also, we have $(x_n) \xrightarrow{cv} \bar{x}$ and from the uniqueness of the limit of the convergent sequence, we conclude that $\bar{x}$ is a unique fixed p

Chapter 06: Differentiation:

In this chapter we discuss basic properties of the derivative of a function and several major theorems, including the Mean Value Theorem and L'Hôpital's Rule.

Definition①: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined on the open set D and let $x_0 \in D$.

1/ We say that f is differentiable at x_0 if the limit $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. In this case, the limit is called the derivative of f at x_0 and denoted by $f'(x_0)$, and f is said to be differentiable at x_0.

Thus, if f is differentiable at x_0, then $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$.

2/ We say that f is differentiable on D if f is differentiable at every point $x_0 \in D$.

. In this case, the function $f': D \to \mathbb{R}$ is called the derivative of f on D.

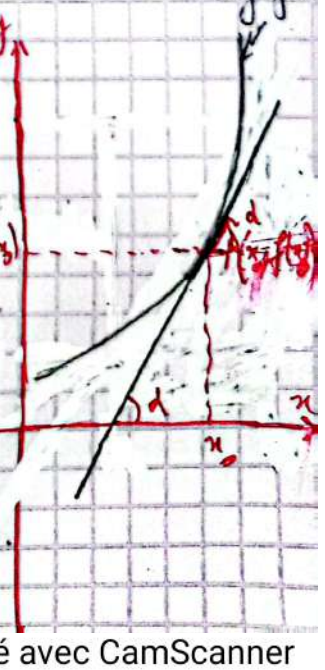
Lemma①: Let D be an open subset of $\mathbb{R}$ and let $f: D \to \mathbb{R}$. Suppose f is differentiable at x_0. Then there exists a function $\varepsilon: D \to \mathbb{R}$ satisfying

$$\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) + \varepsilon(x) \; ; \; \forall x \in D \text{ and } \lim_{x \to x_0} \varepsilon(x) = 0.$$

Remark①: We can define the derivative of f at x_0 as follows

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0).$$

Geometrical meaning Derivative at a point x_0:



Derivative of a function f at a point $x = x_0$ is the slope (or gradient) of the tangent to the curve $f(x) = y$ at the point $p(x_0; f(x_0))$.

Then derivative of the function f at a point x_0 is

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{df}{dx}\right)_{x=x_0} = \tan\alpha = \text{slope or gradient of the tangent.}$$

Example ① Let $f(x) = y = x^2 + 2x + 3$ then $\frac{dy}{dx} = \frac{df}{dx} = 2x + 2$

$\Rightarrow \left(\frac{df}{dx}\right)_{x=3} = 2.3 + 2 = 8.$

② the definition of derivative (or differentiation) to compute the derivative of $f(x) = \sqrt{x}$ at $x_0 = 2$, also for $f(x) = |x|$, $x_0 = 0$.

solution: $\lim\limits_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} = \lim \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{2\sqrt{2}} \Rightarrow f'(2) = \frac{1}{2\sqrt{2}}.$

$\lim\limits_{x \overset{>}{\to} 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \overset{>}{\to} 0} \frac{|x| - |0|}{x - 0} = \lim\limits_{x \overset{>}{\to} 0} \frac{x}{x} = 1$

$\lim\limits_{x \overset{<}{\to} 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \overset{<}{\to} 0} \frac{|x| - |0|}{x - 0} = \lim\limits_{x \overset{<}{\to} 0} \frac{-x}{x} = -1$ $\Rightarrow f$ hasn't a derivative at $x_0 = 0$.

Theorem ①: Let D be an open subset of $\mathbb{R}$ and let f be defined on D. If f is differentiable at $x_0 \in D$, then f is continuous at this point.

Proof: for $x \in D - \{x_0\}$, we have the following identity:

$$f(x) = f(x) - f(x_0) + f(x_0)$$

$$= \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0),$$

thus,

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0)\right]$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0}(x - x_0) + f(x_0)$$

$$= f'(x_0) \cdot 0 + f(x_0).$$

Therefore, f is continuous at x_0 $\left(\lim\limits_{x \to x_0} f(x) = f(x_0)\right)$.

Remark ②: the converse of this theorem is not true.

for example: the function $f(x) = |x|$ is continuous at 0, but it isn't differentiable at this point (expl ② above).

Algebraic operations: Let D be an open subset of $\mathbb{R}$ and $f, g: D \to \mathbb{R}$ suppose both f and g are differentiable at $x_0 \in D$. Then the following hold.

a/ the function $f + g$ is differentiable at a and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$

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b/ for a constant λ, the function λf is differentiable at x_0 and

$$(\lambda f)'(x_0) = \lambda f'(x_0).$$

c/ The function fg is differentiable at x_0 and

$$(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0).$$

d/ Suppose additionally that $g(x_0) \neq 0$. Then the function $\frac{f}{g}$ is differentiable at x_0 and

$$\left(f/g\right)'(x_0) = \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{(g(x_0))^2}.$$

Theorem ②: (Chain rule)

Let $f : D_1 \to \mathbb{R}$ and let $g : D_2 \to \mathbb{R}$, where D_1 and D_2 are two open subsets of $\mathbb{R}$ with $f(D_1) \subset D_2$. Suppose f is differentiable at x_0 and g is differentiable at $f(x_0)$. Then the function $g \circ f$ is differentiable at x_0 and

$$(g\circ f)'(x_0) = g'(f(x_0))\, f'(x_0).$$

proof: we have

$$(g\circ f)'(x_0) = \lim_{x\to x_0} \frac{(g\circ f)(x) - (g\circ f)(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \cdot \frac{f(x) - f(x_0)}{f(x) - f(x_0)}$$

$$= \lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{f(x)\to f(x_0)} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$x \to x_0 \Rightarrow f(x) \to f(x_0)$ since f is continuous at x_0

$$= g'(f(x_0)) \cdot f'(x_0)$$

[g, f are differentiable at $f(x_0)$ and x_0; respectively]

Remark: we can prove a/, b/, c/ and d/ by the same way, i.e.,

c/ $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$ is proven as follows

$$(fg)'(x_0) = \lim_{x\to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{g(x)\left[f(x) - f(x_0)\right] + f(x_0)\left[g(x) - g(x_0)\right]}{x - x_0}$$

$$= \lim_{x\to x_0} g(x) \cdot \left[\frac{f(x) - f(x_0)}{x - x_0}\right] + \lim_{x\to x_0} f(x_0) \left[\frac{g(x) - g(x_0)}{x - x_0}\right] = g(x_0) f'(x_0) + f(x_0) g'(x_0)$$

2nd way: we can also prove that $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$ as follows:

f is diff at $x_0 \Leftrightarrow \exists \varepsilon_1$ continuous at x_0 : $f(x) = f(x_0) + (x-x_0)f'(x_0) + (x-x_0)\varepsilon_1(x)$ / $\varepsilon_1(x) \xrightarrow[x \to x_0]{} 0$ (1)

g " " " " $\Leftrightarrow \exists \varepsilon_2$ continuous at x_0 : $g(x) = g(x_0) + (x-x_0)g'(x_0) + (x-x_0)\varepsilon_2(x)$ / $\varepsilon_2(x) \xrightarrow[x \to x_0]{} 0$ (2)

$\Rightarrow$ multiplying (1) and (2), we get:

$$f(x)g(x) = f(x_0)g(x_0) + (x-x_0)[f'(x_0)g(x_0) + f(x_0)g'(x_0)] + (x-x_0)\underbrace{[f(x_0)\varepsilon_2(x) + g(x)\varepsilon_1(x) + \dots]}_{\varepsilon_3(x)}.$$

since ε_1 and ε_2 are continuous at x_0, then ε_3 is too and we have

$$\lim_{x \to x_0} \varepsilon_3(x) = \lim [f(x_0)\varepsilon_2(x) + g(x)\varepsilon_1(x) + \dots] = 0 \quad (\varepsilon_1, \varepsilon_2 \xrightarrow[x \to x_0]{} 0)$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Hence $(f \cdot g)$ is differentiable at x_0 and its derivative is $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$

d/ Let $h = \frac{f}{g}$ such that $g(x_0) \neq 0$, then

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x)} + \frac{f(x_0)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\frac{1}{g(x)}[f(x) - f(x_0)] + \frac{1}{g(x)g(x_0)}(f(x_0)g(x_0) - f(x_0)g(x))}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{1}{g(x)} \cdot \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{f(x_0)}{g(x)g(x_0)}\left[\frac{g(x_0) - g(x)}{x - x_0}\right]\right]$$

$$\lim_{x \to x_0} \frac{1}{g(x)g(x_0)} \left[g(x_0) \cdot \frac{f(x) - f(x_0)}{x - x_0} + f(x_0) \cdot \frac{g(x_0) - g(x)}{x - x_0}\right]$$

$$= \frac{1}{[g(x_0)]^2} [g(x_0)f'(x_0) - f(x_0) \cdot g'(x_0)]$$

$$= \frac{g(x_0)f'(x_0) - g'(x_0)f(x_0)}{[g(x_0)]^2}.$$

Hence $h = \frac{f}{g}$ is differentiable at x_0 and its derivative is: $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - g'(x_0)f(x_0)}{[g(x_0)]^2}$

Theorem 3 (Inverse function theorem): suppose f is differentiable on $I =]a,b[$ and $f'(x) \neq 0, \forall x \in]a,b[$. Then f is one-to-one, $f(I)$ is an open interval, and the inverse function (when exists) $f^{-1} : f(I) \to I$ is diff

Moreover, $(f^{-1})'(y) = \frac{1}{f'(x)}$, where $f(x)=y$.

Thus: $\boxed{(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))} \, , \forall y \in f(I)}$

e.g: Let $f(x)=x^n \Rightarrow \forall x \in \mathbb{R}, \; f'(x)=n x^{n-1}$

on $\mathbb{R}_+$, f has its inverse function: $f^{-1}(y)=\sqrt[n]{y}$, $n \geq 2$.

let $g(y)=\sqrt[n]{y}$, then g is differentiable on $]0,+\infty[= \mathbb{R}_+^*$ and we have

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{n x^{n-1}} = \frac{1}{n(\sqrt[n]{y})^{n-1}} = \frac{1}{n(y^{\frac{1}{n}})^{n-1}} = \frac{1}{n y^{1-\frac{1}{n}}}, \forall y \in \mathbb{R}_+^*$$

(since $y = x^n \Leftrightarrow x = \sqrt[n]{y}$)

[we can obtain $g'(y)$ from its expression, i.e., $g'(y) = (\sqrt[n]{y})' = n(y^{\frac{1}{n}-1})$.

Leibniz Rule: Let $f, g : D \to \mathbb{R}$ be two real functions.

If f and g are differentiable n times individually at x_0, then their product $f.g$ is also differentiable n times at x_0, and we have:

$$p(n): (fg)^{(n)}(x_0) = \sum_{k=0}^{n} C_n^k f^{(k)}(x_0)\, g^{(n-k)}(x_0)$$

$$= C_n^0 f^{(0)}(x_0) g^{(n)}(x_0) + C_n^1 f'(x_0) g^{(n-1)}(x_0) + \dots + C_n^{n-1} f^{(n-1)}(x_0) g'(x_0) + C_n^n f^{(n)}(x_0) g^{(0)}(x_0)$$

where $C_n^0 = C_n^n = 1$ and $C_n^{n-1} = C_n^1 = n$, $C_n^k = \frac{n!}{k!(n-k)!}$.

proof: by induction,

for $n=0$, we have: $(fg)^{(0)}(x_0) = (fg)(x_0) = C_0^0 f^{(0)}(x_0) g^{(0)}(x_0) = f(x_0) g(x_0)$

$\Rightarrow p(0)$ is true

suppose that $p(n)$ is true and prove that $p(n+1)$ is true too.

$$p(n+1): (fg)^{(n+1)}(x_0) = ((fg)^{(n)})'(x_0) \overset{p(n)}{=} \left[\sum_{k=0}^{n} C_n^k f^{(k)}(x_0)\, g^{(n-k)}(x_0)\right]'$$

$$= \sum_{k=0}^{n} C_n^k \left(f^{(k)}(x_0)\, g^{(n-k)}(x_0)\right)'$$

$$\overset{c/}{=} \sum_{k=0}^{n} C_n^k \left(f^{(k+1)}(x_0)\, g^{(n-k)}(x_0) + f^{(k)}(x_0)\, g^{(n-k+1)}(x_0)\right)$$

$$= \sum_{k=0}^{n} C_n^k f^{(k+1)}(x_0)\, g^{(n-k)}(x_0) + \sum_{k=0}^{n} C_n^k f^{(k)}(x_0)\, g^{(n+1-k)}(x_0)$$

$$= \sum_{k=1}^{n} C_n^{k-1} f^{(k)}(x_0)\, g^{(n+1-k)}(x_0) + C_n^n f^{(n+1)}(x_0)\, g^{(0)}(x_0) + \sum_{k=1}^{n} C_n^k f^{(k)}(x_0)\, g^{(n+1-k)}(x_0) + C_n^0 f^{(0)}(x_0)\, g^{(n+1)}(x_0)$$

$$= \sum_{k=1}^{n} \underbrace{(C_n^{k-1} + C_n^k)}_{= C_{n+1}^k} f^{(k)}(x_0)\, g^{(n+1-k)}(x_0) + f^{(n+1)}(x_0)\, g^{(0)}(x_0) + f^{(0)}(x_0)\, g^{(n+1)}(x_0)$$

$$= \sum_{k=1}^{n} C_{n+1}^k f^{(k)}(x_0)\, g^{(n+1-k)}(x_0) + C_{n+1}^{n+1} f^{(n+1)}(x_0)\, g^{(0)}(x_0) + C_{n+1}^0 f^{(0)}(x_0)\, g^{(n+1)}(x_0)$$

$$= \sum_{k=0}^{n+1} C_{n+1}^k f^{(k)}(x_0)\, g^{(n+1-k)}(x_0) \Rightarrow p(n+1) \text{ is true.}$$

expl: $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$, $f^{-1} [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

sinus and arcsinus: $x \longmapsto \sin x = y$ $\qquad$ $y \longmapsto \arcsin y = x$.

sinus is continuous and increasing strictly on $]-\frac{\pi}{2}, \frac{\pi}{2}[$ and differentiable on $]-\frac{\pi}{2}, \frac{\pi}{2}[$ with $\sin' x = \cos x \neq 0 \Rightarrow$ it has an inverse function denoted by arcsin which is differentiable on $]-1, 1[$ and we have:

$$\boxed{\forall y \in]-1, 1[: (\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}, \forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[}$$

$\cos^2 x + \sin^2 x = 1 \Rightarrow |\cos x| = \sqrt{1-\sin^2 x}$, where $|\cos x| = \cos x$, $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

$\Rightarrow \cos x = \sqrt{1-y^2}$ $\quad$ ($\sin x = y$)

② Cosinus and arc cosinus:

$f : [0, \pi] \longrightarrow [-1, 1]$, $f^{-1} : [-1, 1] \longrightarrow [0, \pi]$

$x \longmapsto \cos x = y$ $\qquad$ $y \longmapsto \arccos y = x$

cosinus is continuous on $[0, \pi]$ and strictly decreasing on $]0, \pi[$ and differ on $]0, \pi[$ with $\cos' x = -\sin x \neq 0$, $\forall x \in]0, \pi[\Rightarrow$ it has its inverse fun denoted by arccos which is differentiable on $]-1, 1[$ and we have:

$$\boxed{\forall y \in]-1, 1[: (\arccos y)' = \frac{1}{(\cos x)'} = \frac{-1}{\sin x} = \frac{-1}{\sqrt{1-\cos^2 x}} = \frac{-1}{\sqrt{1-y^2}}, \forall x \in]0, \pi[}$$

Where $\cos^2 x + \sin^2 x = 1 \Rightarrow |\sin x| = \sin x = \sqrt{1-\cos^2 x} = \sqrt{1-y^2}$, $\cos x = y$, $\forall x \in]$

③ tangent and arctangent:

$$f:]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R} \qquad , \qquad f^{-1}: \mathbb{R} \longrightarrow]-\frac{\pi}{2}, \frac{\pi}{2}[$$
$$x \longmapsto tg\,x = y \qquad\qquad y \longmapsto arctg\,y = x$$

tangent is continuous on $]-\frac{\pi}{2}, \frac{\pi}{2}[$ and strictly increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$ and diff on $]-\frac{\pi}{2}, \frac{\pi}{2}[$ with $(tg\,x)' = \frac{1}{\cos^2 x}, \forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ $\Rightarrow$ it has its inverse function denoted by arctang = arctg which is differentiable on $\mathbb{R}$ and we have

$$\forall y \in \mathbb{R}: (arctg\,y)' = \frac{1}{(tg\,x)'} = \cos^2 x = \frac{1}{1+y^2}, \quad \forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[\; tg\,x = y$$

$$\cos^2 x + \sin^2 x = 1 \Rightarrow 1 + tg^2 x = \frac{1}{\cos^2 x} \Rightarrow \cos^2 x = \frac{1}{1+tg^2 x} = \frac{1}{1+y^2}.$$

④ cotangent and arccotangent:

$$f:]0, \pi[\longrightarrow \mathbb{R} \qquad , \qquad f^{-1}: \mathbb{R} \longrightarrow]0, \pi[$$
$$x \longmapsto f(x) = cotg\,x = y \qquad\qquad y \longmapsto arctg\,y = x.$$

cotangent is continuous on $]0, \pi[$ and strictly decreasing on $]0, \pi[$ and diff on $]0, \pi[$ with $(cotg\,x)' = \frac{-1}{\sin^2 x}, \forall x \in]0, \pi[$ $\Rightarrow$ it has its inverse function denoted by arccotg = arccotan which is differentiable on $\mathbb{R}$ and we have:

$$\forall y \in \mathbb{R}: (arccotg\,y)' = \frac{1}{(cotg\,x)'} = -\sin^2 x = \frac{-1}{1+y^2}, \quad \forall x \in]0, \pi[, \; cotg\,x = y$$

$$\cos^2 x + \sin^2 x = 1 \Rightarrow cotg^2 x + 1 = \frac{1}{\sin^2 x} \Rightarrow \sin^2 x = \frac{1}{1+cotg^2 x} = \frac{1}{1+y^2}.$$

⑤ sinus hyperbolic and argsinus hyperbolic and cosinus hyperbolic and argcosh

from the previous cours, we have seen that each function defined on symmetric interval can be written as a sum of two functions, one even and the other odd, which is the case of e^x, $\forall x \in \mathbb{R}$, i.e.,

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = ch\,x + sh\,x, \quad \forall x \in \mathbb{R}$$

$sh\,x = \frac{e^x - e^{-x}}{2}$ is called sinus hyperbolic function and denoted by sh or sinh and $ch\,x = \frac{e^x + e^{-x}}{2}$ is called cosinus hyperbolic function and denoted by ch or cosh.

a/ sinh/argsh:

$$\mathbb{R} \longrightarrow \mathbb{R} =]-\infty, +\infty[\qquad argsh:]-\infty, +\infty[= \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto sh\,x = y \qquad\qquad y \longmapsto argsh\,y = x$$

sh is continuous and strictly increasing on $\mathbb{R}$ and differentiable on $\mathbb{R}$ with: $(\text{sh}\,x)' = \text{ch}\,x,\ \forall x \in \mathbb{R} \Rightarrow$ it has its inverse function denoted by arg sh and we have

$\forall y \in]-\infty,+\infty[=\mathbb{R} : (\text{arg sh}\,y)' = \dfrac{1}{(\text{sh}\,x)'} = \dfrac{1}{\text{ch}\,x} = \dfrac{1}{\sqrt{1+y^2}},\ \forall x \in \mathbb{R}$

$\text{ch}^2x - \text{sh}^2x = 1 \Rightarrow |\text{ch}\,x| = \sqrt{\text{sh}^2x+1} = \sqrt{1+y^2},\ 1 \leq |\text{ch}\,x| = \text{ch}\,x,\ \forall x \in \mathbb{R}$

b/ ch / argch : $\text{ch} : [0,+\infty[\longrightarrow [1,+\infty[$, $x \longmapsto \text{ch}\,x = y$, $\text{argch} : [1,+\infty[\longrightarrow [0,+\infty[$, $y \longmapsto \text{argch}\,y = x$

ch is continuous and strictly increasing on $\mathbb{R}_+$ or [continuous and strictly decreasing on $\mathbb{R}_-$] and differentiable on $\mathbb{R}$ with $(\text{ch}\,x)' = \text{sh}\,x,\ \forall x \in \mathbb{R}$ $\Rightarrow$ it has its inverse function on the half on interval $\mathbb{R}$ [i.e, on $\mathbb{R}_+$ or $\mathbb{R}_-$] and its inverse is denoted by arg ch and we have;

$$\boxed{\forall y \in [1,+\infty[: (\text{arg ch}\,y)' = \dfrac{1}{(\text{ch}\,x)'} = \dfrac{1}{\text{sh}\,x} = \dfrac{1}{\sqrt{y^2-1}},\ \forall x \in [0,+\infty[}$$

$\text{ch}^2x - \text{sh}^2x = 1 \Rightarrow |\text{sh}\,x| = \sqrt{\text{ch}^2x-1} = \sqrt{y^2-1},\ |\text{sh}\,x| = \text{sh}\,x,\ \forall x \in \mathbb{R}_+$

⑥ tangent hyperbolic / argtangent hyperbolic / cotangent hyperbolic and arg coth

as a trigonometric function, the functions th and coth are defined as

$\text{th}\,x = \dfrac{\text{sh}\,x}{\text{ch}\,x} = \dfrac{e^{2x}-1}{e^{2x}+1},\ \forall x \in \mathbb{R}$

$\text{coth}\,x = \dfrac{\text{ch}\,x}{\text{sh}\,x} = \dfrac{e^{2x}+1}{e^{2x}-1},\ \forall x \in \mathbb{R}^*$

a/ th / argth;

$\text{th} : \mathbb{R} \longrightarrow]-1,1[$, $x \longmapsto \text{th}\,x = y$, $\text{arg th} :]-1,1[\longrightarrow \mathbb{R}$, $y \longmapsto \text{argth}\,y = x$

th is continuous and strictly increasing on $\mathbb{R}$ and differentiable on $\mathbb{R}$ with $(\text{th}\,x)' = \dfrac{1}{\text{ch}^2x},\ \forall x \in \mathbb{R} \Rightarrow$ it has its inverse function which is defined on $]-1,1[$ and denoted by arg th, with:

$\forall y \in]-1,1[: (\text{arg th}\,y)' = \dfrac{1}{(\text{th}\,x)'} = \text{ch}^2x = \dfrac{1}{1-y^2},\ \forall y \in]-1,1[, \forall x \in$

$\text{ch}^2x - \text{sh}^2x = 1 \Rightarrow 1 - \text{th}^2x = \dfrac{1}{\text{ch}^2x} \Rightarrow \text{ch}^2x = \dfrac{1}{1-\text{th}^2x} = \dfrac{1}{1-y^2}.$

b/ coth / argcoth: $\text{coth} : \mathbb{R}^* \longrightarrow]-\infty,-1[\cup]1,+\infty[$, $x \longmapsto \text{coth}\,x = y$; $\text{arg coth} :]-\infty,-1[\cup]1,+\infty[\longrightarrow \mathbb{R}$, $y \longmapsto$ arg

b) coth / argcoth : $\coth : \mathbb{R}^* \to]-\infty,-1[\cup]1,+\infty[$, $\text{argcoth}:]-\infty,-1[\cup]1,+\infty[\to \mathbb{R}^*$
$x \mapsto \coth x = y$, $y \mapsto \text{argcoth} y =$

coth is continuous and strictly decreasing on $\mathbb{R}^*$ and differentiable on $\mathbb{R}^*$, with :
$(\coth x)' = \frac{-1}{sh^2 x}$, $\forall x \in \mathbb{R}^*$ $\Rightarrow$ it has its inverse function which is defined on $]-\infty,-1[\cup]1,+\infty[$ and denoted by argcoth , with :

$$\boxed{\forall y \in]-\infty,-1[\cup]1,+\infty[: (\text{Arg}\coth y)' = \frac{1}{(\coth x)'} = -sh^2 x = \frac{-1}{y^2-1}, \ \forall x \in \mathbb{R}^*}$$

$ch^2 x - sh^2 x = 1 \Rightarrow \coth^2 x - 1 = \frac{1}{sh^2 x} \Rightarrow sh^2 x = \frac{1}{ch^2 x - 1} = \frac{1}{y^2-1}$.

---

theorems on differentiable (or derivative) function on an interval :

Theorem ①: (Rolle's theorem)

Let $a,b \in \mathbb{R}$ with $a<b$ and $f: [a,b] \to \mathbb{R}$. Suppose f is continuous on $[a,b]$ and differentiable on $]a,b[$ with $f(a) = f(b)$. Then $\exists c \in]a,b[: f'(c) = 0$

proof :

- If f is a constant function on $[a,b]$, then it is clear that $\forall c \in]a,b[: f'(c) = 0$.
- If f isn't a constant function on $[a,b]$. Suppose that f takes a values greater than $f(a)$, hence greater than $f(b)$ (since $f(a) = f(b)$)

since f is continuous on $[a,b]$ closed and bounded, then from theorem ⑧ (extrem value thm) f reaches its supremum and infimum, hence
$\exists c \in]a,b[: f(c) = \sup_{x \in [a,b]} f(x)$, (especially sup)
on the other hand,
f is differentiable on $]a,b[\Rightarrow f$ is differentiable at $c \Rightarrow$

$\lim_{x \overset{>}{\to} c} \frac{f(x)-f(c)}{x-c} = f'(c+0) = \frac{\text{negative}}{\text{positive}} \leq 0$ $\left(f(c) = \sup f \Rightarrow f(x) \leq f(c), \forall x \right.$; $x \overset{>}{\to} c \Rightarrow x-c > 0$

$\lim_{x \overset{<}{\to} c} \frac{f(x)-f(c)}{x-c} = f'(c-0) = \frac{\text{negative}}{\text{negative}} \geq 0$ $\left(f(x) \leq f(c), \forall x \in [a,b], x-c \leq 0 \right.$

f is differentiable at $c \Leftrightarrow f'(c+0) = f'(c-0) = 0 \Leftrightarrow \boxed{f'(c) = 0}$

Theorem ② (Mean value theorem)

Let $a,b \in \mathbb{R}$ with $a<b$ and $f: [a,b] \to \mathbb{R}$. suppose f is continuous on $[a,b]$

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and differentiable on $]a,b[$. Then $\exists c \in]a,b[: f'(c) = \frac{f(b)-f(a)}{b-a}$.

proof: applying Rolle's thm on the function:

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

g is continuous on $[a,b]$ (sum of two continuous functions on $[a,b]$)

g is differentiable on $]a,b[$ (sum of two differentiable functions on $]a,b[$)

$g(a) = 0 = g(b) \Rightarrow$ from Rolle's thm $\exists c \in]a,b[: g'(c) = 0$

$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \Rightarrow g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \Rightarrow g'(c) = 0 \Leftrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

eg ① let $f(x) = \ln x$, $[n, n+1]$, $n > 0$, prove that $\exists c \in]n, n+1[: \frac{1}{c} = \ln\left(\frac{n+1}{n}\right)$

f is continuous on $[n, n+1]$ and differentiable on $]n, n+1[$.

$\Rightarrow$ from the Mean value thm, $\exists c \in]n, n+1[: f'(c) = \frac{f(n+1)-f(n)}{1}$

$\Rightarrow \exists c \in]n, n+1[: f'(c) = \ln(n+1) - \ln n = \frac{1}{c}$ / $f'(x) = \frac{1}{x}$.

$\Rightarrow \exists c \in]n, n+1[: \frac{1}{c} = \ln\left(\frac{n+1}{n}\right)$.

② show that: $|\sin x| \leq |x|, \forall x \in \mathbb{R}$.

Let $f(x) = \sin x$, $\forall x \in \mathbb{R}$, then $f'(x) = \cos x$ ⓐ Now, fix $x \in \mathbb{R}$, $x > 0$.

By the Mean value thm applied to f on $[0, x]$, $\exists c \in]0, x[: \cos c = \frac{\sin x - \sin 0}{x - 0}$

Therefore, $\frac{|\sin x|}{|x|} = |\cos c| \leq 1$, since $|\cos c| \leq 1$, we conclude that

$|\sin x| \leq |x|$, $\forall x > 0$. ($x \in \mathbb{R}_+^*$).

ⓑ Next, suppose $x < 0$, by the Mean value thm applied to f on $[x, 0]$, $\exists c \in]x, 0[$

$\frac{\sin 0 - \sin x}{0 - x} = \cos c$, then again $\frac{|\sin x|}{|x|} = |\cos c| \leq 1$. It follows that

$|\sin x| \leq |x|$, $\forall x < 0$ ($x \in \mathbb{R}_-^*$).

ⓒ for $x = 0$, the inequality is obvious $|\sin 0| = |0| \leq |0|$.

Hence from ⓐ, ⓑ and ⓒ, we conclude that: $|\sin x| \leq |x|$, $\forall x \in \mathbb{R}$

Theorem ③: (Cauchy's theorem or Generalized Mean value theorem)

Let $a, b \in \mathbb{R}$ with $a < b$. suppose f and g are continuous on $[a,b]$ and differentiable on $]a,b[$ such that $g'(x) \neq 0$, $\forall x \in]a,b[$ (or $f'(x) \neq 0$, $\forall x \in]a,b[$).

Then, there exists $c \in]a,b[$: $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Proof:

Define: $h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x)$.

Then, we have: $h(a) = f(b)g(a) - f(a)g(b) = h(b)$ and h satisfies the assumptions of Rolle's thm (h is continuous on $[a,b]$ and differentiable on $]a,b[$).

Thus, $\exists c \in]a,b[: h'(c) = 0 \Leftrightarrow f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$

$\Leftrightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$. this completes the proof

Theorem (4): (L'Hopital's Rule) $\Leftrightarrow [g(b)-g(a)]f'(c) = [f(b)-f(a)]g'(c)$.

suppose f and g are continuous on $[a,b]$ and differentiable on $]a,b[$. suppose $f(\bar{x}) = g(\bar{x}) = 0$, where $\bar{x} \in [a,b]$. suppose further that:

$\exists \delta > 0 : g'(x) \neq 0, \forall x \in B(\bar{x}, \delta) \cap [a,b]$, $x \neq \bar{x}$.

If $\lim\limits_{x \to \bar{x}} \frac{f'(x)}{g'(x)} = \ell$, then $\lim\limits_{x \to \bar{x}} \frac{f(x)}{g(x)} = \ell$.

proof: Let (x_k) be a sequence in $[a,b]$ that converges to $\bar{x}$ and such that $x_k \neq \bar{x}$ for every k. By Cauchy's Thm (thm 3), $\forall k, \exists c_k \in]x_k, \bar{x}[$, such that:

$$[f(x_k) - f(\bar{x})]\, g'(c_k) = [g(x_k) - g(\bar{x})]\, f'(c_k)$$

Since $f(\bar{x}) = g(\bar{x}) = 0$ and $g'(c_k) \neq 0$ for sufficiently large k, we have

$$\frac{f(x_k)}{g(x_k)} = \frac{f'(c_k)}{g'(c_k)}.$$

$(x_k) \xrightarrow[k \to +\infty]{cv} \bar{x}$ hence, by the squeeze thm $(c_k) \xrightarrow[k \to +\infty]{cv} \bar{x}$ $\quad (x_k < c_k < \bar{x})$

Thus, $\lim\limits_{k \to +\infty} \frac{f(x_k)}{g(x_k)} = \lim\limits_{k \to +\infty} \frac{f'(c_k)}{g'(c_k)} = \lim\limits_{x \to \bar{x}} \frac{f'(x)}{g'(x)} = \ell$. the proof is completed.

e.g: compute $\lim\limits_{x \to 0} \frac{\sin x}{x}$ and $\lim\limits_{x \to 0} \frac{\ln(1+x)}{x}$ by using L'Hopital's Rule.

$\lim\limits_{x \to 0} \frac{\sin x}{x} = \lim\limits_{x \to 0} \frac{\cos x}{1} = \boxed{1}$, $\quad \lim\limits_{x \to 0} \frac{\ln(1+x)}{x} = \lim\limits_{x \to 0} \frac{\frac{1}{1+x}}{1} = \boxed{1}$

Remark:

- L'Hopital's Rule can be applied many times on the function $\frac{f}{g}$.

e.g. $\lim\limits_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \frac{0}{0} = \lim\limits_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = \frac{0}{0} = \lim\limits_{x \to 0} \frac{e^x - e^{-x}}{\sin x} = \frac{0}{0} = \lim\limits_{x \to 0} \frac{e^x + e^{-x}}{\cos x} = \boxed{2}$

- L'Hopital's Rule can be applied when $x \to \infty$, it is sufficient to make the following variable change ($t = \frac{1}{x}$, $x \to \infty$, $t \to 0$).

e.g: $\lim_{x\to+\infty} \frac{\sin\frac{k}{x}}{\frac{1}{x}} = \frac{0}{0} = \lim_{t\to 0} \frac{k\sin t}{t} = k \lim_{t\to 0}\frac{\cos t}{1} = \boxed{k}$ $\left(t=\frac{k}{x} \Rightarrow x=\frac{k}{t}\right)$

Definition: (absolute minimum (global minimum) and absolute maximum (globale), local minimum and local maximum)

Let $f: D \to \mathbb{R}$ be a function and $x_0 \in D$. we say that the function f has an $\underset{\text{globale}}{\text{absolute}}$ minimum (resp. $\underset{\text{global maximum}}{\text{absolute maximum}}$) at x_0 if :

$$f(x) \geq f(x_0),\ \forall x \in D \quad (\text{resp. } f(x) \leq f(x_0),\ \forall x \in D)$$; local maximal extrem

Similarly, we say that f has a $\underset{\text{local minimal extreme}}{\text{local minimum}}$ (resp local maximum) at x_0 if there exists neighborhood of x_0 ($]x_0-\delta, x_0+\delta[$, with $\delta>0$) such that:

$\forall x \in]x_0-\delta, x_0+\delta[\cap D: f(x) \geq f(x_0)$ (resp. $f(x) \leq f(x_0)$, $\forall x \in \underbrace{]x_0-\delta, x_0+\delta[}_{B(x_0,\delta)} \cap D$).

Theorem ⑥: If f has a local extreme (maximal or minimal) at x_0 and the derivative of f at x_0 exists, then $f'(x_0)=0$

Proof: Suppose that f has a local maximum (local maximal extreme) at x_0 and f is differentiable at x_0, hence

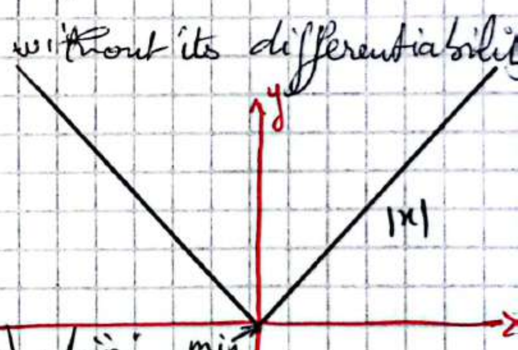
$$f'(x_0+0) = \lim_{x \overset{>}{\to} x_0} \frac{f(x)-f(x_0)}{x-x_0} = \frac{\text{negative}}{\text{positive}} \leq 0 \quad \left(f(x) \leq f(x_0),\ \forall x \in \underbrace{]x_0-\delta, x_0+\delta[}_{B(x_0,\delta)} \cap D\right)$$

$$f'(x_0-0) = \lim_{x \overset{<}{\to} x_0} \frac{f(x)-f(x_0)}{x-x_0} = \frac{\text{negative}}{\text{negative}} \geq 0$$

f is differentiable at $x_0 \Leftrightarrow f'(x_0+0) = f'(x_0-0) \Rightarrow f'(x_0)=0$.

Remark: f can admits an extreme value at x_0 without its differentiability at x_0.

eg: $f(x)=|x|$



we have seen that f isn't differentiable at "0".

but f has a global minimum (minimal extreme value) at "0"

Taylor's formulas

The goal of this part, is to approximate differentiable functions by polynomials

Definition: we have seen that, if f is differentiable at x_0, then we can write:

$$\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) + \varepsilon(x), \text{ where } \varepsilon(x) \xrightarrow[x\to x_0]{} 0$$

$$\Leftrightarrow f(x) = f(x_0) + (x-x_0) f'(x_0) + (x-x_0)\varepsilon(x)$$

this implies that: $f(x)=f(x_0)+(x-x_0)f'(x_0)+R(x)$ / $R(x)=(x-x_0)\varepsilon(x)\xrightarrow[x\to x_0]{}0$

that is, any differentiable function at x_0 can be approximated by a polynomial of degree 1 in the neighborhood of this point with $R(x)\xrightarrow[x\to x_0]{}0$

Theorem (6): (Taylor's theorem)

Let $a,b\in\mathbb{R}$ with $a<b$ and n be a positive integer. suppose $f:[a,b]\to\mathbb{R}$ is a function such that $f^{(n)}$ is continuous on $[a,b]$, and $f^{(n+1)}(x)$ exists for all $x\in]a,b[$. Let $x_0\in[a,b]$. Then for any $x\in[a,b]$ with $x\neq x_0$, $\exists c\in]x_0,x[$ (or $c\in]x,x_0[$) such that:

$$f(x)=f(x_0)+(x-x_0)\frac{f'(x_0)}{1!}+(x-x_0)^2\frac{f''(x_0)}{2!}+\cdots+(x-x_0)^n\frac{f^{(n)}(x_0)}{n!}+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

thus: $$\boxed{f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.}$$

Remark:

1) this formula is called Taylor's formula and $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ is called the remainder of Lagrange.

2) if $x_0=0$, we obtain the Macc-Laurin's developement with the remainder of Lagrange which is written as follows:

$\forall x\in[a,b],\ x\neq 0$: $f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$

$$\Rightarrow \boxed{f(x)=f(0)+\frac{f'(0)}{1!}x+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1},\ \text{where } 0<\theta<1}$$

eg: $f(x)=e^x \Rightarrow f^{(n)}(x)=e^x \Rightarrow f(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\frac{e^{\theta x}}{(n+1)!}x^{n+1}$, $0<\theta<1$

$g(x)=\sin x \Rightarrow g'(x)=\cos x \Rightarrow g''(x)=-\sin x \Rightarrow g^{(3)}(x)=-\cos x \Rightarrow g^{(4)}(x)=\sin x.$

the Macc-Laurin formula with the remainder of Lagrange for the function g is as follows:

$$g(x)=\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{(-1)^k}{(2k+1)!}x^{2k+1}+\frac{\sin^{(2k+3)}(\theta x)}{(2k+3)!}x^{2k+3},\ 0<\theta<1$$

Theorem (7) (Taylor Bis' Theorem)

Let $a,b\in\mathbb{R}$, with $a<b$ and n be a positive integer. supposose $f:[a,b]\to\mathbb{R}$ differentiable function on $]a,b[$ n times and $f^{(n+1)}(x_0)$ exists, $x_0\in[a,b]$, then

$\forall x\neq x_0\in]a,b[$: $f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+$

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$$+\frac{(x-x_0)^{n+1}}{(n+1)!}\left(f^{(n+1)}(x_0)+\varepsilon(x)\right) \text{ where } \varepsilon(x)\xrightarrow[x\to x_0]{} 0 \quad (*)$$

Remark:

1/ $R_n(x)=\frac{(x-x_0)^{n+1}}{(n+1)!}\left(f^{(n+1)}(x_0)+\varepsilon(x)\right)$ is called the remainder of Young. and the expression (*) is called the taylor's depelopement with remainder of young.

2/ if $x_0=0$, we get the developement of Macc-Laurin's with remainder of young.

eg ① Find the Macc-Laurin's developement with the remainder of Lagrange for the function $f(x)=\cos x$ for the degree $n=4$.

② Deduce the value of the limit $\lim_{x\to 0}\frac{2\cos x-2+x^2}{x^4}$.

Solution: we have: $\cos\in C^{\infty}(\mathbb{R})$ and

$$\cos^{(n)}(x)=\cos\left(x+n\frac{\pi}{2}\right)\Rightarrow \cos^{(n)}(0)=\begin{cases}0, & \text{if } n=2k+1\\ (-1)^k, & \text{if } n=2k\end{cases}$$

hence, the developement of Macc-Laurin with the remainder of Lagrange is given as follows:

$$\cos x=\cos(0)+\frac{\cos^{(1)}(0)}{1!}x^1+\frac{\cos^{(2)}(0)}{2!}x^2+\frac{\cos^{(3)}(0)}{3!}x^3+\frac{\cos^{(4)}(0)}{4!}x^4+\frac{\cos^{(5)}(\theta x)}{5!}x^5,\ 0<\theta<1$$

$$\Rightarrow \cos x=1-\frac{\sin(0)}{1}x-\frac{\cos(0)}{2!}x^2+\frac{\sin(0)}{3!}x^3+\frac{\cos(0)}{4!}x^4-\frac{\sin(\theta x)}{5!}x^5,\ 0<\theta<1$$

$$\Rightarrow \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{\sin(\theta x)}{5!}x^5,\ 0<\theta<1$$

②
$$\lim_{x\to 0}\frac{2\cos x-2+x^2}{x^4}=\lim_{x\to 0}\frac{2\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{\sin(\theta x)}{5!}x^5\right)-2+x^2}{x^4}$$

$$=\lim_{x\to 0}\frac{2-x^2+\frac{x^4}{12}-2\frac{\sin(\theta x)}{5!}x^5-2+x^2}{x^4}$$

$$=\lim_{x\to 0}\left[\frac{\cancel{x^4}}{12}\times\frac{1}{\cancel{x^4}}-\frac{2}{5!}\frac{\sin(\theta x)}{\cancel{x^4}}x^{\cancel{5}}\right]$$

$$=\frac{1}{12}-\lim_{x\to 0}\frac{2}{5!}(\sin(\theta x)\cdot x)=\boxed{\frac{1}{12}}$$

• The nth derivatives:

Definition: Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \to \mathbb{R}$ be continuous on $[$

and differentiable on $]a,b[$. Then the nth derivative of the fun

f, if it exists, is defined by the following reccurent relationship:
progressive

$\forall n \in \mathbb{N}^* : f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$

where $f^{(0)}(x) = f(x)$, $f'(x) = f^{(1)}(x)$; $f''(x) = f^{(2)}(x) = (f^{(1)}(x))'$, $f^{(3)}(x) = (f^{(2)}(x)$

• We say that a function is from a class of $C^n(D)$ iff:

1/ f is differentiable n times on D.

2/ the function $f^{(n)}$ is continuous on D.

That is, $C^n(D)$ is the set of functions that are: continuously differentiable n times on D. (or differentiable by continuity n times