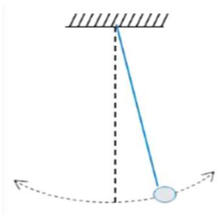
Corrigé Travaux dirigés N°3

Exercice 01

L´equation de Lagrange des systèmes suivants :



$E_c = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}ml^2\dot{\theta}^2$ et $E_p = -mgh = -mglcos\theta$

Pour les faibles oscillations $sin\theta \approx \theta$ et $cos\theta = 1-\theta^2/2$

L'énergie potentielle s'écrit :

$$E_p = -mgl + \frac{mgl}{2}\theta^2$$

Et le Lagrangien :

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\theta^2 + mgl$$

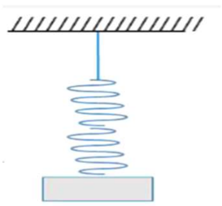
$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\theta$$

En remplaçant dans l'équation de Lagrange :

$$ml^2\ddot{\theta} + mgl\theta = 0 \quad \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

Et la pulsation libre $\omega_0 = \sqrt{\frac{g}{l}}$



$\boldsymbol{E_C = \frac{1}{2}m\dot{y}^2}$ $\boldsymbol{E_P = \frac{1}{2}ky^2}$ $\boldsymbol{L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2}$

$\boldsymbol{\ddot{y} + \frac{\kappa}{m}y = 0}$ $\boldsymbol{\omega_0 = \sqrt{\frac{\kappa}{m}}}$ $\boldsymbol{f_0 = \frac{1}{2\pi}\sqrt{\frac{\kappa}{m}}}$

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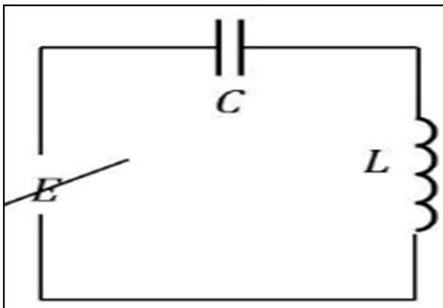

$$E_c = E_{mag} = V_L dq = \int L\frac{di}{dt}dq = \int Lidi = \frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2$$

$$E_p = E_{elec} = V_c dq = \int \frac{q}{c}dq = \frac{1}{c}q^2$$

Le Lagrangien : $L = \frac{1}{2}L\dot{q}^2 - \frac{1}{q}q^2$

L'équation de Lagrange s'écrit :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$\frac{\partial L}{\partial \dot{q}} = L\dot{q} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = L\ddot{q}$$

$$\frac{\partial L}{\partial q} = -\frac{1}{c}q$$

L'équation différentielle de mouvement :

$$L\ddot{q} + \frac{q}{c} = 0 \quad \Rightarrow \quad \ddot{q} + \frac{1}{Lc}q = 0$$

Et la pulsation propre : $\omega_0 = \sqrt{\frac{1}{Lc}}$

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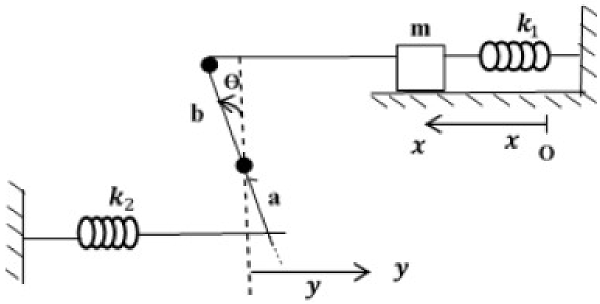

L'équation du mouvement

•L'énergie cinétique

$$T = \frac{1}{2}m\dot{x}^2$$

•L'énergie potentielle:

$$U = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2$$

Pour les petites oscillations :

$$x \simeq b\theta, \quad y \simeq a\theta$$

$$\frac{y}{x} \simeq \frac{a}{b} \Rightarrow y = \frac{a}{b}x$$

Donc : $U = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2\left(\frac{a}{b}x\right)^2 = \frac{1}{2}\left(k_1 + k_2\frac{a^2}{b^2}\right)x^2$

- Le lagrangien :

$$L = T - U$$

Donc : $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\left(k_1 + k_2\frac{a^2}{b^2}\right)x^2$

Le formalisme de Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \\ \frac{\partial L}{\partial x} = -\left(k_1 + k_2\frac{a^2}{b^2}\right)x \end{cases}$$

D'où l'équation du mouvement s'écrit : $m\ddot{x} + \left(k_1 + k_2\frac{a^2}{b^2}\right)x = 0$

$\Rightarrow$ La pulsation propre : $\omega_0 = \sqrt{\frac{k_1 + k_2\frac{a^2}{b^2}}{m}}$

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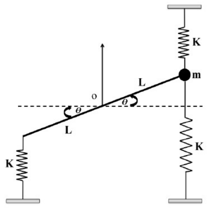

$$E_C = E_{CRot} = \frac{1}{2} J_m \dot{\theta}^2,\ J_m = m(L)^2 = mL^2$$

$$E_C = \frac{1}{2}\ mL^2\dot{\theta}^2$$

$$E_P = E_{P(K)} + E_{P(K)} + E_{P(K)}$$

$$E_p = \frac{1}{2}Kx^2 + \frac{1}{2}Kx^2 + \frac{1}{2}Kx^2$$

Faibles amplitudes $\theta \ll \Rightarrow sin\theta \approx \theta$

$x = L\sin\theta \Rightarrow x = L\,\theta$

$$E_p = \frac{1}{2}K(L\theta)^2 + \frac{1}{2}K(L\theta)^2 + \frac{1}{2}K(L\theta)^2$$

$$E_p = \frac{3}{2}KL^2\theta^2$$

Le Lagrangien:

$$L = E_c - E_P \Rightarrow L = \frac{1}{2}\ mL^2\dot{\theta}^2 - \frac{3}{2}KL^2\theta^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\left(\frac{\partial L}{\partial\dot{\theta}}\right) = mL^2\dot{\theta}\ \Rightarrow\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) =\ mL^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta} = -\frac{3}{4}KL^2\theta$$

L'équation du mouvement s'écrit :

$$mL^2\ddot{\theta} + \frac{3}{4}KL^2\theta = 0$$

$$\ddot{\theta} + \frac{3K}{4m}\theta = 0$$

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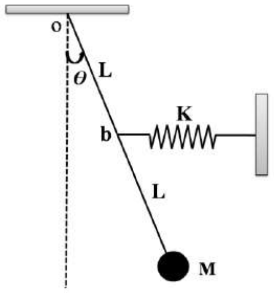


$$E_C = E_{CM} = \frac{1}{2} J_M \dot{\theta}^2, \ J_M = M(2L)^2 = 4ML^2$$

$$E_c = \frac{1}{2}(4ML^2)\dot{\theta}^2 \ \Rightarrow \ E_c = 2ML^2\dot{\theta}^2$$

$$E_P = E_{P(M)} + E_{P(K)}$$

$$E_p = Mgh + \frac{1}{2}Kx^2$$

Faibles amplitudes $\theta \ll \Rightarrow sin\theta \approx \theta, \ \cos\theta = \sqrt{1-\sin^2(\theta)} \approx 1 - \frac{\theta^2}{2}$

$x = L\sin\theta \Rightarrow x = L\theta$

$$h = 2L(1-\cos\theta) \Rightarrow h = 2L\left(1 - 1 + \frac{\theta^2}{2}\right) \Rightarrow h = L\theta^2$$

$$E_p = Mgh + \frac{1}{2}KL^2\theta^2$$

$$E_p = MgL\theta^2 + \frac{1}{2}KL^2\theta^2 \Rightarrow E_p = \frac{1}{2}(2MgL + KL^2)\theta^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \ ou \ \frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{\theta}}\right) + \frac{\partial E_p}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{\theta}}\right) + \frac{\partial E_p}{\partial \theta} = 0$$

$$\frac{dE_c}{d\dot{\theta}} = 4ML^2\dot{\theta} \ ; \ \frac{d}{dt}\left(\frac{dE_c}{d\dot{\theta}}\right) = 4ML^2\ddot{\theta}$$

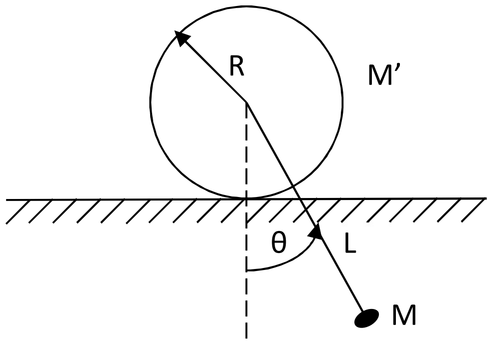
$$\frac{dE_p}{d\theta} = (2MgL + KL^2)\theta$$

$$4ML^2\ddot{\theta} + (2MgL + KL^2)\theta = 0 \ \Rightarrow \ \ddot{\theta} + \frac{(2MgL + KL^2)}{4ML^2}\theta = 0$$

$$\Rightarrow \ \ddot{\theta} + \left(\frac{g}{2L} + \frac{K}{4M}\right)\theta = 0$$

Exercice 02

Trouver la pulsation propre des vibrations des systèmes suivants :



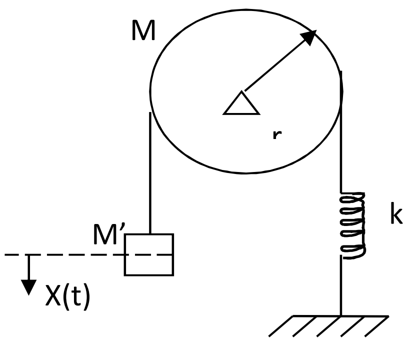
$$J_{M'} = \frac{1}{2} M' R^2$$

$$E_C = \frac{1}{2} J_{M'} \dot{\theta}^2 + \frac{1}{2} M' v^2 + \frac{1}{2} M (L-R)^2 \dot{\theta}^2 = \frac{3}{4} M' R^2 \dot{\theta}^2 + \frac{1}{2} M (L-R)^2 \dot{\theta}^2$$

$$E_P = mgl(1 - \cos\theta) = mgl \frac{\theta^2}{2}$$

$$L = \frac{3}{4} M' R^2 \dot{\theta}^2 + \frac{1}{2} M (L-R)^2 \dot{\theta}^2 - mgl \frac{\theta^2}{2}$$

$$\left(\frac{3}{2} M' R^2 + M(L-R)^2\right) \ddot{\theta} + mgl\theta = 0 \qquad \omega_0 = \sqrt{\frac{mgl}{\frac{3}{2} M' R^2 + M(L-R)^2}}$$



1- l'énergie cinétique E_c et l'énergie potentielle E_p :

$$E_c = E_M + E_m$$

$$E_c = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 \text{ , avec } x = R\theta \Rightarrow \dot{x} = R\dot{\theta}$$

$$E_c = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2$$

$$E_c = \frac{1}{2}\left(\frac{M}{2} + m\right)R^2\dot{\theta}^2$$

$$E_P = E_{P(K)} = \frac{1}{2}kx^2 \Rightarrow E_P = \frac{1}{2}kR^2\theta^2$$

2- Fonction de Lagrange :

$$L = E_c - E_P$$

$$L = \frac{1}{2}\left(\frac{M}{2} + m\right)R^2\dot{\theta}^2 - \frac{1}{2}kR^2\theta^2$$

L'équation différentielle du mouvement :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(\frac{M}{2} + m\right)R^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(\frac{M}{2} + m\right)R^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -kR^2\theta$$

Donc : $\left(\frac{M}{2} + m\right)R^2\ddot{\theta} + kR^2\theta = 0 \Rightarrow \ddot{\theta} + \frac{2k}{M+2m}\theta = 0$

3- la pulsation propre ω_0, la période propre T_0 et la fréquence propre f_0 :

La pulsation propre : $\omega_0^2 = \frac{2K}{M+2m} \Rightarrow \omega_0 = \sqrt{\frac{2K}{M+2m}} = \sqrt{\frac{120}{3}} = 6{,}324 \text{ rad.s}^{-1}$

La période propre T_0 **:** $T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{M+2m}{2K}} = 2\pi\sqrt{\frac{3}{120}} = 0{,}992 \text{ s}$

La fréquence propre f_0 : $f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2K}{M+2m}} = \frac{1}{2\pi}\sqrt{\frac{120}{3}} = 1{,}007 \text{ Hz}$

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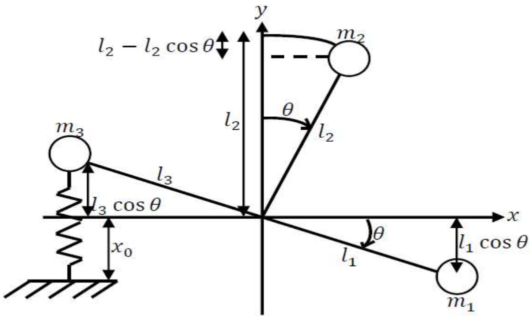


a)Energie potentielle :

$$E_p^{syst} = E_p^{m_1} + E_p^{m_2} + E_p^{m_3} + E_p^{ressort}$$

$$E_p^{m_1} = -m_1 g l_1 \sin\theta \simeq -m_1 g l_1 \theta \quad / \quad \theta \lll$$

$$E_p^{m_2} = -m_2 g l_2 (1-\cos\theta)$$

$$E_p^{m_3} = m_3 g l_3 \sin\theta \simeq m_3 g l_3 \theta \quad / \quad \theta \lll$$

$$E_p^{ressort} = \frac{1}{2}k[(x_0 + l_3\sin\theta)^2]$$

$$\boxed{E_p^{syst} = (m_3 g l_3 - m_1 g l_1)\theta - m_2 g l_2(1-\cos\theta) + \frac{1}{2}k(l_3\theta + x_0)^2}$$ Energie potentielle du système

$$\boxed{E_p^{syst} = -m_2 g l_2(1-\cos\theta) + \frac{1}{2}k{l_3}^2\theta^2 + Cst \Rightarrow E_p^{syst} = -\frac{1}{2}m_2 g l_2\theta^2 + \frac{1}{2}k{l_3}^2\theta^2 + Cst}$$

$$E_c^{syst} = E_c^{m_1} + E_c^{m_2} + E_c^{m_3} = \frac{1}{2}(m_1{l_1}^2 + m_2{l_2}^2 + m_3{l_3}^2)\dot{\theta}^2$$

$$\boxed{\mathcal{L} = \frac{1}{2}(m_1{l_1}^2 + m_2{l_2}^2 + m_3{l_3}^2)\dot{\theta}^2 + m_2 g l_2(1-\cos\theta) - \frac{1}{2}k{l_3}^2\theta^2 - Cst}$$

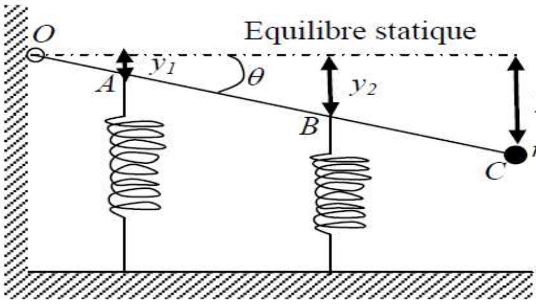
e)$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \Rightarrow$équation du mouvement

$$\Rightarrow (m_1{l_1}^2 + m_2{l_2}^2 + m_3{l_3}^2)\ddot{\theta} - m_2 g l_2\sin\theta + k{l_3}^2\theta = 0$$

$\Rightarrow \boxed{(m_1{l_1}^2 + m_2{l_2}^2 + m_3{l_3}^2)\ddot{\theta} + (k{l_3}^2 - m_2 g l_2)\theta = 0}$ équation de mouvement

$$\Rightarrow \ddot{\theta} + \frac{k{l_3}^2 - m_2 g l_2}{m_1{l_1}^2 + m_2{l_2}^2 + m_3{l_3}^2}\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + {\omega_0}^2\theta = 0$$

$$\Rightarrow \quad {\omega_0}^2 = \frac{k{l_3}^2 - m_2 g l_2}{m_1{l_1}^2 + m_2{l_2}^2 + m_3{l_3}^2} > 0$$



1- Energie cinétique et potentielle :

$$E_c = \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 \quad \text{avec } y = l\sin\theta \approx l\theta$$

$$E_p = E_{p1} + E_{p2} = \frac{1}{2}k_1 y_1^2 + \frac{1}{2}k_2 y_2^2 \quad \text{avec } y_1 = l_1\sin\theta \approx l_1\theta, \quad y_2 = l_2\sin\theta \approx l_2\theta$$

$$E_p = \frac{1}{2}k_1(l_1\theta)^2 + \frac{1}{2}k_2(l_2\theta)^2$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}k_1 l_1^2\theta^2 - \frac{1}{2}k_2 l_2^2\theta^2$$

2- L'équation de Lagrange est donnée par :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

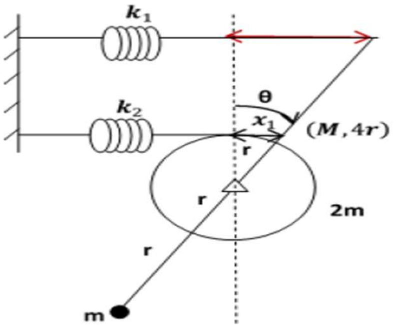
$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -k_1 l_1^2\theta - k_2 l_2^2\theta$$

3- L'équation du mouvement s'écrit :

$$ml^2\ddot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{(k_1 l_1^2 + k_2 l_2^2)}{ml^2}\theta = 0$$

Et la pulsation propre : $\omega_0 = \sqrt{\frac{(k_1 l_1^2 + k_2 l_2^2)}{ml^2}}$



- $T_m = \frac{1}{2} J_m \dot{\theta}^2$ où $J_m = m(2r)^2 = 4mr^2$

Donc : $T_m = \frac{1}{2} m(2r)^2 \dot{\theta}^2 = \frac{1}{2}(4m) r^2 \dot{\theta}^2$

- $T_{2m} = \frac{1}{2} J_{2m} \dot{\theta}^2$ où $J_{2m} = \frac{1}{2}(2m) r^2 = mr^2$

Donc : $T_{2m} = \frac{1}{2}(m) r^2 \dot{\theta}^2$

- $T_M = \frac{1}{2} J_M \dot{\theta}^2$ où $J_M = \frac{1}{12} ML^2 = \frac{1}{12} M(4r)^2 = \frac{4}{3} Mr^2$

Donc : $T_M = \frac{1}{2}\left(\frac{4}{3}M\right) r^2 \dot{\theta}^2$

$$T_{tot} = \frac{1}{2}\left(4m + m + \frac{4}{3}M\right) r^2 \dot{\theta}^2 = \frac{1}{2}\left(5m + \frac{4}{3}M\right) r^2 \dot{\theta}^2$$

- L'énergie potentielle :

$$U_{tot} = U_m + \overbrace{U_{2m}}^{0} + \overbrace{U_M}^{0} + U_{k_1} + U_{k_2}$$

- $U_m = -mgh_m$ avec $h_m = 2r\cos\theta$

Donc : $U_m = -2mgr\cos\theta$

- $U_{k_1} = \frac{1}{2} k_1 x_1^2$ avec $x_1 = r\theta$

Donc : $U_{K_1} = \frac{1}{2} k_1 r^2 \theta^2$

- $U_{K_2} = \frac{1}{2} k_2 x_2^2$ avec $x_2 = 2r\theta$

Donc : $U_{k_2} = \frac{1}{2}(4k_2) r^2 \theta^2$

$$U_{tot} = \frac{1}{2}(k_1 + 4k_2) r^2 \theta^2 - 2mgr\cos\theta$$