Chapter 5 continuity of functions

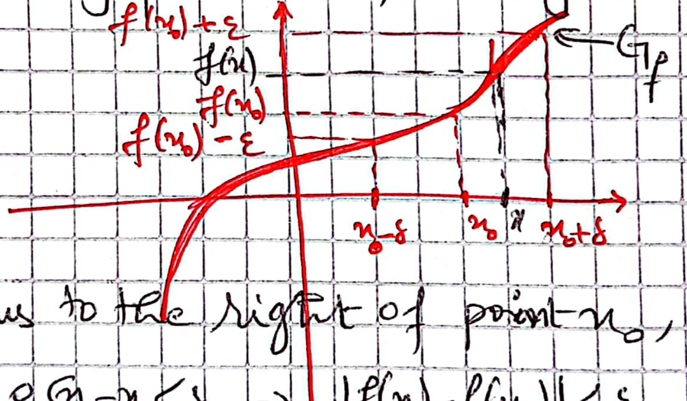
Definition ①: (continuity at a point x_0 and continuity on D)

Let D be a nonempty subset of R and let $f: D \to \mathbb{R}$ be a function. The function:

- f is said to be continous at $x_0 \in D$, if:

$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D: 0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

- If f is continous at every point $x \in D$, we say that f is continous on D.



- f is said to be continous to the right of point x_0, if:

$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D: 0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

- f is said to be continous to the left of point x_0, if:

$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D: 0 < x_0 - x < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Theorem ①: Let $f: D \to R$ and let $x_0 \in D$ be a limit point of D. Then f is continous at x_0 if and only if: $\lim_{x \to x_0} f(x) = f(x_0)$.

Example: Let $f: R \to R$ be given by $f(x) = 3x^2 - 5x + 2$.

fix $x_0 \in R$. Since, from the results of the previous theorem, we have:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (3x^2 - 5x + 2) = 3x_0^2 - 5x_0 + 2 = f(x_0).$$

It follows that f is continous at x_0.

Theorem ②: Let $f: D \to \mathbb{R}$ and let $x_0 \in D$. Then f is continous at x_0 iff for any sequence $(x_n)_n$ in D that converges to x_0, the sequence $(f(x_n))_n$ converges to $f(x_0)$. that is, we have the following equivalence:

$$\left[\lim_{x \to x_0} f(x) = f(x_0)\right] \Leftrightarrow \left[\forall (x_n)_n \in D : \lim_{n \to +\infty} x_n = x_0 \Rightarrow \lim_{n \to +\infty} f(x_n) = f(x_0)\right]$$

Definition ②: (extension by continuity)

Let $f: D - \{x_0\} \to \mathbb{R}$ be a function, and we assume that f has a finite limit ℓ at x_0. Then, the function $\tilde{f}$ (tild f) defined for all $x \in D$ by:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in D - \{x_0\} \quad (x \neq x_0) \\ \ell, & \text{if } x = x_0 \end{cases}$$

is continous at x_0.

- We say that $\tilde{f}$ constitutes the extension by continuity of f at x_0.

Example ① Let $f: \mathbb{R}^* \to \mathbb{R}$ given by $f(x) = \frac{\sin x}{x}$.

then $$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}, \quad \text{since } \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Thus $\tilde{f}$ is continous at $0 = x_0$ or f admits an extension by continuity at 0.

Example ②: Find the values of a and b such that f is continous on $\mathbb{R}$ with

$$f(x) = \begin{cases} a \sin x + b \cos x, & \text{if } x \leq \frac{\pi}{2}, \\ \pi - x, & \text{if } \frac{\pi}{2} < x \leq \pi, \\ x^2 - b, & \text{if } x > \pi. \end{cases}$$

f is continous on $\mathbb{R}$, if it is continous at points $x_0 = \frac{\pi}{2}$ and $x_0 = \pi$.

(since, f is a polynomial function on $]\frac{\pi}{2}, +\infty[\Rightarrow f$ continous on $]\frac{\pi}{2}, +\infty[$, or, f is a trigonometric function on $]-\infty, \frac{\pi}{2}] \Rightarrow f$ continous on $]-\infty, \frac{\pi}{2}]$)

$$\left.\begin{aligned} \lim_{x \overset{<}{\to} \frac{\pi}{2}} f(x) &= \lim_{x \overset{<}{\to} \frac{\pi}{2}} (a \sin x + b \cos x) = a \\ \lim_{x \overset{>}{\to} \frac{\pi}{2}} f(x) &= \lim_{x \overset{>}{\to} \frac{\pi}{2}} (\pi - x) = \frac{\pi}{2} \end{aligned}\right\} \Rightarrow \boxed{a = \frac{\pi}{2}}$$

$$\left.\begin{aligned} \lim_{x \overset{<}{\to} \pi} f(x) &= \lim_{x \overset{<}{\to} \pi} (\pi - x) = 0 \\ & \quad (x^2 - b) = \pi^2 - b \end{aligned}\right\} \Rightarrow \boxed{b = \pi^2}$$

$$\Rightarrow \boxed{f \text{ is continous on } \mathbb{R} \text{ if } a = \frac{\pi}{2} \text{ and } b = \pi^2}$$

$\lim f(x) = \lim_{x \overset{>}{\to} \pi}$

Theorem ③: Let $f, g : D \to \mathbb{R}$ and let $x_0 \in D$. Suppose f and g are continous at x_0. Then

1/ $f+g$ and $f \cdot g$ are continuous at x_0.

2/ λf is continous at x_0 for any constant $\lambda \in \mathbb{R}$.

3/ If $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.

Theorem ④: Let $f : D \to \mathbb{R}$ and let $g : E \to \mathbb{R}$ with $f(D) \subset E$. If f is continous at x_0 and g is continous at $f(x_0)$, then $g \circ f$ is continous at x_0.

Remark: The set of continous functions on the interval D is denoted by $C(D)$.

Definition ③: (uniformly continous)

Let D be a nonempty subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is called uniformly continous on D, if:

$\forall \varepsilon > 0, \exists \delta > 0 : \text{if } u, v \in D : |u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon$.

Theorem ⑤: If $f : D \to \mathbb{R}$ is uniformly continous on D, then f is continous at every point $x_0 \in D$.

Remark: ① Uniform continuity is always on an interval.

② In Uniform continuity the choice of δ depends only on ε, but in simple continuity the choice of δ depends on ε and x_0.

Theorems on continuity:

Theorem ⑥ (Heine theorem): Any continous function on a closed interval $[a, b]$ is uniformly continous on this interval.

Example ① Let $f : [1/2, 1] \to \mathbb{R}$ be given by $f(x) = \frac{1}{x}$.

Then f is continous on $[1/2, 1] \Rightarrow f$ is uniformly continous on $[1/2, 1]$ (from Heine theorem)

That is, $\forall u, v \in [1/2, 1] : |f(u) - f(v)| = \left|\frac{1}{u} - \frac{1}{v}\right| = \left|\frac{u-v}{uv}\right| < 4|u - v| < \varepsilon$ $\left(x, x_0 \in [1/2, 1] \Rightarrow x, x_0 \geq \frac{1}{2} \Rightarrow x x_0 \geq \frac{1}{4}\right)$

Thus, it is enough to take $\delta = \varepsilon/4$ such that f is uniformly continous on $[1/2, 1]$.

Example ② Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 7x - 2$, then f is uniformly continous on $\mathbb{R}$:

Let $\varepsilon > 0$ and choose $\delta = \frac{\varepsilon}{7}$. Then, if $u, v \in \mathbb{R}$ and $|u - v| < \delta$, we have:

$|f(u)-f(v)| = |7u-2-(7v-2)| = |7(u-v)| = 7|u-v| < 7\delta \leq \varepsilon$.

Therefore, f is uniformly continous on $\mathbb{R} \Rightarrow f$ is continuous at every pt $x \in \mathbb{R}$.

Theorem (7) If the function f is continuous on the closed interval $[a,b]$, then f is bounded on this interval, i.e.,

$\exists M > 0, \forall x \in [a,b] : |f(x)| \leq M$.

Example: the function $f(x) = tg\, x$ is continous on the closed interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$, then f is bounded on $[-\frac{\pi}{4}, \frac{\pi}{4}]$ and we have:

$\forall x \in [-\frac{\pi}{4}, \frac{\pi}{4}] : \quad -1 \leq f(x) \leq 1 \Leftrightarrow \quad |f(x)| \leq 1$.

Theorem (8) (extreme value theorem) If f is a continuous function on D and D is a compact set. Then f has an absolute minimum and an absolute maximum on D.

(Note that D compact set $= D$ closed and bounded set $= [a,b]$). i.e.,

$\exists x_1, x_2 \in [a,b] = D : \quad f(x_1) = \sup_{x \in [a,b]} f(x)$ and $f(x_2) = \inf_{x \in [a,b]} f(x)$

proof) Let f be continuous on $[a,b]$ then f is bounded on $[a,b]$ (from thm (7)) its supremum is denoted by M, i.e., $\sup f = M$, and assume that f does not reach its supremum, i.e., $\forall x \in [a,b] : f(x) < M$ (not $f(x) \leq M$)

Let $g(x) = \frac{1}{M - f(x)}$ be continuous function on $[a,b]$ (f is continuous on $[a,b]$ and $\forall x \in [a,b] : f(x) - M \neq 0$ ($f(x) - M < 0$))

then from theorem (7), g is bounded on $[a,b]$, that is:

$\exists c > 0, \forall x \in [a,b] : g(x) \leq c \Leftrightarrow \frac{1}{M - f(x)} \leq c \Rightarrow f(x) \leq M - \frac{1}{c} = M' < M$.

i.e., $M - \frac{1}{c}$ is an upper bound of f on $[a,b]$, contradiction with M is the least upper bounds of f ($M' < M$)

Therefore f reaches its supremum (least upper bound), that is

$\exists x_1 \in [a,b] : \quad f(x_1) = \sup_{[a,b]} f$

In the same way, we prove that f reaches its greatest lower bounds

Example) Let $f(x) = x$ be continuous on $[0,1[$, then f does not reach its supremum because $f(1) = 1 \notin [0,1[$, but $\inf f = 0$ and $f(0) = 0 \in [0,1[$

$\Rightarrow f$ reaches its infimum.

Theorem (9): (Intermediate value theorem)

If a function f is continuous over a closed interval $[a,b]$ such that $f(a).f(b)<0$, then $\exists\, c \in]a,b[: f(c)=0$

Example: prove that the equation $2x^3-5x+2=0$ has at least one solution on $[0,1]$?

solution: f is continuous and defined on $[0,1]$ and $f(1)=-1<f(0)=2$ then from theorem (I.V.T) $\exists\, c\in]0,1[; f(c)=0 \Leftrightarrow 2x^3-5x+2=0$.

Theorem (10) (second Intermediate value theorem)

Let f be a continuous function defined on $[a,b]$ and let S be a number with $f(a)<S<f(b)$. Then there exists some x between a and b such that $f(x)=S$. $(\exists\, x\in]a,b[: f(x)=S)$

Proof: Let $g(x)=f(x)-S$, then g is continuous on $[a,b]$ since f is continuous on $[a,b]$. Also, we have:

$g(a)=f(a)-S<0$ and $g(b)=f(b)-S>0$ (from $f(a)<S<f(b)$)

$\Rightarrow g(a)g(b)<0$

Thus, from theorem of I.V, $\exists\, x\in]a,b[; g(x)=0 \Leftrightarrow f(x)=S$.

Example: prove that the equation $x^2=7$ has at least one solution on $[1,3]$.

solution: let $g(x)=x^2-7 \Rightarrow g$ is continuous on $[1,3]$ (polynomial function) and $g(1)=-6<0$ and $g(3)=2>0 \Rightarrow g(1)g(3)<0$, then, from IVT (them (9)), $\exists\, c\in]1,3[; g(c)=0 \Leftrightarrow c^2=7$

Remark: (from IVT, we can conclude that)

1. The image of a domain interval $D\subseteq\mathbb{R}$ by a continuous function f is a domain interval.
2. The image of a closed interval $[a,b]$ by a continuous function f is a closed interval.

6/ the inverse function:

Theorem (11): Let $f:[a,b]\to\mathbb{R}$ be strictly increasing and continuous on $[a,b]$. Let $c=f(a)$ and $d=f(b)$. Then f is one-to-one, $f([a,b])=[c,d]$ and the inverse function f^{-1} defined on $[c,d]$ by $f^{-1}(f(x))=x$, where $x\in[a,b]$, is continuous function from $[c,d]$ onto $[a,b]$.

Example: Let $f(x)=x^2-3x+2$, $\forall x\in[a,b]$.

what are the cdts that a and b must verify such that f has its inverse funct? give it?

solution: we have f is continuous defined function on $\mathbb{R}$ (polynomial func) hence f is strictly increasing for $x>\frac{3}{2}$ and strictly decreasing for $x<3/2$, then from the above thm f has the inverse funct on $f([a,b])$ if $a<b<3/2$ or $\frac{3}{2}<a<b$

- The inverse function of f is: $f^{-1}(x)=3/2\pm\sqrt{x+1/4}$, $\forall x\geq -\frac{1}{4}$

since: $f(x)=y \Leftrightarrow x=f^{-1}(y)$ ~~$(x^2-3x+$~~

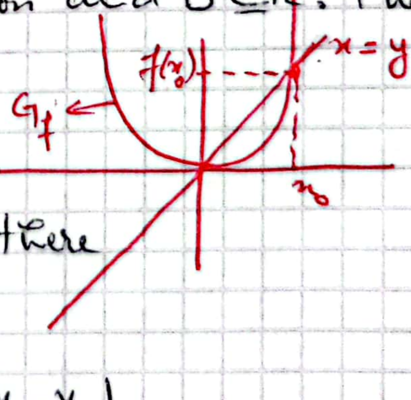
thus: $x^2-3x+2=y \Leftrightarrow (x-3/2)^2=y+\frac{1}{4} \Rightarrow x-3/2=\pm\sqrt{y+1/4}$

$\Leftrightarrow \boxed{x=3/2\pm\sqrt{y+1/4}}$, $\forall y\geq -1/4$

7/ Banach fixed point theorem:

in this section, we will give an important theorem in functional Analysis, which is fixed point theorem.

Definition (1) Let $f:D\to\mathbb{R}$ be a real function and $D\subseteq\mathbb{R}$. The point x_0 is said fixed point if: $f(x_0)=x_0$



Definition (2): Let D be a nonempty subset of $\mathbb{R}$. A function $f:D\to\mathbb{R}$ is said Lipschitzian if there is a constant $k\geq 0$ such that:

$$\forall x_1,x_2\in D: \quad |f(x_1)-f(x_2)|\leq k|x_1-x_2|.$$

Definition (3): Let $D\subseteq\mathbb{R}$, $D\neq\phi$. A function $f:D\to\mathbb{R}$ is called a contraction if there exists $k\in[0,1[$ such that:

$$\forall x_1,x_2\in D: \ |f(x_1)-f(x_2)|\leq k|x_1-x_2|$$

Theorem ①: If a function $f: D \to \mathbb{R}$ is Lipshitzian, then it is uniformly continuous on D. It is enough to take $\delta = \frac{\varepsilon}{k}$.

Remark: ~~All~~ contractive function is Lipschitzian function.

Theorem ② (Banach fixe point theorem).

Let f be a contractive function on the closed interval $[a,b]$ ($f: [a,b] \to [a,b]$), then f has a unique fixed point on $[a,b]$.

proof: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $(x_n) \subset [a,b]$ and $x_{n+1} = f(x_n)$ and let x_0 be an element of $[a,b]$ ($x_0 \in [a,b]$).

f is a contraction then:

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq k|x_n - x_{n-1}|$$
$$= k|f(x_{n-1}) - f(x_{n-2})| \leq k^2 |x_{n-1} - x_{n-2}| \leq \cdots$$
$$\leq k^n |x_1 - x_0|, \ \forall n \in \mathbb{N}.$$

Hence: $|x_{n+1} - x_n| \leq k^n |x_1 - x_0|, \ \forall n \in \mathbb{N}$ ——— $(*)$

$$|x_{n+p} - x_n| = |x_{n+p} - x_{n+p-1} + x_{n+p-1} + \cdots - x_{n+1} + x_{n+1} - x_n|$$
$$\overset{T.I}{\leq} |x_{n+p} - x_{n+p-1}| + \cdots + |x_{n+1} - x_n|$$
$$\overset{(*)}{\leq} \left(k^{n+p-1} + k^{n+p-2} + \cdots + k^{n+1} + k^n\right) |x_1 - x_0|$$
$$= \left(k^{p-1} + k^{p-2} + \cdots + k + 1\right) k^n |x_1 - x_0|$$
$$= \frac{1-k^p}{1-k} k^n |x_1 - x_0|$$
$$\leq \frac{k^n}{1-k} |x_1 - x_0| \quad (1 - k^p \leq 1, \ k \in [0,1[)$$

$\Rightarrow (x_n)$ is a sequence of cauchy $\Rightarrow (x_n)$ is convergent, i.e., $\lim_{n\to+\infty} x_n = \bar{x}$.

• Since $(x_n) \subset [a,b]$ then $\bar{x} \in [a,b]$.

• f is a contractive function $\Rightarrow f$ is Lipschitzian $\Rightarrow f$ is uniformly continuous on $[a,b]$ $\Rightarrow f$ is continuous on $[a,b]$, hence

$\lim_{n\to+\infty} x_n = \bar{x} \Rightarrow \lim_{n\to+\infty} f(x_n) = f(\bar{x})$ ——— ①

on the other hand: $x_{n+1} = f(x_n) \Rightarrow \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} f(x_n)$ ——— ②

from ① and ②, we deduce that $f(\bar{x}) = \bar{x} \Rightarrow \bar{x}$ is a fixed point.

Theorem ①: If a function $f: D \to \mathbb{R}$ is Lipschitzian, then it is uniformly contin[uous] on D. It is enough to take $\delta = \frac{\varepsilon}{k}$.

Remark: ~~all~~ contractive function is Lipschitzian function.

Theorem ② (Banach fixe point theorem).

Let f be a contractive function on the closed interval $[a, b]$ ($f: [a,b] \to [a,b]$), then f has a unique fixed point on $[a,b]$.

proof: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $(x_n) \subset [a,b]$ and $x_{n+1} = f(x_n)$ and let x_0 be an element of $[a,b]$ ($x_0 \in [a,b]$).

f is a contraction then:

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq k|x_n - x_{n-1}|$$
$$= k|f(x_{n-1}) - f(x_{n-2})| \leq k^2|x_{n-1} - x_{n-2}| \leq \cdots$$
$$\leq k^n |x_1 - x_0|, \forall n \in \mathbb{N}.$$

Hence: $|x_{n+1} - x_n| \leq k^n |x_1 - x_0|, \forall n \in \mathbb{N}$ —— (*)

$$|x_{n+p} - x_n| = |x_{n+p} - x_{n+p-1} + x_{n+p-1} + \cdots - x_{n+1} + x_{n+1} - x_n|$$
$$\overset{T.I}{\leq} |x_{n+p} - x_{n+p-1}| + \cdots + |x_{n+1} - x_n|$$
$$\overset{(*)}{\leq} \left(k^{n+p-1} + k^{n+p-2} + \cdots + k^{n+1} + k^n\right)|x_1 - x_0|$$
$$= \left(k^{p-1} + k^{p-2} + \cdots + k + 1\right) k^n |x_1 - x_0|$$
$$= \frac{1-k^p}{1-k} k^n |x_1 - x_0|$$
$$\leq \frac{k^n}{1-k}|x_1 - x_0| \quad \left(1 - k^p < 1 \},\ k \in [0, 1[\right)$$

$\Rightarrow (x_n)$ is a sequence of Cauchy $\Rightarrow (x_n)$ is convergent, i.e, $\lim_{n\to+\infty} x_n = \overline{x}$.

Since $(x_n) \subset [a,b]$ then $\overline{x} \in [a,b]$.

f is a contractive function $\Rightarrow f$ is Lipschitzian $\Rightarrow f$ is uniformly continuous on $[a,b]$ $\Rightarrow f$ is continuous on $[a,b]$, hence

$$\lim_{n\to+\infty} x_n = \overline{x} \Rightarrow \lim_{n\to+\infty} f(x_n) = f(\overline{x}) \text{ —— ①}$$

on the other hand: $x_{n+1} = f(x_n) \Rightarrow \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} f(x_n)$ —— ②

from ① and ②, we deduce that $f(\overline{x}) = \overline{x} \Rightarrow \overline{x}$ is a fixed point.

also, we have $(x_n) \xrightarrow{CV} \overline{x}$ and from the uniqueness of the limit of the convergent sequence, we conclude that $\overline{x}$ is a unique fixed pt.