $\{U_{k+1}\}_{k=1}^{\infty}$ also converges to $\ell$, taking limits on both sides of the equation in $(*)$, we obtain $\ell = \frac{\ell+5}{3}$, therefore, $3\ell = \ell+5$ and, hence, $\ell = 5/2$.

**Theorem (4):** Let $(U_n)$ and $(V_n)$ be sequences of real numbers and let $\alpha$ be a real number. Suppose $(U_n)$ converges to $l$ and $(V_n)$ converges to $l'$. Then the sequences $(U_n+V_n)_n$, $(\lambda U_n)_n$ and $(U_n V_n)_n$ converge and

a/ $\lim\limits_{n}(U_n+V_n) = l+l'$,

b/ $\lim\limits_{n}(kU_n) = kl$, $k\in\mathbb{R}$

c/ $\lim\limits_{n}(U_nV_n) = ll'$

d/ If in addition $l'\neq 0$ and $V_n\neq 0$ for $n\in\mathbb{N}$: $\lim\limits_{n}\left(\dfrac{U_n}{V_n}\right)=\dfrac{l}{l'}$.

**proof:** [p46]

a/ Fix any $\varepsilon>0$, since $(U_n)$ converges to $l$, there exists $N_1\in\mathbb{N}$, such that
$|U_n-l|<\varepsilon/2$, $\forall n\geq N_1$.
Similarly, there exists $N_2\in\mathbb{N}$, such that: $|V_n-l'|<\varepsilon/2$ for all $n\geq N_2$
Let $N=\max(N_1,N_2)$ for any $n\geq N$, one has:
$|U_n+V_n-(l+l')|\leq|U_n-l|+|V_n-l'|<\varepsilon/2+\varepsilon/2=\varepsilon$
Therefore, $\lim\limits_{n\to+\infty}(U_n+V_n)=l+l'$. This proves (a).

b/ If $k=0$, then $kU_n=0$ for all $n$. The conclusion follows immediately.
Suppose next that $k\neq 0$. Given $\varepsilon>0$, let $N\in\mathbb{N}$ be such that $|U_n-l|<\frac{\varepsilon}{|k|}$ for $n\geq N$. Then for $n\geq N$, $|kU_n-kl|=|k||U_n-l|<\varepsilon$.
It follows that $\lim\limits_{n\to+\infty}(kU_n)=kl$ as desired. This proves (b).

c/ Since $(U_n)$ is convergent, it follows from Theorem (2) that is bounded.
Thus, there exists $M>0$ such that: $|U_n|\leq M$ for all $n\in\mathbb{N}$.
For every $n\in\mathbb{N}$, we have the following estimate:
$|U_nV_n-ll'|=|U_nV_n-U_nl'+U_nl'-ll'|\leq|U_n||V_n-l'|+|l'||U_n-l|$ — (1)
Let $\varepsilon>0$, since $(U_n)$ converges to $l$, we may choose $N_1\in\mathbb{N}$ such that
$|U_n-l|<\dfrac{\varepsilon}{2(|l'|+1)}$ for all $n\geq N_1$
Similarly, since $(V_n)$ converges to $l'$, we may choose $N_2\in\mathbb{N}$ such that:
$|V_n-l'|<\dfrac{\varepsilon}{2M}$ for all $n\geq N_2$.

Let $N = \max(N_1, N_2)$. Then, for $n \geqslant N$, it follows from ① that

$$|U_n V_n - \ell\ell'| < M \frac{\varepsilon}{2M} + |\ell| \frac{\varepsilon}{2(|\ell'|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \geqslant N \quad \left(\text{since } \frac{|\ell'|}{|\ell'|+1} < 1\right)$$

therefore, $\lim\limits_{n \to +\infty} U_n V_n = \ell \ell'$. This proves (c).

d/ Let us first show that: $\lim\limits_{n \to +\infty} \frac{1}{V_n} = \frac{1}{\ell'}$.

Since $(V_n)$ converges to $\ell'$, there exists $N_1 \in \mathbb{N}$ such that:

(*) $|V_n - \ell'| < \frac{|\ell'|}{2}$ for $n \geqslant N_1$.

It follows that: $\big||V_n| - |\ell'|\big| \overset{(9)\,\mathrm{TI}\,1}{\leqslant} |V_n - \ell'| < \frac{|\ell'|}{2}$ for such $n$, hence, (by ⑨ of TI)

$$-\frac{|\ell'|}{2} < |V_n| - |\ell'| < \frac{|\ell'|}{2} \iff \frac{|\ell'|}{2} < |V_n| < \frac{3}{2}|\ell'| \text{ and hence,}$$

$$|V_n| > \frac{|\ell'|}{2} \quad \text{—— (**)}$$

For each $n \geqslant N_1$, we have the following estimate:

$$\left|\frac{1}{V_n} - \frac{1}{\ell'}\right| = \left|\frac{V_n - \ell'}{V_n \ell'}\right| = \frac{|V_n - \ell'|}{|V_n||\ell'|} \overset{(**)}{\leqslant} \frac{2|V_n - \ell'|}{(\ell')^2} \quad \text{—— (3*)}$$

Now, let $\varepsilon > 0$, since $\lim\limits_{n \to +\infty} V_n = \ell'$, there exists $N_2 \in \mathbb{N}$ such that

$|V_n - \ell'| < \frac{\ell'^2 \varepsilon}{2}$ for all $n \geqslant N_2$.

Let $N = \max(N_1, N_2)$, by (3*), we have

$\left|\frac{1}{V_n} - \frac{1}{\ell'}\right| \leqslant \frac{2|V_n - \ell'|}{\ell'^2} < \varepsilon$ for all $n \geqslant N$.

It follows that $\lim\limits_{n \to +\infty} \frac{1}{V_n} = \frac{1}{\ell'}$.

Finally, we can apply part (c) and have:

$$\lim_{n \to +\infty} \frac{U_n}{V_n} = \lim_{n \to +\infty} U_n \cdot \frac{1}{V_n} = \frac{\ell}{\ell'}.$$

The proof is now complete.

Theorem ⑤: (Sandwich Theorem) or (the squeeze theorem). Suppose that $(U_n)$, $(V_n)$ and $(W_n)$ are sequences such that: $U_n \leqslant V_n \leqslant W_n$ for all $n \in \mathbb{N}$ and that: $U_n \to \ell$ and $W_n \to \ell$, then $V_n \to \ell$.

proof :

For any $\varepsilon > 0$, since $\lim_n U_n = \ell$, there exists $N_1 \in \mathbb{N}$ such that:

$$\ell - \varepsilon < U_n < \ell + \varepsilon \quad \text{for all } n \geq N_1 .$$

Similarly, since $W_n \xrightarrow[n]{} \ell$, there exists $N_2 \in \mathbb{N}$ such that:

$$\ell - \varepsilon < W_n < \ell + \varepsilon \quad \text{for all } n \geq N_2 .$$

Let $N = \max(N_1, N_2)$. Then, for $n \geq N$, we have

$$\ell - \varepsilon < U_n \leq V_n \leq W_n < \ell + \varepsilon ,$$

which implies $|V_n - \ell| < \varepsilon$. Therefore $\lim_n V_n = \ell$.

Example : ① Since $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$, by sandwich theorem (or the squeeze theorem), we obtain $\lim_{n \to +\infty} \frac{\sin n}{n} = 0$, i.e., $\left(\frac{\sin n}{n}\right)_n \to 0$.

② Let $V_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$. ($(V_n)$ is it cv ?)

We have: $\frac{n \cdot n^2}{n^3+2n} \leq V_n \leq \frac{n \cdot n^2}{n^3+n+1}$, also, we know that

$\lim_{n \to +\infty} \frac{n \cdot n^2}{n^3+2n} = 1$ and $\lim_{n \to +\infty} \frac{n \cdot n^2}{n^3+n+1} = 1$, hence $\lim_{n \to +\infty} V_n = 1$

by the squeeze theorem (or sandwich theorem).

3) Adjacent sequences :

Definition ① Let $(U_n)$ and $(V_n)$ be two sequences of real numbers. We say that $(U_n)$ and $(V_n)$ are <u>adjacent</u> if :

(a) $\underbrace{U_n \leq U_{n+1}}_{\text{increasing}} \leq \underbrace{V_{n+1} \leq V_n}_{\text{decreasing}}$ for all $n \in \mathbb{N}$, and

(b) $\lim_{n \to \infty} (V_n - U_n) = 0$

Example : 1. Let $U_n = 1 - \frac{1}{n^2}$ and $V_n = 1 + \frac{1}{n^2}$.

Then $(U_n)$ and $(V_n)$ are adjacent.

proof :

(a) We can write : $U_{n+1} - U_n = \frac{-1}{(n+1)^2} + \frac{1}{n^2} = \frac{-n^2 + (n+1)^2}{n^2(n^2+1)} > 0$

similarly, $V_{n+1}-V_n = \frac{1}{(n+1)^2} - \frac{1}{n^2} < 0$

(b) we can write : $V_n - U_n = \frac{2}{n^2}$. thus $V_n > U_n$ and $V_n - U_n \to 0$ as $n \to \infty$.

2. Let $X_n = 1 - \frac{1}{2} + \frac{1}{3} + \dots + \frac{(-1)^{n+1}}{n}$. then the sequences given by $U_n = X_{2n}$ and $V_n = X_{2n+1}$ are adjacent

Proof:

(a) we can write : $U_n = X_{2n} = 1 - \frac{1}{2} + \frac{1}{3} + \dots - \frac{1}{2n}$, and

$$U_{n+1} = X_{2(n+1)} = X_{2n+2} = 1 - \frac{1}{2} + \frac{1}{3} + \dots - \frac{1}{2n} + \frac{1}{2n+1} - \frac{1}{2n+2}.$$

It follows that, $U_{n+1} - U_n = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$.

Similarly, we find that : $V_{n+1} - V_n = -\frac{1}{2n+2} + \frac{1}{2n+3} < 0$.

(b) we can write : $V_n - U_n = \frac{1}{2n+1}$. thus $V_n > U_n$ and $V_n - U_n \to 0$ as $n \to \infty$.

theorem ⑥: two adjacent sequences $(U_n)$ and $(V_n)$ converge to the same limit $l$ and $U_n \leq l \leq V_n$ for all $n \in \mathbb{N}$. $[U_0 \leq U_1 \leq \dots \leq U_n \leq V_n \leq V_{n-1} \leq \dots \leq V_0]$

Proof: By assumption, the sequence $(U_n)$ is increasing and bounded from above by $V_0$ (or $V_1$ or any $V_k$). Therefore it is convergent to some limit $l$ (by theorem ③) such that $U_n \leq l$ $(l = \lim_n U_n = \sup_n U_n)$.

Similarly, $(V_n)$ is decreasing and bounded from below by $U_0$ (or any $U_k$). Therefore it is convergent to some limit $l'$ such that $l' \leq V_n$ $(l' = \inf_n V_n)$.

Since both sequences are convergent, we can write :

$\lim_n (V_n - U_n) = \lim_n V_n - \lim_n U_n = l' - l$,

but $\lim_n (V_n - U_n) = 0$ (cdt (b) of adjacent sequences). By uniqueness of limit we get $l' - l = 0$, hence $l = l'$.

According to what we said above, we have $U_n \leq l = l' \leq V_n$, for all $n \in \mathbb{N}$

4- Cauchy convergence criterion:

Definition ②: A sequence $(U_n)$ of $\mathbb{R}$ is called a Cauchy sequence if for any $\varepsilon > 0$, there is a number $n_0$ such that $|U_p - V_q| < \varepsilon$ for all integers $p, q \geq n_0$.

$[\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N} : p > q \geq n_0 \Rightarrow |U_p - U_q| < \varepsilon]$.

Example: Let $U_n = \frac{n+1}{n}$. Then $(U_n)$ is a Cauchy sequence.

$\forall \varepsilon > 0, \overset{?}{\exists} n_0 \in \mathbb{N}^*, \forall p, q \in \mathbb{N}^*: p > q \geq n_0 \Rightarrow |U_p - U_q| < \varepsilon$ (T.I. triangle inequality)

$$|U_p - U_q| = \left|\frac{p+1}{p} - \frac{q+1}{q}\right| = \left|\frac{pq+q-pq-p}{pq}\right| = \left|\frac{q-p}{pq}\right| \leq \left|\frac{p}{pq}\right| + \left|\frac{p}{pq}\right| = \frac{1}{p} + \frac{1}{q}$$
$$< \frac{1}{n_0} + \frac{1}{n_0} = \frac{2}{n_0} < \varepsilon \quad (p > q \geq n_0)$$

$\frac{2}{n_0} < \varepsilon \Rightarrow n_0 > \frac{2}{\varepsilon} \Rightarrow$ it is enough to take $n_0 = \left[\frac{2}{\varepsilon}\right] + 1$ such that

$|U_p - U_q| < \varepsilon$.

Theorem (7): Every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.

proof: Let $(U_n)$ be a convergent sequence, such that $\lim_n U_n = \ell$, i.e.,

$\lim_n U_n = \ell \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 : |U_n - \ell| < \varepsilon/2$ — $(*)$

$(U_n)$ is a Cauchy sequence? $\Leftrightarrow \forall \varepsilon > 0, \overset{?}{\exists} n_1 \in \mathbb{N}, \forall p, q \in \mathbb{N}: p > q \geq n_1 : |U_p - U_q| < \varepsilon$

$$|U_p - U_q| = |U_p - \ell + \ell - U_q| \overset{T.I.}{\leq} |U_p - \ell| + |\ell - U_q| = |U_p - \ell| + |U_q - \ell|$$
$$\overset{(*)}{<} \varepsilon/2 + \varepsilon/2 = \varepsilon, \ (\text{if } n_1 = n_0), \text{ hence:}$$

$\forall \varepsilon > 0, \exists n_1 = n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}, p > q \geq n_1 : |U_p - U_q| < \varepsilon$,

which proves that $(U_n)$ is a Cauchy sequence.

Theorem (8): Every Cauchy sequence in $\mathbb{R}$ is convergent.

proof: Let $(U_n)$ be a Cauchy sequence and let

$$S = \{y \in \mathbb{R},\ U_n \geq y \text{ for all } n \text{ large enough}\}.$$

Claim 1.: $S \neq \phi$ (why?)

Let $\varepsilon > 0$ be given. Let $N$ be such that $|U_p - U_q| < \varepsilon$ for $p, q \geq N$.

In particular,

$$(\alpha) \text{ — } U_N - \varepsilon < U_n < U_N + \varepsilon, \text{ for all } n \geq N.$$

It follows: $U_N - \varepsilon \in S$, hence $S \neq \phi$.

Claim 2.: $y \in S \Rightarrow y < U_N + \varepsilon$. Indeed, let $y \in S$. Then $U_n \geq y$ for $n$ large enough. If $y \geq U_N + \varepsilon$, then $U_n \geq U_N + \varepsilon$ for $n$ large enough, which contradicts $(\alpha)$. Therefore $y < U_N + \varepsilon$.

This means that $U_N + \varepsilon$ is an upper bound of $S$. Let now, $\ell = \sup S$. It follows from what we said above that:

$U_N - \varepsilon \leq \ell \leq U_N + \varepsilon$, that is $|U_N - \ell| \leq \varepsilon$. Now, if $q \geq N$, then $|U_q - \ell| \leq |U_q - U_N| + |U_N - \ell| \leq 2\varepsilon$. Since $\varepsilon$ was arbitrary, it follows that $U_n \longrightarrow \ell$. (add the 2nd proof version arabic)

Remarks:

① From the previous two theorems, we conclude the equivalence between the convergent sequence and the Cauchy sequence.

② Cauchy's criterion is used to prove the convergence of a sequence without knowing its limit.

③ The convergence criterion is valid for proving the convergence of a sequence in $\mathbb{R}$ and is invalid in $\mathbb{Q}$ [e.g: the sequence $U_n = \left(1 + \frac{1}{n}\right)^n$ is a cauchy sequence in $\mathbb{Q}$ but it converges only in $\mathbb{R}$ and we have: $\lim\limits_{n} U_n = \lim\limits_{n} \left(1 + \frac{1}{n}\right)^n = e \notin \mathbb{Q}$].

④ We say that a space $\mathbb{R}$ is a complete space if every Cauchy sequence in $\mathbb{R}$ is convergent in this space.

[ ~~the~~ space $\mathbb{Q}$ is not complete (is an incomplete space) because the sequence previous at the general term $U_n$ is a Cauchy sequence in $\mathbb{Q}$ but convergent in $\mathbb{R}$ ].

5/ Special cases: progressive sequences (reccurent)

Definition: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a given function

a sequence $(U_n)$ is said to be reccurent (regressive) if it is defined by its first term $U_0 \in D$ and the following reccurent relation:

$$\forall n \in \mathbb{N}^* (n \geq 1) : U_{n+1} = f(U_n)$$

Study of monotomic reccurent sequence:

the study of the monotonicity of this type of sequences is based essentially to study the variation sens of the function $f$.

a) If $f$ is an increasing function ($f' \geq 0$), then $(U_n)$ is a monotonic sequence and we have:

1) $(U_n)$ is strictly increasing if: $U_0 < U_1$

2) $(U_n)$ " " decreasing if: $U_0 > U_1$.

3) $(U_n)$ is constant if $U_0 = U_1$.

b/ • If f is a decreasing function ($f'<0$), then $(U_n)$ is non-monotonic, because the term has not a fixed signe $\Rightarrow (U_n)$ is divergent

Example: Let $(U_n)$ be a numerical sequence, defined as follows:

$$\begin{cases} U_0 = 3 \\ U_{n+1} = \sqrt{U_n + 2}, \ \forall n \in \mathbb{N}^* \end{cases}$$

Q: study the monotonicity of the sequence.

Solution:

let $f(x) = \sqrt{2+x} \Rightarrow f'(x) = \frac{1}{\sqrt{2+x}} > 0, \forall x \in ]-2, +\infty[$.

$\Rightarrow f$ is strictly increasing on $]-2, +\infty[$ and continous on $[-2, +\infty[$

also, we have $U_0 = 3$ and $U_1 = \sqrt{5} = 2{,}7... \Rightarrow U_1 < U_0 \Rightarrow (U_n) \searrow$

Q: What we say about the monotonicity of $(U_n)$, when $U_0 = 1$?

and when $U_0 = 2$?

Solution: if $U_0 = 1$, then $U_1 = \sqrt{3} \Rightarrow U_1 > U_0 \Rightarrow (U_n)$ is strictly increasing.

for $U_0 = 2$, we obtain $U_1 = \sqrt{4} = 2 \Rightarrow U_0 = U_1 \Rightarrow (U_n)$ is a constant sequence

• Calcul. the limit of reccurent sequence

we will see in the next chapter (chapter 04) that for all function f continous in a point $x_0$, we have

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Thus, all reccurent sequence $(U_n)$ which is convergent to $\ell$ satisfies: $\lim_n f(U_n) = f(\lim_n U_n) \Rightarrow \lim_n U_{n+1} = f(\lim_n U_n) \Rightarrow \ell = f(\ell)$

hence, to obtain the limit of the reccurent sequence $(U_n)$ it is enough to solve the following equation: $f(\ell) = \ell$.

That is, in the previous example it is enough to solve: $l = \sqrt{l+2}$, i.e.

$l = \sqrt{l+2} \Leftrightarrow l^2 - l - 2 = 0 \Leftrightarrow (l+1)(l-2) = 0 \Rightarrow \begin{cases} l = 2 \text{ (accepted)} \\ \vee \\ l = -1 \text{ (rejected)} \end{cases}$

Because: $U_{n+1} = \sqrt{2+U_n} \geq 0$ [all terms of $(U_n)$ are positive]

$\Rightarrow \boxed{\lim_n U_n = 2}$

• The sequences that lead to infinity (i.e., $\pm\infty$)

We saw above that among the divergent sequences, the sequences that lead to $\pm\infty$. We will give in this section the mathematical definition of sequences that lead to $\pm\infty$, i.e., $\lim_n U_n = \pm\infty$.

Definition: Let $(U_n)$ be a numerical sequence, then

1/ $\lim_n U_n = +\infty \Leftrightarrow \forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow U_n > A$

2/ $\lim_n U_n = -\infty \Leftrightarrow \forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}: n \geq n_0 \Rightarrow U_n < -A$

3/ $\lim_n U_n = \infty \Leftrightarrow \forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}: n \geq n_0 \Rightarrow |U_n| > A$

example: prove that $\lim_n U_n = +\infty$, where $U_n = 2n^2 - 1, \forall n \in \mathbb{N}^*$.

$\lim_n U_n = +\infty \Leftrightarrow \forall A > 0, \exists ? n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow U_n > A$

$U_n > A \Leftrightarrow 2n^2 - 1 > A \Leftrightarrow 2n^2 > A + 1 \Leftrightarrow n^2 > \frac{1+A}{2} \Leftrightarrow |n| > \sqrt{\frac{1+A}{2}}$

$\Leftrightarrow \begin{cases} n > \sqrt{\frac{1+A}{2}} & \text{(accepted value)} \\ \vee \\ n < -\sqrt{\frac{1+A}{2}} & \text{(rejected value)}. \end{cases}$

Hence, it is enough to take $n_0 = \left[\sqrt{\frac{1+A}{2}}\right] + 1$ such that $U_n > A$.

Some properties of sequences that lead to $\pm\infty$:

1/ If $\lim_n U_n = +\infty$ with $U_n \neq 0, \forall n \in \mathbb{N}$, then $\lim_n \frac{1}{U_n} = 0$.

2/ If $\lim_n U_n = +\infty$, then $\lim_n (\lambda U_n) = \text{sign}(\lambda)\infty$.

6/ The extracted subsequences

Definition: Let $(U_n)$ be a numerical sequence and let $f: \mathbb{N} \to \mathbb{N}$ be a strictly increasing application. We can define a new sequence $(V_n)$ as follows: $\forall n \in \mathbb{N}: V_n = f(U_n)$.

$(U_{\varphi(n)})$ is called the subsection extracted from $(U_n)$ by the application $\varphi$.

Example ①: Let $U_n = \frac{1}{n}, \forall n \in \mathbb{N}^*$ then we can extract many subsequences from $(U_n)$, for example for:

- $\varphi(n) = 2n$, we get the following subsequence $(U_{\varphi(n)}) = \left(\frac{1}{2n}\right)_{n \in \mathbb{N}^*}$ and its terms are: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots, \frac{1}{2n}, \dots$
- $\varphi(n) = 2n+1$, we get: $U_{\varphi(n)} = \frac{1}{2n+1}, \forall n \in \mathbb{N}$, and its terms are: $1, \frac{1}{3}, \frac{1}{5}, \dots, \frac{1}{2n+1}, \dots$
- $\varphi(n) = 3n$, we get: $U_{\varphi(n)} = \frac{1}{3n}, \forall n \in \mathbb{N}^*$, and its terms are: $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{3n}$

② the alternative sequence $U_n = (-1)^n, \forall n \in \mathbb{N}$. from this sequence we can extract. for $\varphi(n) = 2n$, the sequence that its general term is $(-1)^{2n} = 1$, that is, all terms of this subsequence equal to 1.

- the same for $\varphi(n) = 2n+1$, we get the sequence that its term is $(-1)^{2n+1} = -1$, i.e., all terms of this sub sequence equal to $-1$.

Remark: we have: $\varphi(n) \geq n, \forall n \in \mathbb{N}$.

Theorem ⑧: If a sequence $(U_n)$ converges to $\ell$, then any subsequence $(U_{\varphi(n)})$ (or $(U_{n_k})$) also converges to the same limit $\ell$.

Proof: $\lim_n U_n = \ell \iff \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 \Rightarrow |U_n - \ell| < \varepsilon/2$

we need to prove $\lim_n U_{\varphi(n)} \overset{?}{=} \ell \iff \forall \varepsilon > 0, \exists n_1 \in \mathbb{N}, \forall n \geq n_1 \Rightarrow |U_{\varphi(n)} - \ell| < \varepsilon$?

Since $\varphi(n) \geq n \geq n_0$, it is enough to take $n_1 = n_0$ such that:

$$\forall \varepsilon > 0, \exists n_1 = n_0 \in \mathbb{N}, \forall n \geq n_1 \Rightarrow |U_{\varphi(n)} - \ell| = |U_{\varphi(n)} - U_n + U_n - \ell| \leq$$
$$\leq \underbrace{|U_{\varphi(n)} - U_n|}_{(U_n) \text{ is cauchy}} + \underbrace{|U_n - \ell|}_{(U_n) \text{ is cv}}$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow \lim_{n \to +\infty} U_{\varphi(n)} = \ell \Rightarrow (U_{\varphi(n)})$ is convergent to $\ell$.

Example: Let $U_n = \frac{1}{n}, \forall n \in \mathbb{N}^*$,

we have $(U_n)$ is a convergent sequence and $\lim_n U_n = 0$, then all extracted subsequence from $(U_n)$ are converge to $\ell = 0$, that is, $\left(\frac{1}{2n}\right), \left(\frac{1}{2n+1}\right), \left(\frac{1}{3n}\right), \dots$ converge to $0$.

**Corollary:** Let $(U_n)$ be a sequence of real numbers.
If $(U_n)$ is a divergent ~~sub~~sequence or if it has two subsequences that converge to two different limits, then this sequence $(U_n)$ is divergent.

**Example:** Let $U_n = (-1)^n$, $\forall n \in \mathbb{N}$, then $(U_n)$ is divergent. Indeed, the subsequence $(U_{2n}) = \{(-1)^{2n}, n \in \mathbb{N}\} = \{1\}$ converges to $\ell = 1$, but the subsequence $(U_{2n+1}) = \{(-1)^{2n+1}, n \in \mathbb{N}\} = \{-1\}$ converges to $\ell' = -1$
hence $(U_n)$ is divergent.

**Theorem (Nested intervals theorem):**
Let $(I_n)_{n \geq 1}$ be a sequence of nonempty closed bounded intervals satisfying
- $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$ and $I_n = [a_n, b_n]$.

Then, the following hold:
1/ $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$, If, in addition, the lengths of the intervals $I_n$ converg to zero, i.e.,
$\lim_n |I_n| = \lim_n (b_n - a_n) = 0$ [with $I_n = [a_n, b_n]$]
then $\bigcap_{n=1}^{\infty} I_n$ consists of a single point, i.e., $\bigcap_{n=0}^{\infty} I_n = \{\ell\}$, $\ell \in \mathbb{R}$.

**Proof:** for all $n \in \mathbb{N}$, we have: $a_n \leq a_{n+1} \leq \dots \leq b_{n+1} \leq b_n$ $(I_{n+1} \subseteq I_n)$
with $\lim_n (b_n - a_n) = 0$, then the sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are adjacent, hence they converge to the same limit $\ell$, that is
$\ell = \sup\{a_n, n \in \mathbb{N}\} = \inf\{b_n, n \in \mathbb{N}\}$, thus:
$\forall n \in \mathbb{N}: a_n \leq \ell \leq b_n \Rightarrow \ell \in I_n = [a_n, b_n], \forall n \in \mathbb{N} \Rightarrow \ell \in \bigcap_{n \in \mathbb{N}} I_n$
Now, we need to prove that if $x \in \bigcap I_n$, then $x = \ell$.
Let $x \in \bigcap_{n \in \mathbb{N}} I_n$, then $\forall n \in \mathbb{N}: |x - \ell| \leq b_n - a_n$ $(x, \ell \in \bigcap_{n \geq 0} I_n)$
and from the supposition: $\lim_n (b_n - a_n) = 0$, then $|x - \ell| < \varepsilon$, because
$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0: |b_n - a_n| = b_n - a_n < \varepsilon$ $[\underbrace{(b_n - a_n)_n}_{\text{sequence}} \xrightarrow{cv} 0]$
hence: $\forall n \in \mathbb{N}: |x - \ell| < \varepsilon \Leftrightarrow x = \ell \Rightarrow \boxed{\bigcap_{n \in \mathbb{N}} I_n = \{\ell\}}$

**Remark:** the Nested intervals theorem can applied when all of its conditions are verified.

**Example:** Let $(I_n)_{n\geq 0}$ be a sequence of open bounded intervals such that : $\forall n\in\mathbb{N}: I_n = ]0, \frac{1}{n+1}[$.

**Sol:** $(I_n)_{n\in\mathbb{N}}$ is decreasing and we have $\lim_n |I_n| = \lim \frac{1}{n+1} = 0$ but $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$

• **Theorem (Bolzano-Weierstrass Theorem)**

Every bounded sequence $(U_n)$ of real numbers has a convergent subsequence $(U_{\varphi(n)})$.

**proof:** the proof is based essentially on the dichotomy principal's. From the hypothesis $(U_n)$ is bounded, ~~so~~ the set of values of the sequence $(U_n)$ is contained in an interval $[a,b]$. ($S\subset[a,b]$, $S=\{U_n, n\in\mathbb{N}\}$)

The main idea is to construct a subsequence $(U_{\varphi(n)})$ such that each its term is included in an interval $[a_n,b_n]\subset[a_{n-1},b_{n-1}]$ such as

• $b_n - a_n = \frac{b_{n-1}-a_{n-1}}{2} = \frac{b_{n-2}-a_{n-2}}{2^2} = \dots = \frac{b_0-a_0}{2^n} = \frac{b-a}{2^n}$ $(U_{\varphi(n)}\in[a_n,b_n], \forall n\in\mathbb{N})$

• $[a_0,b_0] = [a,b]$.

Hence, we have from this construction: $a_n \leq U_{\varphi(n)} \leq b_n, \forall n\in\mathbb{N}$.

we note that the sequences $(a_n)$ and $(b_n)$ are adjacente.

indeed, $a_n\leq b_n$ and $a_0\leq \dots \leq a_{n-1} \leq a_n \leq b_n \leq b_{n-1} \leq \dots \leq b_0 \Rightarrow (b_n)\searrow \wedge (a_n)\nearrow$

also $\lim_n (b_n - a_n) = \lim_n \left(\frac{b-a}{2^n}\right) = 0 \Rightarrow (a_n)_n$ and $(b_n)_n$ are adjacente

therefore $(a_n)_n$ and $(b_n)$ converge to the same limit $\ell$.

Hence, $\lim_n a_n = \ell$ and $\lim_n b_n = \ell$ with $a_n \leq U_{\varphi(n)} \leq b_n, \forall n\in\mathbb{N}$

then from the sandwich theorem or the squeeze theorem we conclude that $(U_{\varphi(n)})$ converges to $\ell$.

**Example:** Let $U_n = (-1)^n$, we have $|U_n| \leq 1, \forall n\in\mathbb{N}$ ($(U_n)$ is bounded) hence, we can extract a convergent subsequence. (at least one subseq)

• $U_{2n} = (-1)^{2n} = 1$ is the constant sequence $= 1 \Rightarrow (U_{2n})$ is convergent.

• $U_{2n+1} = (-1)^{2n+1} = -1$, is the constant sequence $= -1 \Rightarrow (U_{2n+1})$ is convergent