Chapter 04: Real functions of a real variable.

1/ Generality on the functions:

Definition ①: We call real function of a real variable, any application $\begin{pmatrix} f: E \to F \\ x \mapsto f(x) \end{pmatrix}$ f defined from E to F such as $E, F \subseteq \mathbb{R}$, which associate to each element x of E $(x \in D)$ the image $f(x) \in F$

- E is called the starting set.
- F is called the arrival set.

Remark: If $E = \mathbb{N}$, then any application from $E = \mathbb{N}$ to $\mathbb{R}$ is a real sequence.
So, a real sequence is a real function whose starting set is $\mathbb{N}$ or a subset of $\mathbb{N}$. D_f denote the definition set of f and $f(D_f)$ its direct image under f that is $f(E) = f(D_f) = \{ y \in F \,/\, \exists x \in D_f \,/\, f(x) = y \}$

- Given an element y in F, there may exist elements in E with y as their image. If $f(x) = y$, then x is called a preimage of y or an inverse image of y.
The set of all inverse images of y is called the inverse image of y, that is:

$$f^{-1}(y) = \{ x \in E \,/\, f(x) = y \}.$$

Example ① Let $f(x) = x^2 + 2x + 3$, then $D_f = \mathbb{R}$ and the direct image under f is $f(D_f) = [2, +\infty[$. indeed,

x	$-\infty$		-1		$+\infty$
f'		$-$	0	$+$	
f	$+\infty$	$\searrow$	$f(-1)=2$	$\nearrow$	$+\infty$

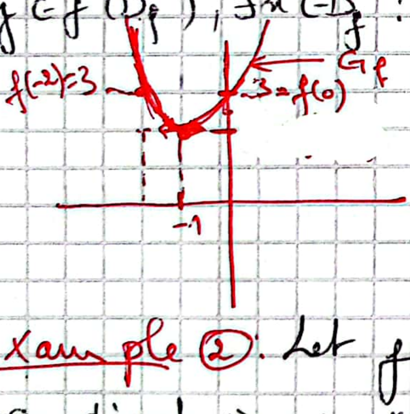
$\forall y \in f(D_f), \exists x \in D_f : y = f(x) \Leftrightarrow y = x^2 + 2x + 3$

$\Leftrightarrow y - 2 = x^2 + 2x + 1$

$\Leftrightarrow y - 2 = (x+1)^2 \geq 0$

$\Rightarrow y - 2 \geq 0$

$\Rightarrow y \geq 2 \Rightarrow y \in [2, +\infty[= f(D_f)$



Example ②: Let $f(x) = \sqrt{x^2 - 1}$, then $D_f =]-\infty, -1] \cup [1, +\infty[$ and the direct image of D_f under f is $f(D_f) = [0, +\infty[$

x	$-\infty$		-1		1		$+\infty$
f'		$-$				$+$	
$f(x)$	$+\infty$	$\searrow$	0		0	$\nearrow$	$+\infty$

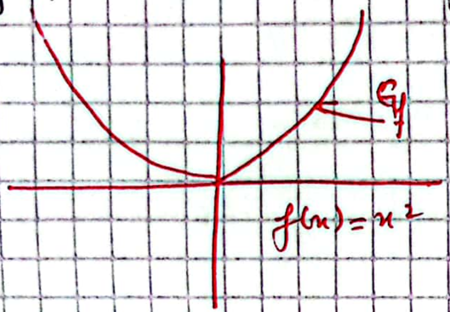
Definition ②: Let D be an interval from $\mathbb{R}$.
D is called symmetrical interval iff: $\forall x \in D$; $(-x) \in D$, that is: D is symmetrical if zero is the center of D, e.g: $[-1, 1]$, $\mathbb{R}$, $]-2, 2[$ ...

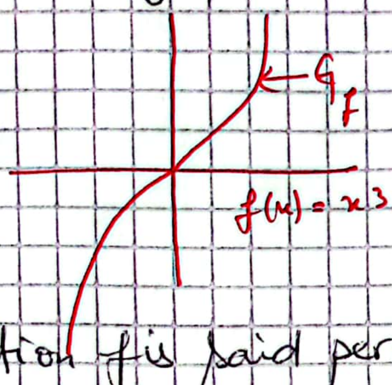
Definition ③: Let f be a real function defined on D symmetric, we have

- f is called even iff : $\forall x \in D : f(-x) = f(x)$
- f is called odd iff : $\forall x \in D : f(-x) = -f(x)$

Example: • $f(x) = x^2$ is a function and its graphe is

- $f(x) = x^3$ is a function and its graphe is





Definition (4): the real function f is said periodic if:

$\exists T > 0, \forall x \in D_f : f(x+T) = f(x)$ --- $(*)$

in this case, the smallest number that satisfies $(*)$ is called the period of f which is denoted by T.

Remark: The trigonometric functions $\sin x$ and $\cos x$ are periodic and its period is $T = 2\pi$, i.e, $\cos(x+2\pi) = \cos x \wedge \sin(x+2\pi) = \sin x$.

Definition (5): Let $f : D \longrightarrow \mathbb{R}$ be a real function $(D \subseteq \mathbb{R})$.

- f is bounded above if the set $f(D)$ is bounded above, that is :
  $\exists M \in \mathbb{R}, \forall x \in D : f(x) \leq M$. ($M$ is an upper bound of f)
- similarly, f is bounded below if the set $f(D)$ is bounded below, that is
  $\exists m \in \mathbb{R}, \forall x \in D : f(x) \geq m$ (m is a lower bound of f)
- f is bounded if $f(D)$ is bounded, that is:
  $\exists m, M \in \mathbb{R}, \forall x \in D, \quad m \leq f(x) \leq M$ or $\exists M_1 > 0, \forall x \in D : |f(x)| \leq M_1$.
- M is a maximum value of f if M is an upper bound of f and $\exists x \in D : f(x) = M$
- m is a minimum value of f if m is a lower bound of f and $\exists x \in D : f(x) = m$.
- M is the least upper bound, or supremum of f if
  1. $f(x) \leq M, \forall x \in D$ (M is an upper bound of $f(D)$)
  2. if $M' < M$, then M' isn't an upper bound of $f(D)$.

In this case, we write $M = \sup\limits_{D} f(x)$ or $M = \sup\{f(x), x \in D\}$ or $M = \sup\limits_{x \in D} f(x)$.

or $M = \sup f$.

• Similarly, m is the greatest lower bound or the infimum of f, if:

① $m \leq f(x)$, $\forall x \in D$ (m is a lower bound of $f(D)$)

② if $m' > m$, then m' isn't a lower bound of $f(D)$.

In this case, we write $m = \inf_D f$ or $m = \inf_D f(x)$ or $m = \inf\{f(x), x \in D\}$ or $m = \inf_{x \in D} f(x)$.

Example: Let f be the function defined by $f(x) = \frac{1}{x^2}$, $x \in [1, 4[$.

• Determine the least upper bound and the greatest lower bound of f on $[1,4[$

solution: $f(x) = \frac{1}{x^2} \Rightarrow f'(x) = \frac{-2}{x^3} < 0$, hence f is a decreasing function on the interval $[1, 4[$. Since $1 \leq x < 4$, so that $1 \leq x^2 < 16$ and we have $\frac{1}{16} < f(x) = \frac{1}{x^2} \leq 1$. Hence f is bounded above by 1 and bounded below by $\frac{1}{16}$. Next, since $f(1) = 1$ and 1 is an upper bound for f on $[1, 4[$ it follows that f has least upper bound 1 on $[1, 4[$ and $\max_{[1,4[} f(x) = 1$

Finally, $\frac{1}{16}$ is a lower bound of f on the interval $[1, 4[$ but there is no point $x \in [1, 4[$ for which $f(x) = \frac{1}{16}$. So $\min_{[1,4[} f(x)$ cannot exists.

Now, we prove that: $\inf f(x) = \frac{1}{16}$ if $\forall \varepsilon > 0, \exists x_0 \in [1, 4[: f(x_0) < \frac{1}{16} + \varepsilon$

$f(x_0) < \frac{1}{16} + \varepsilon \Leftrightarrow \frac{1}{x_0^2} < \frac{1}{16} + \varepsilon \Rightarrow x_0^2 > \frac{16}{1+16\varepsilon} \Rightarrow |x_0| = x_0 > \sqrt{\frac{16}{1+16\varepsilon}}$, $x_0 \in [1, 4[$

Hence, it is enough to take $x_0 \in \left] \sqrt{\frac{16}{1+16\varepsilon}}, +\infty \right[$ for which: $f(x) < \frac{1}{16} + \varepsilon$

Therefore, $\frac{1}{16}$ is the greatest lower bound of f on $[1, 4[$.

Definition ⑥: (Algebraic operations of the functions)

Let $f, g : D \to \mathbb{R}$ be two real function, then the functions: $f \pm g$, fg, $\lambda f \,(\lambda \in \mathbb{R})$, $\frac{f}{g}\,(g \neq 0)$ are defined as follows:

- $\forall x \in D$: $(f \pm g)(x) = f(x) \pm g(x)$.
- $\forall x \in D$: $(fg)(x) = f(x)\, g(x)$
- $\forall x \in D$: $(\lambda f)(x) = \lambda \cdot f(x)$, $\forall \lambda \in \mathbb{R}$.
- $\forall x \in D$: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$

• we denote by $\mathcal{F}(D, \mathbb{R})$ the set of real functions defined from D to $\mathbb{R}$.

Definition ⑦: Let $f : D \to \mathbb{R}$ be a real function, then f is said:

1) Increasing iff, $\forall x_1, x_2 \in D$: $x_1 \geq x_2 \Rightarrow f(x_1) \geq f(x_2)$.

2) Decreasing iff, $\forall x_1, x_2 \in D$: $x_1 \geq x_2 \Rightarrow f(x_1) \leq f(x_2)$.

Definition (8): Let $f: A \to B$ and $g: B \to C$ be two functions with the property that the range of f is a subset of the domain of g (i.e., $f(A) \subseteq D_g$). Define a new function $g \circ f: A \to C$ as follows:

$(g \circ f)(x) = g(f(x))$ for all $x \in A$,

where "$g \circ f$" is read "g circle f" and $g(f(x))$ is read "g of f of x". the function $g \circ f$ is called the composition of f and g.

Example: Let $f: \mathbb{Z} \to \mathbb{Z}$, $n \mapsto f(n) = n+1$ and let $g: \mathbb{Z} \to \mathbb{Z}$, $n \mapsto g(n) = n^2$

a/ Find the compositions $g \circ f$ and $f \circ g$.

b/ Is $g \circ f = f \circ g$? explain.

Solution: a/ the functions $g \circ f$ and $f \circ g$ are defined as follows:

$(g \circ f)(n) = g(f(n)) = g(n+1) = (n+1)^2; \forall n \in \mathbb{Z}$

$\wedge$

$(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1, \forall n \in \mathbb{Z}$

b/ the two functions $f \circ g$ and $g \circ f$ are not equal: $f \circ g \neq g \circ f$.

As conclusion, the composition of functions isn't a commutative operation.

2/ Limits and continuity

2-1 limits of functions:

2-1/ The finite limits (when x towards to x_0, i.e., $x \to x_0$)

Definition (1): Let $f: D \to \mathbb{R}$, and let x_0 be a limit point of D ($x_0 \in \bar{D}$).

We say that f has a limit ℓ at x_0, iff: (i.e., $\lim_{x \to x_0} f(x) = \ell$)

$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D: 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$.

remark: • Note that the limit point x_0 in the definition of limit may or may not be an element of the domain D.

• The inequality $|f(x) - \ell| < \varepsilon$ needs only be satisfied by elements of D.

Example (1): Let: $f: \mathbb{R} \to \mathbb{R}$ be given by: $f(x) = 5x + 2$. prove that $\lim_{x \to 1} f(x) = 7$

solution: $\lim_{x \to 1} f(x) = 7 \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \mid \forall x \in \mathbb{R}: 0 < |x - 1| < \delta \Rightarrow |f(x) - 7| < \varepsilon$

$|f(x) - 7| = |5x + 2 - 7| = |5x - 5| = 5|x - 1| < \varepsilon \Rightarrow |x - 1| < \varepsilon/5$

this suggests the choice $\delta = \varepsilon/5$. Then, if $|x-1|<\delta$ we have:

$|f(x)-7| = 5|x-1| < 5\delta = \varepsilon$

Example (2) Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{3x-5}{x^2+3}$. prove that $\lim_{x\to 1} f(x) = -\frac{1}{2}$

solution: $\lim_{x\to 1} f(x) = -\frac{1}{2} \Leftrightarrow \forall \varepsilon>0, \exists \delta>0, \forall x\in\mathbb{R}: 0<|x-1|<\delta \Rightarrow |f(x)+\frac{1}{2}|<\varepsilon$

$|f(x)+1/2| = \left|\frac{3x-5}{x^2+3}+\frac{1}{2}\right| = \left|\frac{x^2+6x-7}{2(x^2+3)}\right| = \frac{|x-1||x+7|}{2(x^2+3)} \leqslant \frac{1}{6}|x-1||x+7|$ $(x^2+3\geq 3)$ — (*)

Now, we have: $x\to 1 \Rightarrow 0<x<2 \Rightarrow \begin{cases} 7<x+7<9 \Rightarrow |x+7|<9 \text{ — (1)} \\ -1<x-1<1 \Leftrightarrow |x-1|<1 \text{ — (2)} \end{cases}$

replacing (1) in (*), we get: $|f(x)+1/2| < \frac{9}{6}|x-1| = \frac{3}{2}|x-1| < \varepsilon$

$\Rightarrow |x-1| < \frac{2\varepsilon}{3}$ — (3)

from (2) and (3), we choose $\delta = \min\{1, \frac{2\varepsilon}{3}\}$. Hence, it follows that if $|x-1|<\delta$ we get $|f(x)+\frac{1}{2}| \leqslant \frac{|x+7|}{6}|x-1| < \frac{3}{2}\delta \leqslant \varepsilon$.

theorem (1): Let $f: D\to\mathbb{R}$ be a real function and let x_0 be a limit point of D. If f has a limit at x_0, then this limit is unique.

proof: suppose by contradiction that f has two different limits ℓ_1 and ℓ_2 and we arrive at $\ell_1=\ell_2$.

$\lim_{x\to x_0} f(x) = \ell_1 \Leftrightarrow \forall\varepsilon>0, \exists\delta_1>0, \forall x\in D: 0<|x-x_0|<\delta_1 \Rightarrow |f(x)-\ell_1|<\varepsilon/2$ — (1)

$\lim_{x\to x_0} f(x) = \ell_2 \Leftrightarrow \forall\varepsilon>0, \exists\delta_2>0, \forall x\in D: 0<|x-x_0|<\delta_2 \Rightarrow |f(x)-\ell_2|<\varepsilon/2$ — (2)

Let $\delta_1=\delta_2$, then:

$0 \leqslant |\ell_1-\ell_2| = |\ell_1-f(x)+f(x)-\ell_2| \overset{T.I}{\leqslant} |f(x)-\ell_1|+|f(x)-\ell_2| \overset{(1),(2)}{<} \varepsilon/2+\varepsilon/2 = \varepsilon$

hence $\ell_1=\ell_2$.

theorem (2): (Sequential characterization of limits)

Let $f: D\to\mathbb{R}$ be a real function and let x_0 be a limit point of D. Then, we have the following equivalence

(1) $\lim_{x\to x_0} f(x) = \ell$

(2) $\lim_{n\to+\infty} f(x_n) = \ell$, for every sequence $(x_n)_n$ in D such that $x_n\neq x_0$ for every n and $(x_n)_n$ converges to x_0. (i.e., $\forall (x_n)_n \in D: x_n \xrightarrow[n]{} x_0 \Rightarrow f(x_n) \xrightarrow[n]{} \ell$)

proof ① $\Rightarrow$ ② ? we suppose that $\lim_{x\to x_0} f(x)=\ell$ and we prove that $\lim_{n\to+\infty} f(x_n)=\ell$ for every $(x_n)_n\in D$: $x_n \xrightarrow[n]{} x_0$ ($x_n\neq x_0$, $\forall n\in\mathbb{N}$).

$\lim_{x\to x_0} f(x)=\ell \Leftrightarrow \forall\varepsilon>0, \exists\delta>0, \forall x\in D : 0<|x-x_0|<\delta \Rightarrow |f(x)-\ell|<\varepsilon$ — ⓐ

Let (x_n) be a sequence in D with $x_n\neq x_0$, $\forall n$ and such that $x_n \xrightarrow[n]{} x_0$.

Thus, we can write : $\forall\varepsilon'>0, \exists n_0\in\mathbb{N}, \forall n>n_0 \Rightarrow |x_n-x_0|<\varepsilon'$ — ⓑ

in particular when $\varepsilon'=\delta$, we obtain from ⓐ and ⓑ that:

$|x_n-x_0|<\delta \overset{ⓐ}{\Rightarrow} |f(x_n)-\ell|<\varepsilon$, hence :

$\forall\varepsilon>0, \exists n_0\in\mathbb{N}, \forall n>n_0 : |f(x_n)-\ell|<\varepsilon \Rightarrow \lim_{n\to+\infty} f(x_n)=\ell$.

Conversely, suppose that ① is false. [Now, we need to prove that ② $\Rightarrow$ ① ?
(i.e; $\lim_n f(x_n)=\ell \Rightarrow \lim_{x\to x_0} f(x)=\ell$?)], that is, $\lim_{x\to x_0} f(x)\neq\ell \Leftrightarrow$

$\exists\varepsilon>0, \forall\delta>0, \exists x\in D : 0<|x-x_0|<\delta \wedge |f(x)-\ell|\geq\varepsilon$ — (*)

Let $\delta=\frac{1}{n}$, then (from (*)) for every $n\in\mathbb{N}$, there exists $x_n\in D$ with : $0<|x_n-x_0|<\frac{1}{n}$ (α)
and $|f(x_n)-\ell|\geq\varepsilon$.

By the squeeze theorem, the sequence $(x_n)_n$ converges to x_0.
[since : $0<|x_n-x_0|<\frac{1}{n}\to 0 \Rightarrow \lim|x_n-x_0|=0 \Leftrightarrow x_n \xrightarrow[n]{} x_0$]

Moreover, $x_n\neq x_0$, $\forall n\in\mathbb{N}$. this shows that ② is false.

It follows that (2) implies ① and the proof is completed.

Remark: this theorem is usually used to prove that a function f hasn't a limit ℓ at x_0 [i.e, f doesn't have a limit ℓ at x_0 iff there exists a sequence (x_n) in D such that $x_n\neq x_0$, $\forall n\in\mathbb{N}$, $x_n \xrightarrow[n]{} x_0$ and $(f(x_n))$ doesn't converge.

Example ①: Let $f:\mathbb{R}^*\to\mathbb{R}$ given by $f(x)=\sin\frac{1}{x}$. then $\lim_{x\to 0} f(x)$ does not exists. sol Indeed, let $x_n=\dfrac{1}{\frac{\pi}{2}+n\pi}$ such that $x_n \xrightarrow[n\to+\infty]{} 0$

Moreover, $\left[f(x_n)=\sin\left(\frac{\pi}{2}+n\pi\right)=(-1)^n\right]$ $(f(x_n))_n$ doesn't converge.

Example ②: consider the Dirichlet function $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q}\\ 0, & \text{if } x\notin\mathbb{Q} \ (x\in\mathbb{Q}^c)\end{cases}$$

then $\lim_{x \to x_0} f(x)$ doesn't exist for any $x_0 \in \mathbb{R}$.

solution: Indeed, fix $x_0 \in \mathbb{R}$ and choose two sequences $(y_n)_n, (s_n)_n$ converging to x_0 such that $y_n \in \mathbb{Q}$ and $s_n \notin \mathbb{Q}$, $\forall n \in \mathbb{N}$.

Define a new sequence (x_n) by:

$$x_n = \begin{cases} y_k, & \text{if } n = 2k \\ s_k, & \text{if } n = 2k+1 \end{cases}$$

It is clear that (x_n) converges to x_0. $[(y_n) \wedge (s_n) \to x_0]$. Moreover, since $(f(y_n))_n$ converges to 1 and $(f(s_n))_n$ converges to 0. then, from the uniqueness of the limit $(f(x_n))_n$ doesn't converge. It follows from theorem ② (the sequential characterization of limits) that:

$\lim_{x \to x_0} f(x)$ doesn't exists.

Definition ②: we call a neighbor of a point $x_0 \in \mathbb{R}$, all open interval that its center is x_0 and its lenghts is 2α, $\alpha > 0$, that is, the intervals of the form: $]x_0 - \alpha, x_0 + \alpha[$.

[or more precisely, a neighbor of a point x_0 is a set containing points that are close to x_0 according to some specified metric of distance measure]

open set $= N(x_0) = B(x_0, \delta) = \{x \in D \subseteq \mathbb{R} : d(x, x_0) < \delta\} \overset{\text{in } \mathbb{R}}{=} \{|x - x_0| < \delta, \forall x \in D \subseteq \mathbb{R}\}$.

Definition ③ (left limit point and right limit point) Let $B_+(x_0, \delta) =]x_0, x_0+\delta[$ and $B_-(x_0, \delta) =]x_0 - \delta, x_0[$

Let $f : D \to \mathbb{R}$ and let x_0 be a right limit point of D. then

$\lim_{x \overset{>}{\to} x_0} f(x) = \ell \iff \forall \varepsilon > 0, \exists \delta > 0, \forall x \in B_+(x_0, \delta): 0 < x - x_0 < \delta \Rightarrow |f(x) - \ell| < \varepsilon$.

we say that ℓ is the right-handed limit of f at x_0. the left-hand limit of f at x_0 can be defined in a similar way, that is,

$\lim_{x \overset{<}{\to} x_0} f(x) = \ell \iff \forall \varepsilon > 0, \exists \delta > 0, \forall x \in B_-(x_0, \delta): 0 < x_0 - x < \delta \Rightarrow |f(x) - \ell| < \varepsilon$.

Example ① Consider the function: $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} x + 4, & \text{if } x < -1 \\ x^2 - 1, & \text{if } x \geq -1 \end{cases}$$

solution: we have:

$\lim_{x \overset{>}{\to} -1} f(x) = \lim_{x \overset{>}{\to} -1} (x^2 - 1) = 0$, and $\lim_{x \overset{<}{\to} -1} f(x) = \lim_{x \overset{<}{\to} -1} (x + 4) = 3$

Theorem ③ Let $f: D \to \mathbb{R}$ be a real function and let x_0 be both a left limit point of D and a right limit point of D. Then

$$\lim_{x\to x_0} f(x) = \ell \iff \lim_{x \overset{<}{\to} x_0} f(x) = \ell \text{ and } \lim_{x \overset{>}{\to} x_0} f(x) = \ell.$$

2-2/ The infinite limits (when x tend towards to x_0, i.e., $x \to x_0$)

Definition ④: Let $f: D \to \mathbb{R}$ and let x_0 be a limit point of D. Then,

$$\lim_{x\to x_0} f(x) = +\infty \iff \forall A > 0, \exists \delta > 0, \forall x \in B(x_0,\delta): 0 < |x - x_0| < \delta \Rightarrow f(x) > A$$

with: $B(x_0,\delta) = B_+(x_0,\delta) \cup B_-(x_0,\delta) =]x_0, x_0+\delta[\cup]x_0-\delta, x_0[=]x_0-\delta, x_0+\delta[\setminus \{x_0\}$

Similarly,

$$\lim_{x\to x_0} f(x) = -\infty \iff \underbrace{\forall A > 0}_{A\in\mathbb{R}}, \exists \delta > 0, \forall x \in B(x_0,\delta)\ 0<|x-x_0| < \delta \Rightarrow \underbrace{f(x) < -A}_{f(x) < A}.$$

2-3: Algebraic operations of the limits:

Theorem ④: Let $f, g: D \to \mathbb{R}$ and let $c \in \mathbb{R}$. Suppose x_0 is a limit point of D and $\lim_{x\to x_0} f(x) = \ell$, $\lim_{x\to x_0} g(x) = \ell'$. Then:

1/ $\lim_{x\to x_0} (f+g)(x) = \ell + \ell'$.

2/ $\lim_{x\to x_0} (fg)(x) = \ell \cdot \ell'$.

3/ $\lim_{x\to x_0} (cf)(x) = c\ell$.

4/ $\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{\ell}{\ell'}$ provided that $\ell' \neq 0$.

5/ If $f(x) \leq g(x)$, $\forall x \in B(x_0,\delta)$ then $\ell \leq \ell'$, $x \neq x_0$.

6/ If $\lim_{x\to x_0} f(x) = 0$ and g is a bounded function in $B(x_0,\delta)$ ($|g(x)| \leq M, \forall x \in B$) then $\lim_{x\to x_0} (fg)(x) = 0$.

Example: 1/ Consider $f: \mathbb{R} - \{-7\} \to \mathbb{R}$ given by $f(x) = \frac{x^2+2x-3}{x+7}$, $\lim_{x\to -2} f(x) = ?$

then $\lim_{x\to -2} f(x) = \lim_{x\to 2} \frac{(x^2+2x-3)}{x+7} = \frac{\lim_{x\to -2} (x^2+2x-3)}{\lim_{x\to -2} (x+7)} = \frac{\lim_{-2} x^2 + \lim_{-2} 2x + \lim_{-2}(-3)}{\lim_{-2} x + \lim_{-2} 7}$

$= \frac{\left(\lim_{-2} x \cdot \lim_{-2} x\right) + 2\lim_{-2} x - \lim_{-2} 3}{\lim_{-2} x + \lim_{-2} 7} = \frac{(-2)^2 + 2(-2) - 3}{-2+7} = \boxed{-\frac{3}{5}}$

2/ $g(x) = x^2 \cos\frac{1}{x}$, $x_0 = 0$

We have $\lim_{x\to 0} x^2 = 0$ and $|\cos\frac{1}{x}| \leq 1, \forall x \in B(0,\delta)$, then from (6) $\lim_{x\to 0} x^2 \cos\frac{1}{x} = 0$.

Definition ⑤ (limit of a composition function) Let $f: D_f \to \mathbb{R}$ and $g: D_g \to \mathbb{R}$ be two real functions, with $g(D_g) \subseteq D_f$ then the function $(f \circ g)(x)$ is defined and its the limit at a point x_0 is given by:

$$\lim_{x \to x_0} f(g(x)) = f\left(\lim_{x \to x_0} g(x)\right).$$

This expression holds true if the limit on the right side exists. In other words, if the limit of $g(x)$ at x_0 approaches x_0 exists and $f(x)$ is continous at the point equal to that limit, then the limit $f(g(x))$ as x approaches x_0 exists and is equal to f evaluated at the limit of $g(x)$ as x approaches x_0.

Example: Let: $g(x) = 0, \forall x \in \mathbb{R}$ and $f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$

then: $\forall x \in \mathbb{R}: f \circ g(x) = f(g(x)) = f(0) = 1$.

but $\lim\limits_{x \overset{>}{\to} 0} f(x) = \lim\limits_{x \overset{<}{\to} 0} f(x) = 0$ (without $x = 0$)

2-4: limit and monotony:

Theorem ⑤ Suppose $f:]a,b[\to \mathbb{R}$ is an increasing function and $x_0 \in]a,b[$. Then $\lim\limits_{x \overset{<}{\to} x_0} f(x)$ and $\lim\limits_{x \overset{>}{\to} x_0} f(x)$ exist. Moreover,

$$\sup_{a<x<x_0} f(x) = \lim_{x \overset{<}{\to} x_0} f(x) \leq f(x_0) \leq \lim_{x \overset{>}{\to} x_0} f(x) = \inf_{x_0<x<b} f(x).$$

2-5: Comparaison of functions. (Negligible functions and equivalent functions)

Definition ⑥: (negligible function)

Let f and g be two functions defined in a neighborhood $B(x_0, \delta) = N(x_0)$ of x_0 except perhaps in x_0. Then, f is said negligible compared to g when x tend towards x_0, and we write $f = o(g)$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in N(x_0): 0 < |x - x_0| < \delta \Rightarrow |f(x)| < \varepsilon |g(x)|.$$

• It follows from the definition that if g doesn't vanish in $N(x_0)$ [$g(x) \neq 0, \forall x \in$ ...] then, $f \underset{x_0}{=} o(g) \Leftrightarrow \lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 0$.

Examples: $x^2 \underset{0}{=} o\left(\frac{1}{x}\right)$, $x^2 tg\, x \underset{0}{=} o(x^2)$, $x^2 \sin\frac{1}{x} + x^3 \underset{0}{=} o(tg\, x)$

$x^2 \sin\frac{1}{x} + x^3 \underset{+\infty}{=} o(x^4)$, $\dfrac{1}{x^2} \underset{+\infty}{=} o\left(\frac{1}{x}\right)$.

Definition ⑦ (equivalent function)

Let f and g be two functions defined in a neighborhood $N(x_0)$ of $\{x_0\}$, except perhaps in x_0. Then f is said equivalent to g when $x \to x_0$ and denoted $f \sim g$ if : $f-g = o(f)$, when $x \to x_0$

Note that : $f-g = o(f) \Leftrightarrow f-g = o(g)$.

- It follows from the definition that if g doesn't vanish in $N(x_0)\setminus\{x_0\}$, then

$$f \sim g \text{ at } x_0 \;\left(f \underset{x_0}{\sim} g\right) \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

Examples: $\sin x \underset{0}{\sim} x \underset{0}{\sim} tg\, x$; $1-\cos x \underset{0}{\sim} \frac{x^2}{2}$; $\ln(1+x) \underset{0}{\sim} x$

$e^x - 1 \underset{0}{\sim} x$, $\sin \frac{1}{x} \underset{+\infty}{\sim} \frac{1}{x} \underset{+\infty}{\sim} tg\, \frac{1}{x}$, $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \underset{+\infty}{\sim} a_n x^n, a_n \neq 0$