Numerical sequences and series

1 Numerical Sequences

Definition 1 *A sequence of real numbers (or a sequence on $\mathbb{R}$) is a function defined on the set $\mathbb{N}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers, i.e:*

$$\begin{aligned} u : \mathbb{N} &\rightarrow \mathbb{R} \\ n &\mapsto u_n = u(n). \end{aligned}$$

This application is denoted in index form:

$$(u_n)_{n\in\mathbb{N}} \quad or \quad (u_n)_{n\geq 0}.$$

Example 1 *The formula $u_n = \dfrac{1}{n}$ defines a sequence whose first terms are: $u_1 = 1,\ u_2 = \dfrac{1}{2},\ u_3 = \dfrac{1}{3}, \cdots$*

Definition 2 (Operation on sequences) *We define the following laws on the set of sequences:*

1. $(u_n)_{n\in\mathbb{N}} + (v_n)_{n\in\mathbb{N}} = (u_n + v_n)_{n\in\mathbb{N}}$.

   Example 2

$$\left(\frac{n+1}{n}\right)_{n\in\mathbb{N}^*} + \left(\frac{1}{n}\right)_{n\in\mathbb{N}^*} = \left(\frac{n+1}{n} + \frac{1}{n}\right)_{n\in\mathbb{N}^*} = \left(\frac{n+2}{n}\right)_{n\in\mathbb{N}^*}.$$

2. $\lambda(u_n)_{n\in\mathbb{N}} = (\lambda u_n)_{n\in\mathbb{N}}$.

   Example 3

$$4\left(\frac{n+1}{n}\right)_{n\in\mathbb{N}^*} = \left(\frac{4(n+1)}{n}\right)_{n\in\mathbb{N}^*} = \left(\frac{4n+4}{n}\right)_{n\in\mathbb{N}^*}.$$

3. $(u_n)_{n\in\mathbb{N}}(v_n)_{n\in\mathbb{N}} = (u_n v_n)_{n\in\mathbb{N}}$.

Example 4

$$\left(\frac{n+1}{n}\right)_{n\in\mathbb{N}^*}\left(\frac{1}{n}\right)_{n\in\mathbb{N}^*}=\left(\frac{n+1}{n}\,\frac{1}{n}\right)_{n\in\mathbb{N}^*}=\left(\frac{n+1}{n^2}\right)_{n\in\mathbb{N}^*}.$$

4. $\dfrac{(u_n)_{n\in\mathbb{N}}}{(v_n)_{n\in\mathbb{N}}}=\left(\dfrac{u_n}{v_n}\right)_{n\in\mathbb{N}},\qquad v_n\neq 0,\ \forall\ n\in\mathbb{N}.$

Example 5

$$\frac{\left(\frac{n+1}{n}\right)_{n\in\mathbb{N}^*}}{\left(\frac{1}{n}\right)_{n\in\mathbb{N}^*}}=\left(\frac{\frac{n+1}{n}}{\frac{1}{n}}\right)_{n\in\mathbb{N}^*}=(n+1)_{n\in\mathbb{N}^*}.$$

5. $\dfrac{1}{(u_n)_{n\in\mathbb{N}}}=\left(\dfrac{1}{u_n}\right)_{n\in\mathbb{N}},\qquad u_n\neq 0,\ \forall\ n\in\mathbb{N}.$

Example 6

$$\frac{1}{\left(\frac{1}{n}\right)_{n\in\mathbb{N}^*}}=\left(\frac{1}{\frac{1}{n}}\right)_{n\in\mathbb{N}^*}=(n)_{n\in\mathbb{N}^*}.$$

Definition 3 (Bounded sequences)

- *We say that* $(u_n)_{n\in\mathbb{N}}$ *is bounded above* $\Leftrightarrow \exists\ M\in\mathbb{R},\ \forall\ n\in\mathbb{N}:\ u_n\leq M.$
- *We say that* $(u_n)_{n\in\mathbb{N}}$ *is bounded below* $\Leftrightarrow \exists\ m\in\mathbb{R},\ \forall\ n\in\mathbb{N}:\ u_n\geq m.$
- *We say that* $(u_n)_{n\in\mathbb{N}}$ *is bounded* $\Leftrightarrow$ $(u_n)_{n\in\mathbb{N}}$ *is bounded below and above in the same time.*

Example 7 *Consider the sequence* $(u_n)_{n\in\mathbb{N}}$*, such that:* $u_n=\dfrac{1}{n}$*. Then,* $u_n\leq 1$*, for all* $n\in\mathbb{N}^*$*.*
Thus, $(u_n)_{n\in\mathbb{N}}$ *is bounded above by* 1*.*
On the other hand, $u_n>0$*, for all* $n\in\mathbb{N}^*$*. Then,* $(u_n)_{n\in\mathbb{N}}$ *is bounded below by* 0*.*
Therefore, $0<u_n\leq 1$*, for all* $n\in\mathbb{N}^*$*.* $(u_n)_{n\in\mathbb{N}}$ *is bounded.*

Proposition 1 *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is bounded* $\Leftrightarrow \exists\ M\in\mathbb{R},\ \forall\ n\in\mathbb{N}:\ |u_n|\leq M.$

Example 8

1. *The sequence* $((-1)^n)_{n\in\mathbb{N}}$ *is bounded:*

$$\forall\ n\in\mathbb{N}:\ |(-1)^n|=1\leq 1.$$

2. *The sequence* $(\sin(n))_{n\in\mathbb{N}}$ *is bounded:*

$$\forall\ n\in\mathbb{N}:\ |\sin(n)|\leq 1.$$

Definition 4 (Monotonic sequences)

- *We say that* $(u_n)_{n\in\mathbb{N}}$ *is increasing (strictly increasing) if:*

$$\forall\, n \in \mathbb{N}: \quad u_{n+1} - u_n \geq 0, \qquad (\forall\, n \in \mathbb{N}: \quad u_{n+1} - u_n > 0).$$

- *We say that* $(u_n)_{n\in\mathbb{N}}$ *is decreasing (strictly decreasing) if:*

$$\forall\, n \in \mathbb{N}: \quad u_{n+1} - u_n \leq 0, \qquad (\forall\, n \in \mathbb{N}: \quad u_{n+1} - u_n < 0).$$

- *We say that* $(u_n)_{n\in\mathbb{N}}$ *is monotonic if it is increasing or decreasing.*

Example 9

1. *Let* $u_n = n + 1,\ \forall\, n \in \mathbb{N}$. *Then,*

$$u_{n+1} - u_n = (n+2) - (n+1) = n + 2 - n - 1 = 1 > 0.$$

Thus, $(u_n)_{n\in\mathbb{N}}$ *is strictly increasing.*

2. *Let* $u_n = 1 + \dfrac{1}{n},\ \forall\, n \in \mathbb{N}^*$. *Then,*

$$u_{n+1} - u_n = 1 + \frac{1}{n+1} - 1 - \frac{1}{n} = -\frac{1}{n(n+1)} < 0.$$

Thus, $(u_n)_{n\in\mathbb{N}}$ *is strictly decreasing.*

Definition 5 (Limits, convergent and divergent of sequences)

- *We say that* $(u_n)_{n\in\mathbb{N}}$ *is convergent to* $l \in \mathbb{R}$, *if*

$$\forall\, \varepsilon > 0,\ \exists\, p \in \mathbb{N},\ \forall\, n \geq p:\ |u_n - l| < \varepsilon$$

and we write: $\lim\limits_{n\to+\infty} = l$ *or* $u_n \underset{n\to+\infty}{\longrightarrow} l$.

- *If there is no real number* l *satisfying the above property, we say that the sequence is divergent.*

Example 10 *The sequence* $u_n = \dfrac{1}{n}$, *for all* $n \in \mathbb{N}^*$ *is convergent to* 0.

Proposition 2 *If* $(u_n)_{n\in\mathbb{N}}$ *has a limit, then this limit is unique.*

Theorem 1 *Let* $(u_n)_{n\in\mathbb{N}}$ *be an increasing (resp. decreasing) sequence. Then,*

$$(u_n)_{n\in\mathbb{N}} \text{ is convergent } \Longleftrightarrow (u_n)_{n\in\mathbb{N}} \text{ is bounded above (resp. bounded below).}$$

Example 11 *Let* $u_n = \dfrac{1}{n^2+1}$, *which is decreasing and bounded below by* 0. *Then,* $(u_n)_{n\in\mathbb{N}}$ *is convergent.*

Theorem 2 *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences converging to l and l' respectively, i.e: $\lim_{n\to+\infty} u_n = l$ and $\lim_{n\to+\infty} v_n = l'$.*

- *The sequence $(u_n + v_n)$ converges to $l + l'$.*
- *The sequence $(u_n \times v_n)$ converges to $l \times l'$.*
- *The sequence (λu_n) converges to λl, $\lambda \in \mathbb{R}$.*
- *The sequence $(|u_n|)$ converges to $|l|$.*
- *$\forall n \in \mathbb{N},\ u_n \neq 0$ and $l \neq 0$, the sequence $\left(\frac{1}{u_n}\right)$ converges to $\frac{1}{l}$.*
- *If (u_n) converges to l and $u_n \geq 0$, then $l \geq 0$.*
- *If (u_n) converges to l and $u_n \leq 0$, then $l \leq 0$.*

1.1 Convergence Theorems

Theorem 3 (Comparaison Rule)

- *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two convergent sequences, such that $\forall n \in \mathbb{N}:\ u_n \leq v_n$. Then, $\lim_{n\to+\infty} u_n \leq \lim_{n\to+\infty} v_n$.*
- *If $u_n \leq v_n,\ \forall\ n \in \mathbb{N}$. Then, $\lim_{n\to+\infty} u_n = +\infty \Longrightarrow \lim_{n\to+\infty} v_n = +\infty$.*

Theorem 4 (Gendarme Theorem) *Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be three sequences, such that*

$$\forall n \in \mathbb{N}:\ u_n \leq w_n \leq v_n \quad \text{and} \quad \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = l \in \mathbb{R}.$$

Then, $(w_n)_{n\in\mathbb{N}}$ is convergent and we have:

$$\lim_{n\to+\infty} w_n = l.$$

1.2 Specific sequences

(*a*) **Constant sequences:** The sequence $(u_n)_{n\in\mathbb{N}}$ is constant if

$$\forall n \in\in \mathbb{N}:\ u_{n+1} = u_n.$$

(*b*) **Arithmetic sequences:** The sequence $(u_n)_{n\in\mathbb{N}}$ is said to be arithmetic sequence if there exists $r \in \mathbb{R}$, such that

$$\forall n \in \mathbb{N}:\ u_{n+1} = u_n + r.$$

The general term of the sequence can be obtained based on its first term u_0 and r: $u_n = u_0 + nr$.
The sum of the first n term of the sequence is given by:

$$S_n = u_0 + u_1 + u_2 + \cdots + u_{n-1} = \sum_{k=0}^{n-1} u_k = \frac{n}{2}(u_0 + u_n).$$

(c) **Geometric sequences:** The sequence $(u_n)_{n\in\mathbb{N}}$ is said to be geometric sequence if there exists $q \in \mathbb{R}$, such that

$$\forall n \in \mathbb{N} : \ u_{n+1} = qu_n.$$

The general term of the sequence can be obtained based on its first term u_0 and q: $u_n = u_0 q^n$.
The sum of the first n term of the sequence is given by:

$$S_n = u_0 + u_1 + u_2 + \cdots + u_{n-1} = \sum_{k=0}^{n-1} u_k = u_0 \sum_{k=0}^{n-1} q^k = u_0 \frac{1-q^n}{1-q}.$$

2 Numerical Series

2.1 Real term series

Definition 6 *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence, we put*

$$S_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^{n} u_k.$$

To study the series of general term u_n is to study the sequence (S_n).
(S_n) is called the sequence of partial sums of the series.

Notations: A series of general term u_n is noted by:

$$\left(\sum_n u_n\right) \quad \text{or} \quad \left(\sum_{n\geq 0} u_n\right).$$

(a) Convergence:

Definition 7 *A series of general term u_n is said to be convergent if the sequence $(S_n)_n$ is convergent.*
In this case, the limit of the sequence $(S_n)_n$ is called the sum of the series and we note:

$$\lim_{n\to+\infty} S_n = \sum_{n=0}^{+\infty} u_n.$$

A series that is not convergent is said to be divergent. On the other words, if we note: $l = \lim_{n\to+\infty} S_n$, then we have

$$\left(\sum_{n\geq 0} u_n\right) \quad \textit{is convergent towards} \quad l \Longleftrightarrow \lim_{n\to+\infty} S_n = l,$$

which is equivalent to

$$\forall \varepsilon > 0, \ \exists \ p \in \mathbb{N}, \ \forall \ n \in \mathbb{N}, \ n \geq p \Longrightarrow |S_n - l| < \varepsilon.$$

Example 12

1. *Geometric series: the general term has the form* $u_n = aq^n,\ a \neq 0$. *Then,*

$$\begin{aligned} S_n &= u_0 + u_1 + u_2 + \cdots + u_n = a + aq + aq^2 + \cdots + aq^n = a(1 + q + q^2 + \cdots q^n) \\ &= \begin{cases} a\dfrac{1-q^{n+1}}{1-q}, & q \neq 1 \\ a(n+1), a & q = 1. \end{cases} \end{aligned}$$

To calculate $\lim_{n\to+\infty} S_n$ *we get 3 cases:*

- *If* $q = 1 :\ \lim_{n\to+\infty} S_n = +\infty$, *then* $(S_n)_n$ *is divergent.*
- *If* $-1 < q < 1,\ \lim_{n\to+\infty} S_n = \dfrac{a}{1-q}$, *then* $(S_n)_n$ *is convergent.*
- *If* $|q| > 1 :\ \lim_{n\to+\infty} S_n = -\infty$, *then* $(S_n)_n$ *is diveregent.*

2. *The general term series* $u_n = \dfrac{1}{n(n+1)},\ n \geq 1$. *We can write* $(u_n)_n$ *like:* $u_n = \dfrac{1}{n} - \dfrac{1}{n+1}$.
Then, the partial sum of the series is given by:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n-2} - \frac{1}{n-1}) + (\frac{1}{n-1} - \frac{1}{n}) + (\frac{1}{n} - \frac{1}{n+1}) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Then,

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} (1 - \frac{1}{n+1}) = 1.$$

Thus, $(\sum u_n)_n$ *is convergent.*

Proposition 3 *Let* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *are two series. We assume that these two series differ only by one finite number of terms (i.e;* $\exists\ p \in \mathbb{N},\ \forall n \geq p$, *we have* $u_n = v_n$*), then* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *are of the same nature (convergent or divergent).*

Remark 1 *The above proposition allows us to say that the series are of the same nature but in the case of convergence, they do not necessarily have the same sum.*

Proposition 4

- $(\sum u_n)_n$ *is convergent* $\Rightarrow \lim_{n\to+\infty} u_n = 0$. *The receprocal is false; (i.e;* $\lim_{n\to+\infty} u_n = 0 \not\Rightarrow (\sum u_n)_n$ *is convergent).*
- $\lim_{n\to+\infty} u_n \neq 0 \Rightarrow (\sum u_n)_n$ *is divergent.*

Example 13 *Let* $u_n = \frac{1}{n}$, $n \geq 1$, *we have* $\lim_{n\to+\infty} \frac{1}{n} = 0$, *but* $(\sum \frac{1}{n})$ *is divergent (harmonic series).*

(b) Operations on series:

Theorem 5 *Let* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *be two series. Then, we have the following properties:*

- *If* $(\sum u_n)_n$ *is convergent to* S_1 *and if* $(\sum v_n)_n$ *is convergent to* S_2*. Then,* $(\sum u_n + v_n)_n$ *is convergent to* $S_1 + S_2$.
- *If* $(\sum u_n)_n$ *is convergent to* S_1 *and if* $a \in \mathbb{R}$, *then,* $(\sum au_n)_n$ *is convergent to* aS_1.
- *If* $(\sum u_n)_n$ *is convergent and* $(\sum v_n)_n$ *is divergent, then* $(\sum u_n + v_n)_n$ *is divergent.*
- *If* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *are divergent, then we can't conclude anytying about the nature of the series* $(\sum u_n + v_n)_n$.

2.2 Positive term series

Definition 8 *A series* $(\sum u_n)_n$ *is called a series with positive terms if* $u_n \geq 0$, *for all* $n \in \mathbb{N}$.

Proposition 5 *Let* $(\sum u_n)_n$ *be a series with positive terms, then*

$$(\sum u_n)_n \quad \text{is convergent} \iff (S_n)_n \quad \text{is bounded above.}$$

Theorem 6 (Comparaison Rule) *Let* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *be two series with positive terms. Suppose that:* $0 \leq u_n \leq v_n$, $\forall\ n \in N$. *Then,*

- $(\sum v_n)_n$ *is convergent, then* $(\sum u_n)_n$ *is convergent.*
- $(\sum u_n)_n$ *is divergent, then* $(\sum v_n)_n$ *is divergent.*

Example 14 *Let* $\sum_{n=0}^{+\infty} u_n = \sum_{n=0}^{+\infty} \sin(\frac{1}{2^n})$. *We have:* $0 \leq \sin(\frac{1}{2^n}) \leq \frac{1}{2^n}$ *and since* $(\sum \frac{1}{2^n})$ *is a geometric series with* $q = \frac{1}{2}$. *Then, it is convergent, thus* $(\sum \sin(\frac{1}{2^n}))$ *is convergent.*

Theorem 7 (Logarithmic comparaision Rule) *Let* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *be two series with strictly positive terms. Suppose that* $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$. *Then,*

- $(\sum v_n)_n$ *is convergent, then* $(\sum u_n)_n$ *is convergent.*
- $(\sum u_n)_n$ *is divergent, then* $(\sum v_n)_n$ *is divergent.*

Theorem 8 (Equivalent criteria) *Let* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *be two series with strictly positive terms. Suppose that:* $\lim_{n\to+\infty} \frac{u_n}{v_n} = l$, *such that* $l \neq 0$ *or* $l \neq \pm\infty$. *Then,* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *have the same nature.*
If $u_n \sim v_n$ *(i.e;* $\lim_{n\to+\infty} \frac{u_n}{v_n} = 1$*), then* $(\sum u_n)_n$ *and* $(\sum v_n)_n$ *have the same nature.*

Example 15

1. *Let* $u_n = \ln(1+\frac{1}{2^n})$ *and* $v_n = \frac{1}{2^n}$. *We have:* $\lim\limits_{n\to+\infty} \frac{u_n}{v_n} = \lim\limits_{n\to+\infty} \frac{\ln(1+\frac{1}{2^n})}{\frac{1}{2^n}} = 1$.

   Since $(\sum \frac{1}{2^n})$ *is convergent (geometric sereis), then* $\Big(\sum \ln(1+\frac{1}{2^n})\Big)$ *is also convergent.*

2. *Let* $u_n = \frac{1}{n}$ *and* $v_n = \ln(1+\frac{1}{n})$, *we have:* $\lim\limits_{n\to+\infty} \frac{u_n}{v_n} = \lim\limits_{n\to+\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = 1$.

   Since $(\sum \frac{1}{n})$ *is divergent (harmonic sereis), then* $\Big(\sum \ln(1+\frac{1}{n})\Big)$ *is also divergent.*

Convergence usuelle rules:

Definition 9 (Riemann series) *We called a Riemann series a series with positive terms* $(\sum\limits_{n\geq 1} \frac{1}{n^\alpha})$, *where* $\alpha \in \mathbb{R}$.

Theorem 9

$$(\sum_{n\geq 1} \frac{1}{n^\alpha}) \quad \text{is convergent} \iff \alpha > 1.$$

$$(\sum_{n\geq 1} \frac{1}{n^\alpha}) \quad \text{is divergent} \iff \alpha \leq 1.$$

Proposition 6 (Riemann Rule) *Let* $(\sum\limits_{n\geq 1} u_n)$ *be a series with positive terms and* $\alpha \in \mathbb{R}$, *such that* $\lim\limits_{n\to+\infty} n^\alpha u_n = l \in \mathbb{R}$.

- *If* $l = 0$ *and* $\alpha > 1$, *then* $(\sum\limits_{n\geq 1} u_n)$ *is convergent.*
- *If* $l = +\infty$ *and* $\alpha \leq 1$, *then* $(\sum\limits_{n\geq 1} u_n)$ *is divergent.*
- *If* $l \neq 0$ *and* $l \neq +\infty$, *then* $(\sum\limits_{n\geq 1} u_n)$ *and* $(\sum\limits_{n\geq 1} \frac{1}{n^\alpha})$ *have the same nature.*

Example 16 *Let* $u_n = \frac{1}{n^2 \ln(n)}$. *Then,* $\lim\limits_{n\to+\infty} n^2 u_n = \lim\limits_{n\to+\infty} \frac{1}{\ln(n)} = 0$. *Thus,* $\Big(\sum\limits_{n\geq 1} \frac{1}{n^2 \ln(n)}\Big)$ *is convergent with* $\alpha = 2 \geq 1$.

Proposition 7 (D'Alembert Rule) *Let* $(\sum\limits_{n\geq 1} u_n)$ *be a series with strictly positive terms, such that* $\lim\limits_{n\to+\infty} \frac{u_{n+1}}{u_n} = l \in \mathbb{R}$.

- *If* $l < 1$, *then* $(\sum_{n\geq 1} u_n)$ *is convergent.*
- *If* $l > 1$, *then* $(\sum_{n\geq 1} u_n)$ *is divergent.*
- *If* $l = 1$, *then we can't say anything.*

Example 17

1. *Let* $u_n = \dfrac{1}{n!}$, *then* $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to+\infty} \dfrac{1}{n+1} = 0 < 1$. *Then,* $\left(\sum\limits_{n\geq 1} \dfrac{1}{n!}\right)$ *is convergent.*
2. *Let* $u_n = \dfrac{n^n}{n!}$. *Then,* $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to+\infty} \dfrac{(n+1)^{n+1}}{(n+1)!} \dfrac{n!}{n^n} = \lim\limits_{n\to+\infty} \left(\dfrac{n+1}{n}\right)^n = e > 1$. *Thus,* $\left(\sum\limits_{n\geq 1} \dfrac{n^n}{n!}\right)$ *is divergent.*

Proposition 8 (Cauchy Rule) *Let* $(\sum_{n\geq 1} u_n)$ *be a series with strictly positive terms, such that* $\lim\limits_{n\to+\infty} = \sqrt[n]{u_n} = \lim\limits_{n\to+\infty} (u_n)^{\frac{1}{n}} = l \in \mathbb{R}$.

- $l < 1$, *then* $(\sum_{n\geq 1} u_n)$ *is convergent.*
- $l > 1$, *then* $(\sum_{n\geq 1} u_n)$ *is divergent.*
- $l = 1$, *then we can't say anything.*

Example 18 *Let* $u_n = \left(3 + \dfrac{1}{n^4}\right)^n$, $\forall\, n \in \mathbb{N}^*$. *Then,*

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \left(3 + \frac{1}{n^4}\right) = 3 > 1.$$

Then, $(\sum_{n\geq 1} u_n)$ *is divergent.*

Dr. Kicha Abir