Double and triple integrals

1 Double integrals

Definition 1 *Let $f : D \to \mathbb{R}$ be a function defined on $D \subset \mathbb{R}^2$. The integral over D of f is called a double integral and we denoted by:*

$$\int\int_D f = \int\int_D f(x,y)dxdy.$$

Theorem 1 (Fubini's Theorem) *Let $f : D \to \mathbb{R}$ be a continuous function.*

1. ***Case 1:*** *where $D = [a,b] \times [c,d]$, $a \leq b$ and $c \leq d$. Then,*

$$\int\int_D f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

2. ***Case 2:*** *where $D = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b;\ u(x) \leq y \leq v(x)\}$, where $u, v : [a,b] \to \mathbb{R}$ are continuous functions. Then,*

$$\int\int_D f(x,y)dxdy = \int_a^b \left(\int_{u(x)}^{v(x)} f(x,y)dy \right) dx.$$

Proposition 1 *If $f(x,y) = g(x)h(y)$ with $D = [a,b] \times [c,d]$, where $a \leq b$ and $c \leq d$, then*

$$\int\int_D f(x,y)dxdy = \left(\int_a^b g(x)dx \right) \left(\int_c^d h(y)dy \right).$$

Example 1 *Calculate the following integral:*

$$I = \int\int_D xydxdy, \qquad \text{where} \quad D = [1,2] \times [3,5].$$

Then,

$$\begin{aligned} I &= \left(\int_1^2 xdx \right) \left(\int_3^5 ydy \right) = \left[\frac{1}{2}x^2 \right]_1^2 \left[\frac{1}{2}y^2 \right]_3^5 \\ &= \left(2 - \frac{1}{2} \right) \left(\frac{25}{2} - \frac{9}{2} \right) = 12. \end{aligned}$$

Or,

$$\begin{aligned} I &= \int_1^2 \left(\int_3^5 yxdy \right) dx = \int_1^2 \left[\frac{1}{2} xy^2 \right]_3^5 dx \\ &= \int_1^2 \left(\frac{25}{2} - \frac{9}{2} \right) xdx \\ &= 8 \left[\frac{1}{2} x^2 \right]_1^2 = 12. \end{aligned}$$

Proposition 2 *Let $D \subset \mathbb{R}^2$.*

1. ***Area (D):*** *If $f(x,y) = 1$, then*

$$Area(D) = \int\int_D dxdy.$$

2. ***Positivity:*** *If $f(x,y) \geq 0$, for all $(x,y) \in D$. Then,*

$$\int\int_D f(x,y)dxdy \geq 0.$$

3. ***Linearity:*** *for all $\lambda, \nu \in \mathbb{R}$, we have*

$$\int\int_D \Big(\lambda f(x,y) + \nu g(x,y) \Big) dxdy = \lambda \int\int_D f(x,y)dxdy + \nu \int\int_D g(x,y)dxdy.$$

4. ***Monotonic:*** *Ig $f(x,y) \leq g(x,y)$, for all $(x,y) \in D$. Then,*

$$\int\int_D f(x,y)dxdy \leq \int\int_D g(x,y)dxdy.$$

Exercise 1 *Calculate the following double integrals:*

1. $I = \int\int_D dxdy, \qquad$ *where* $D = \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, \ -2 \leq y \leq -1\}.$

2. $I = \int_0^{\frac{\pi}{2}} \int_0^{\pi} \Big(\sin(x+y) + \cos(x-2y) \Big) dxdy.$

3. $I = \int_1^2 \int_2^3 \frac{x^2}{y} dxdy.$

2 Triple integrals

Definition 2 *Let $f : V \to \mathbb{R}$ be a function defined on $V \subset \mathbb{R}^3$. The integral over V of f is called a triple integral and we denoted by:*

$$\int\int\int_V f = \int\int_V f(x,y,z)dxdydz.$$

Theorem 2 (Fubini's Theorem) *Let $f : V \to \mathbb{R}$ be a continuous function.*

1. ***Forme 1:*** *case where $V = [a,b] \times [c,d] \times [s,t]$, $a \leq b$, $c \leq d$ and $s \leq t$. Then,*

$$\begin{aligned}\int\int\int_V f(x,y,z)dxdydz &= \int_a^b \left[\int_c^d \left(\int_s^t f(x,y)dz\right)dy\right]dx \\ &= \int_s^t \left[\int_c^d \left(\int_a^b f(x,y)dx\right)dy\right]dz = \cdots\end{aligned}$$

2. ***Forme 2:*** *case where $V = \{(x,y,z) \in \mathbb{R}^3 : \ (x,y) \in \Omega, \ u(x,y) \leq z \leq w(x,y)\}$, where Ω is the orthogonal projection of V on (OXY). Then,*

$$\int\int\int_V f(x,y,z)dxdydz = \int\int_\Omega \left(\int_{u(x,y)}^{w(x,y)} f(x,y,z)dz\right)dxdy.$$

3. ***Forme 3:*** *case where $V = \{(x,y,z) \in \mathbb{R}^3 : s \leq z \leq t, \ (x,y) \in \Omega(z)\}$, where $\Omega(z)$ is the intersection of V with $z = cte$. Then,*

$$\int\int\int_V f(x,y,z)dxdydz = \int_s^t \left(\int\int_{\Omega(z)} f(x,y,z)dxdy\right)dz.$$

Exercise 2 *Calculate the following triple integrals:*

1. $I = \int\int\int_V (x - y + 2z)dxdydz$,
   where $V = \{(x,y,z) \in \mathbb{R}^3 : \ 0 \leq x \leq 1, \ 0 \leq y \leq 3, \ 2 \leq z \leq 3\}$.

2. $I = \int\int\int_V (x^2 - 2yz)dxdydz$,
   where $V = \{(x,y,z) \in \mathbb{R}^3 : \ 0 \leq x \leq 1, \ 1 \leq y \leq 2, \ 2 \leq z \leq 3\}$.

3. $I = \int\int\int_V \dfrac{x^2y}{z}dxdydz$, *where* $V = [0,1] \times [0,1] \times [1,2]$.

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