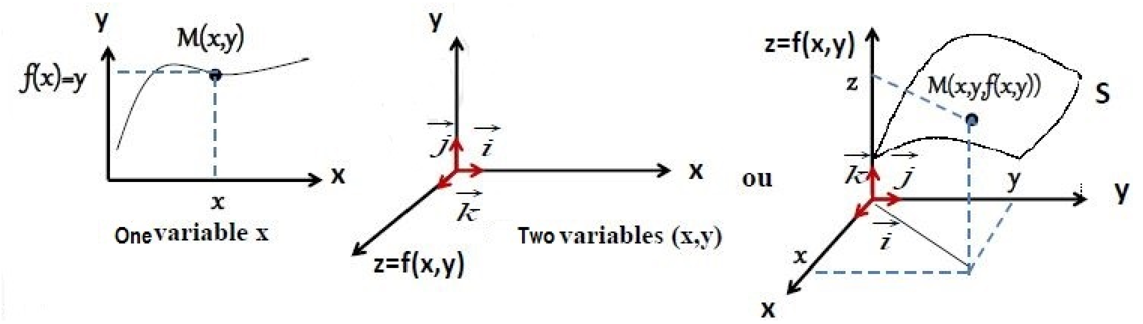
Functions with multiples variables

1 Definitions and General notations

- We called a real valued function of n real variables $(x_1, x_2, \cdots, x_n)$ any application from $D \subseteq \mathbb{R}^n$ to $\mathbb{R}$. We write: $y = f(x_1, x_2, \cdots, x_n)$ with $y \in \mathbb{R}$.
- $\mathbb{R}^n$ is called the cartisien product of $\mathbb{R}$, i.e: $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$.
- We will limit ourselves to the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$.
- We write $u = f(x, y)$ for $n = 2$ and $u = f(x, y, z)$ for $n = 3$.

For $n = 2$**:** the representative surface S of f in a three dimensional orthonormal coordinate system is the set of points $M(x, y, f(x, y))$. We say also S is the surface of the equation $z = f(x, y)$.



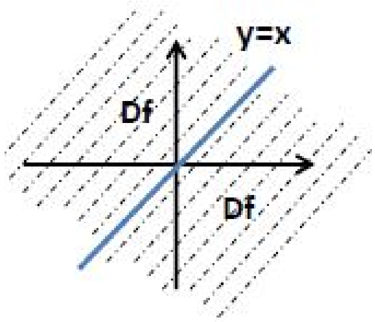
2 Domain of definition

Definition 1 *The domain of the function $f(x_1, x_2, \cdots, x_n)$ is the largest subset $D_f \subseteq \mathbb{R}^n$, such that $(x_1, x_2, \cdots, x_n)$ of D_f, $f(x_1, x_2, \cdots, x_n)$ is well defined. Then, we have: $f : D_f \to \mathbb{R}$.*

Example 1 *Determine the domain of the following functions:*

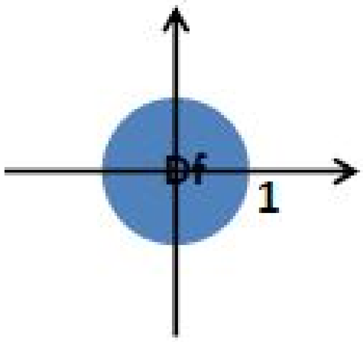
1. *Let f be defined by: $f(x, y) = \dfrac{1}{x - y}\cdot$ Then:*

$$D_f = \{(x, y) \in \mathbb{R}^2 : x - y \neq 0\} = \{(x, y) \in \mathbb{R}^2 : x \neq y\}.$$



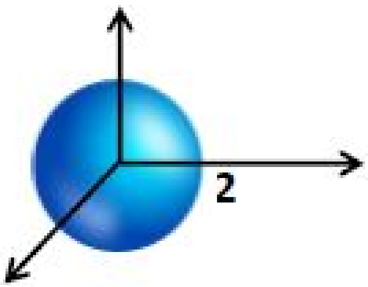
2. Let f be defined by: $f(x,y) = \ln(1 - x^2 - y^2)$. *Then:*

$$D_f = \{(x,y) \in \mathbb{R}^2 : 1 - x^2 - y^2 > 0\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$



3. Let f be defined by: $f(x,y,z) = \sqrt{4 - x^2 - y^2 - z^2}$. *Then:*

$$D_f = \{(x,y,z) \in \mathbb{R}^3 : 4 - x^2 - y^2 - z^2 \geq 0\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + z^2 \leq 4\}.$$

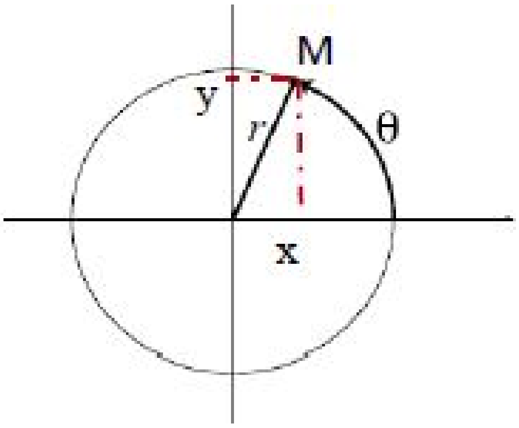


3 Coordinates systems

(a) Polar coordinates (r, θ)**:**

$$x = r\cos\theta$$
$$y = r\sin\theta$$

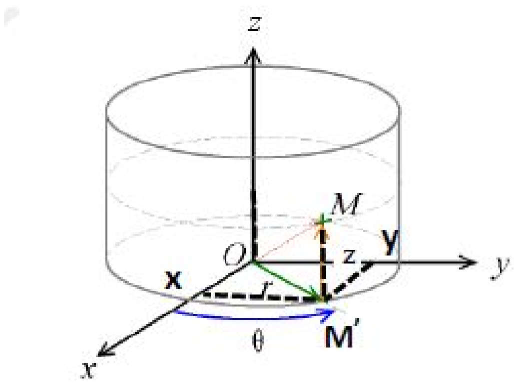
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arccos\frac{x}{\sqrt{x^2+y^2}}, \quad 0 \le \theta \le 2\pi$$



(b) Cylindrical coordinates (r, θ, z) :

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

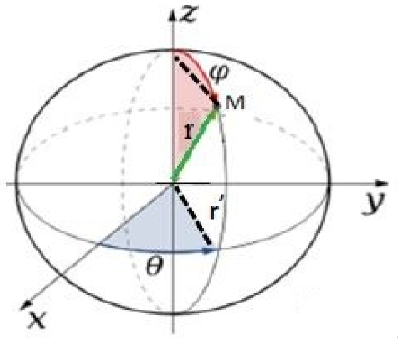
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arccos\frac{x}{\sqrt{x^2+y^2}}$$
$$z = z$$



(c) Spherical coordinates (r, θ, φ) :

$$x = r\cos\theta\sin\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\varphi$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arccos\frac{x}{\sqrt{x^2 + y^2}}$$
$$\varphi = +\arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}, \quad 0 \leq \varphi \leq \pi$$



4 Limits and continuity

Definition 2

- *We say that* f *admits a finite limit* $l \in \mathbb{R}$ *at the point* $a \in \mathbb{R}^2$ *or* $a \in \mathbb{R}^3$ *if:*

$$\lim_{t \to a} f(t) = l, \qquad or \qquad \lim_{(x,y)\to(a_1,a_2)} f(x,y) = l.$$

- *The limit, if it is exixts, is unique.*
- *The properties of the limit in* $\mathbb{R}$ *remain valid in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$.

Example 2 *Determine if the following functions have limits at the point* $a = (0,0)$:

1. *Let* $f(x,y) = \begin{cases} \dfrac{xy}{xy + (x-y)^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$

   We have:

$$\lim_{\substack{(x,y)\to(0,0) \\ x=y}} f(x,y) = \lim_{\substack{(x,y)\to(0,0) \\ x=y}} \frac{xy}{xy + (x-y)^2} = \lim_{x\to 0}\frac{x^2}{x^2} = 1.$$

$$\lim_{\substack{(x,y)\to(0,0) \\ x=-y}} f(x,y) = \lim_{\substack{(x,y)\to(0,0) \\ x=-y}} \frac{xy}{xy + (x-y)^2} = \lim_{x\to 0}\frac{-y^2}{3y^2} = -\frac{1}{3}.$$

   The limits are differents, then $\lim_{(x,y)\to(0,0)} f(x,y)$ *is not exixts.*

2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1+x^2+y^2}{y}\sin y = \lim\limits_{(x,y)\to(0,0)} (1+x^2+y^2)\dfrac{\sin y}{y} = 1.$

3. *Let* $f(t,x) = \dfrac{x^2-y^2}{x^2+y^2}\cdot$ *We use the polar coordinates to calculate the limit:*

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{r\to 0}(\cos^2\theta - \sin^2\theta).$$

The limit depends to θ *(where* $0 \le \theta \le 2\pi$*). Then, there is no fixed limit at the point* $(0,0)$.

Definition 3

- *We say that* f *is continuous at the point* $a \in \mathbb{R}^2$ *or* $a \in \mathbb{R}^3$ *if:*

$$\lim_{t\to a} f(t) = f(a), \qquad or \qquad \lim_{(x,y)\to(a_1,a_2)} f(x,y) = f(a_1,a_2).$$

- *The properties of the continuity in* $\mathbb{R}$ *remain valid in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$.

Example 3 *Study the continuity of the function* f *on* $\mathbb{R}^2$:

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

- f *is continuous on* $\mathbb{R}^2 - \{(0,0)\}$.
- *The continuity at the point* $(0,0)$: *we use the polar coordinates:*

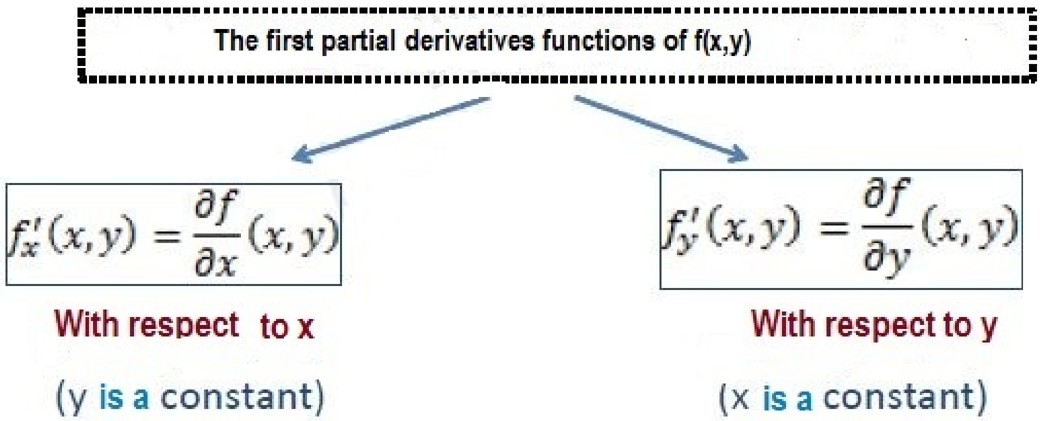
$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0 = f(0,0).$$

Then, f *is continuous at* $(0,0)$. *Therefore,* f *is continuous on* $\mathbb{R}^2$.

5 Partial derivatives

5.1 The first order partial derivatives:

(i) For $n = 2$:



The first partial derivatives functions of $f(x, y)$ at a point (x_0, y_0):

with respect to x: $$f'_x(x_0,y_0)=\frac{\partial f}{\partial x}(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}=\lim_{h\to 0}\frac{f(h+x_0,y_0)-f(x_0,y_0)}{h}.$$ ($x-x_0$ = h)

with respect to y: $$f'_y(x_0,y_0)=\frac{\partial f}{\partial x}(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}=\lim_{k\to 0}\frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}.$$ ($y-y_0$ = k)

Example 4

1. *Calculate the first partial derivatives functions of:* $f(x, y) = x^3e^y + \sin^2 y + 3x$.

$$\frac{\partial f}{\partial x}(x, y) = 3x^2e^y + 0 + 3, \qquad \frac{\partial f}{\partial y}(x, y) = x^3e^y + 2\sin y \cos y + 0.$$

2. *Study the continuity of the function f at the point $(0, 0)$ and calculate the first partial derivatives functions for all $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$, then at $(x, y) = (0, 0)$.*

$$f(x,y) = \begin{cases} x + y + \dfrac{x^3y}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

- *The continuity: we use the polar coordinates:*

$$\lim_{r\to 0} f(r, \theta) = \lim_{r\to\theta}\left(r\cos\theta + r\sin\theta + \frac{r^4\cos^3\theta\sin\theta}{r^2\cos\theta + r^2\sin\theta}\right) = 0 = f(0, 0).$$

Then, f is continuous at the point $(0, 0)$.

- *The partial derivatives functions at $(x,y)\neq(0,0)$:*

$$\frac{\partial f}{\partial x}(x,y)=1+0+\frac{3x^2y(x^2+y^2)+2x^4y}{(x^2+y^2)^2}.$$
$$\frac{\partial f}{\partial y}(x,y)=0+1+\frac{x^3(x^2+y^2)+2x^3y^2}{(x^2+y^2)^2}.$$

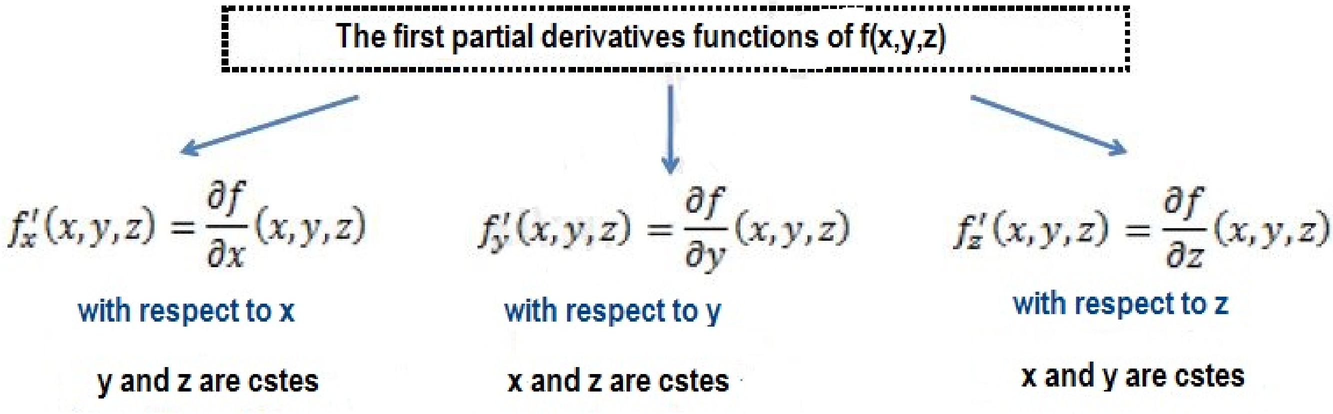
- *The partial derivatives functions at $(x,y)=(0,0)$:*

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=1.$$
$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=1.$$

Proposition 1 *We define the gradiant of f at the point $a=(a_1,a_2)\in\mathbb{R}^2$ by:*

$$\nabla f(a)=grad_a f:=\left(\frac{\partial f}{\partial x}(a),\frac{\partial f}{\partial y}(a)\right).$$

(ii) For $n=3$:



Example 5

1. $f(x,y,z)=\frac{xy}{z}$, $\qquad D_f=\{(x,y,z)\in\mathbb{R}^3/z\neq 0\}$.

- *The partial derivatives functions at $(x,y,z)\in D_f$:*

$$\frac{\partial f}{\partial x}(x,y)=\frac{y}{z},\qquad \frac{\partial f}{\partial y}(x,y)=\frac{x}{z},\qquad \frac{\partial f}{\partial z}(x,y)=\frac{-xy}{z^2}.$$

- *The partial derivatives functions at $(x,y,z)=(0,0,1)$*

$$\frac{\partial f}{\partial x}(0,0,1)=\lim_{x\to 0}\frac{f(x,0,1)-f(0,0,1)}{x-0}=\lim_{h\to 0}\frac{f(h,0,1)-f(0,0,1)}{h}=0.$$

$$\frac{\partial f}{\partial y}(0,,0,1) = \lim_{y\to 0}\frac{f(0,y,1)-f(0,0,1)}{y-0} = \lim_{k\to 0}\frac{f(0,k,1)-f(0,0,1)}{k} = 0.$$

$$\frac{\partial f}{\partial z}(0,0,1) = \lim_{z\to 1}\frac{f(0,0,z)-f(0,0,1)}{z-1} = \lim_{l\to 0}\frac{f(0,0,l+1)-f(0,0,1)}{l} = 0.$$

2. $f(x,y,z) = \begin{cases} x - 3y + \dfrac{x^4}{x^2+y^2+z^2} & (x,y,z) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0). \end{cases}$

Study the differetiability of f on $\mathbb{R}^3$.

- *f is differetiable on $\mathbb{R}^3 - \{(0,0,0)\}$.*
- *At the point $(0,0,0)$:*

$$\frac{\partial f}{\partial x}(0,0,0) = \lim_{x\to 0}\frac{f(x,0,1)-f(0,0,0)}{x-0} = \lim_{h\to 0}\frac{f(h,0,0)-f(0,0,0)}{h} = \lim_{h\to 0}\frac{h+h^2}{h} = 1.$$

$$\frac{\partial f}{\partial y}(0,,0,0) = \lim_{y\to 0}\frac{f(0,y,0)-f(0,0,0)}{y-0} = \lim_{k\to 0}\frac{f(0,k,0)-f(0,0,0)}{k} = \lim_{k\to 0}\frac{-3k}{k} = -3.$$

$$\frac{\partial f}{\partial z}(0,0,0) = \lim_{z\to 1}\frac{f(0,0,z)-f(0,0,0)}{z-0} = \lim_{l\to 0}\frac{f(0,0,l)-f(0,0,0)}{l} = 0.$$

Then, f is differetiable at the point $(0,0,0)$. Therefore, f is differetiable on $\mathbb{R}^3$.

5.2 The second order partial derivatives:

$$f'_x \to \begin{cases} f''_{xx} = f''_{x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}. \\ f''_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}. \end{cases} \quad f'_y \to \begin{cases} f''_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}. \\ f''_{yy} = f''_{y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}. \end{cases}$$

Remark 1 *The 2^{nd} PD for $n = 3$ are 3 for each of the variables, will be 9 PD.*

Proposition 2 *f is differentiable $\Longrightarrow$ f is continuous.*

Example 6 *Calculate the second partial derivatives functions of $f(x,y) = x + y^2 + \ln(xy)$. In the beginning, we calculate the first partial derivatives functions of f, after that we calculate the derivatives of the first partial derivatives to obtain the second ones.*

$$\frac{\partial f}{\partial x}(x,y) = 1 + 0 + y.\frac{1}{xy} = 1 + \frac{1}{x}$$

$$\rightarrow \frac{\partial^2 f}{\partial x^2}(x,y) = -\frac{1}{x^2}$$

$$\rightarrow \frac{\partial^2 f}{\partial y \partial x}(x,y) = 0$$

$$\frac{\partial f}{\partial y}(x,y) = 0 + 2y + x.\frac{1}{xy} = 2y + \frac{1}{y}$$

$$\rightarrow \frac{\partial^2 f}{\partial y^2}(x,y) = 2 - \frac{1}{y^2}$$

$$\rightarrow \frac{\partial^2 f}{\partial x \partial y}(x,y) = 0$$

5.3 Total derivative:

<u>For n=2</u>

$$\boxed{Df(x,y) = \frac{\partial f}{\partial x}(x,y)dx + \frac{\partial f}{\partial y}(x,y)dy}$$

Example: $f(x,y) = x + y^2 + ln(xy)$ $\quad Df(x,y) = \left(1 + \frac{1}{x}\right)dx + \left(2y + \frac{1}{y}\right)dy$

<u>For n=3</u>

$$Df(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)dx + \frac{\partial f}{\partial y}(x,y,z)dy + \frac{\partial f}{\partial z}(x,y,z)dz$$

Example: $f(x,y,z) = \frac{xy}{z}$ $\quad Df(x,y,z) = \frac{y}{z}dx + \frac{x}{z}dy - \frac{xy}{z^2}dz$