Asymptotic Analysis

Master 2: Functional Analysis

Semestre: 03

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Asymptotic analysis is a local and analytical method used to approximate mathematical functions or the solutions to a problems. This course aims to introduce students to the fundamental principles of asymptotic methods and to develop the essential skills required for a foundational understanding of asymptotic analysis.

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Chaptere 1

Order Relations and Asymptotic Comparison Relations

Introduction

In this chapter, we denote by K either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Let $D \subset K$, and let $x_0 \in D$. Consider two functions f and g defined on D.

Definition

Definition 1.1 (Order Relation – Little-o).

We say that f is a *little-o* of g near x_0 (as $x \to x_0$) if there exists a neighborhood V of x_0 and a function φ defined on $D \cap V$ such that:

$$f(x) = g(x)\varphi(x), \quad \text{and} \quad \lim_{x \to x_0} \varphi(x) = 0. \tag{1.1}$$

In this case, we write:

$$f(x) = o(g(x)) \quad \text{as } x \to x_0. \tag{1.2}$$

Remark

Remark.

We say that f is negligible with respect to g near x_0. In other words, f tends to zero faster than g as $x \to x_0$.

If $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = 0$, then we also have $f = o(g)$.

Equivalent Formulation

If $g(x)$ does not vanish near x_0, the definition above can equivalently be written as:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{1.3}$$

Examples

- $x = o(1)$ as $x \to 0$.
- $\sin(x) = o(x)$ as $x \to 0$.
- $x^2 = o(x)$ as $x \to 0$.

Remark

Interpretation.

The notation $f = o(g)$ expresses that f becomes insignificant compared to g near a certain point. It provides a precise way to compare the rate at which functions approach zero or infinity.

1.1 Remarks and Definitions

Remark 1.4

If g does not vanish on $D \cap V$, the previous definition is equivalent to:

$$f = O(g) \iff \frac{f}{g} \text{ is bounded on } D \cap V.$$

Definition 1.5 – Relation of Equivalence ($\sim$)

We say that f is equivalent to g in the neighborhood of x_0 (or when $x \to x_0$) if there exists a neighborhood V of x_0 and a function ε defined on $D \cap V$ such that:

$$\forall x \in D \cap V : f(x) = (1 + \varepsilon(x))g(x) \quad \text{and} \quad \lim_{x \to x_0} \varepsilon(x) = 0.$$

We then say that f is equivalent to g near x_0, and we write:

$$f \approx g \quad \text{or} \quad f(x) \sim g(x), \quad x \to x_0.$$

Remark 1.6

If the limit $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ exists, the previous definition is equivalent to:

$$f \sim g \iff \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

Remark 1.7

If the functions f and g depend on a parameter $t \in I$, i.e. $f = f(x,t)$ and $g = g(x,t)$, then the previous relations are said to be **uniform** if the function ε and the constant k are independent of t; otherwise, they are said to be **non-uniform**.

2. Examples

Example 1.8

Let $\alpha > \beta > 0$. Consider the polynomial:

$$P(x) = a_0x^2 + a_1x^{2+1} + \cdots + a_nx^{2+n}$$

with $a_0 \neq 0$ and $a_n \neq 0$. We have the following relations:

(a.1) $x^\alpha = o(x^\beta)$	(b.1) $P(x) = O(x^\alpha)$	(c.1) $P(x) \sim_0 a_0x^\alpha$
(a.2) $x^\beta = o(x^\alpha)$	(b.2) $P(x) = O(x^\beta)$	(c.2) $P(x) \sim_\infty a_nx^\beta$
(a.3) $\ln x = o(x^x)$	(b.3) $\sin x = O(1)$	(c.3) $\sin x \sim_0 x$
(a.4) $x^2 = o(e^x)$	(b.4) $\sinh x = O(e^x)$	(c.4) $\sinh x \sim_{+\infty} \frac{1}{2}e^x$

Example 1.9

1. Let $f(x,t) = e^{xt+2}$ and $g(x,t) = e^{x^2+xt}$. Then $f(x,t) = \varepsilon(x)g(x,t)$ where $\varepsilon(x) = e^{2-x^2}$. Since $\lim_{x\to\infty} \varepsilon(x) = 0$ and ε depends only on x, $f(x,t)$ is **uniformly negligible** with respect to $g(x,t)$ near ∞.

2. Let $f(x,t) = x^2 + 2xt$ and $g(x,t) = xt + 2t^2$. Then $f(x,t) = \varepsilon(x,t)g(x,t)$ where $\varepsilon(x,t) = \dfrac{x}{t}$. Since $\lim_{x\to 0} \varepsilon(x,t) = 0$ and ε depends on t, $f(x,t)$ is **negligible** with respect to $g(x,t)$ near 0, but **not uniformly**.

Proposition 1.1.1. ***(Links between order relations)***
Let f, g and h be functions.
In the neighborhood of the point x_0, we have:

1. *If $f(x) = o(g(x))$, then $f(x) = O(g(x))$. (the reverse is false)*
2. *If $f(x) = o(h(x))$ and $g(x) = O(h(x))$, then $f(x) + g(x) = o(h(x))$.*
3. *If $f(x) = O(g(x))$ and $g(x) = o(h(x))$, then $f(x) = o(h(x))$.*
4. *If $f(x) \sim g(x)$, then $f(x) = g(x) + o(g(x))$.*
5. *If $f(x) \sim g(x)$, then $f(x) = O(g(x))$ and $g(x) = O(f(x))$.*
6. *If $f(x) \sim g(x)$ and $g(x) = o(h(x))$, then $f(x) = o(h(x))$.*
7. *If $f(x) = o(g(x))$ and $g(x) \sim h(x)$, then $f(x) = o(h(x))$.*

Proposition 1.1.2. ***(operations with order relations)***.
Let f, g, h and k be functions, and let λ and μ two real numbers.
In the neighborhood of the point x_0, we have:

1. *If $f(x) = o(g(x))$, then $\lambda f(x) = o(\lambda g(x))$.*
2. *If $f(x) = o(g(x))$ and $g(x) = o(h(x))$, then $\lambda f(x) + \mu g(x) = o(h(x))$.*
3. *If $f(x) = o(g(x))$ and $h(x) = o(k(x))$, then $f(x)h(x) = o(g(x)k(x))$.*
4. *If $f(x) = o(g(x))$, then $|f(x)|^\alpha = o(|g(x)|^\alpha)$, $\alpha > 0$.*
5. *Let $x \in D$, if $f(x) = o(g(x))$ and f and g are continuous on D, then*

$$\int_x^{x_0} f(t)\,dt = o\Big(\int_x^{x_0} |g(t)|\,dt\Big).$$

6. *Let $\xi \in]\alpha, \beta[$ a parameter, and $f(x, \xi)$ and $g(x, \xi)$ two functions depend on the parameter ξ. If $f(x, \xi) = o(g(x, \xi))$ uniformly, then*

$$\int_\alpha^\beta f(x,\xi)\,d\xi = o\left(\int_\alpha^\beta |g(x,\xi)|\,d\xi\right).$$

- *The above properties are also valid for (O) and ($\sim$).*

Proposition 1.1.3. .
Let f, g and h are functions verifying $f(x) \sim g(x)$ (f and g nonzero in a neighborhood of a point x_0), then,

1. $f(x)h(x) \sim g(x)h(x)$.
2. $\dfrac{h(x)}{f(x)} \sim \dfrac{h(x)}{g(x)}$.
3. $\dfrac{f(x)}{h(x)} \sim \dfrac{g(x)}{h(x)}$.

Remarks 1.1.1. .

1. *If* $\lim_{x\to x_0} f(x) = l$, *then* $f(x) \sim l$.
2. $f(x) \sim g(x) \Rightarrow \lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$.
3. $f(x) \sim g(x) \Rightarrow \operatorname{sign}(f(x)) = \operatorname{sign}(g(x))$ *in a neighborhood of* x_0.

Proposition 1.1.4. .
Let f *and* g *ions two functions verifying* $f(x) \sim g(x)$, *and* φ *a function defined in a neighborhood of a point* $b \in \mathbb{R}$ *such that* $\varphi(x) \underset{x\mapsto b}{\longrightarrow} a$.
Then,

$$f \circ \varphi(x) \underset{b}{\sim} g \circ \varphi(x).$$

Remarks 1.1.2. .

1. $f_1(x) = o(g_1(x))$ *and* $f_2(x) = o(g_2(x))$ *do not imply that* $f_1(x) + f_2(x) = o(g_1(x) + g_2(x))$.
2. $f(x) = o(g(x))$ *does not imply that* $f'(x) = o(g'(x))$.

- *The two preceding remarks also apply to the relations* O *and* $\sim$.

3. *Never write* $f(x) \sim 0$.
4. $f(x) \sim g(x)$ *does not imply that* $\lim_{x\to x_0}(f(x) - g(x)) = 0$.

1.2 Asymptotic Sequences

Definition 1.2.1 (gauge functions)**.** *. Let $(\delta_n(x))_{n\in\mathbb{N}}$ be a sequence of functions defined on $D \subset \mathbb{R}$, and let $x_0 \in \overline{D}$. We say that $(\delta_n(x))_n$ is an textbfasymptotic sequence (or a sequence of gauge functions) in a neighborhood of x_0 if*

$$\forall n \in \mathbb{N}: \quad \delta_{n+1}(x) = o(\delta_n(x)), \quad x \to x_0.$$

Remark 1.2.1. *If $\delta_{n+1}(x) = o(\delta_n(x))$, then $\delta_{n+k}(x) = o(\delta_n(x))$ for all $k > 0$.*

Examples 1.2.1. *1. $\delta_n(x) = (x - x_0)^n$, $\Big(\delta_n(x)\Big)_{n\in\mathbb{N}}$ is an asymptotic sequence near x_0.*

2. $\delta_n(x) = \dfrac{1}{x^n}$, $\Big(\delta_n(x)\Big)_{n\in\mathbb{N}}$ is an asymptotic sequence near ∞.

3. let g be a function defined on $D \subset \mathbb{R}$ such that $\lim\limits_{x\to x_0} g(x) = 0$, then $\delta_n(x) = (g(x))^n$, $\Big(\delta_n(x)\Big)_{n\in\mathbb{N}}$ is an asymptotic sequence near x_0.

Remark 1.2.2. *An asymptotic sequence is not necessarily convergent.* ***Exemple:*** *$\delta_n = x^{\frac{1}{x}}$, $x \to \infty$*
$\lim\limits_{x\to\infty} \dfrac{\delta_{n+1}}{\delta_n} = \lim\limits_{x\to\infty} \dfrac{x^{\frac{1}{n+1}}}{x^{\frac{1}{n}}} = \lim\limits_{x\to\infty} x^{\frac{-1}{n(n+1)}} = 0$
but, $\lim\limits_{n\to\infty} \delta_n(x) = \lim\limits_{x\to\infty} x^{\frac{1}{n}} = \infty$, i.e $\delta_n(x)$ divergent.

Theorem 1.2.1. *Every subsequence of an asymptotic sequence is itself an asymptotic sequence.*

Proof. let $\alpha_n = \delta_{\varphi(n)}$, where $\varphi : \mathbb{N} \underset{n\mapsto\varphi(n)}{\to} \mathbb{N}$is an increasing function.
$\alpha_{n+1}(x) = \delta_{\varphi(n+1)} = o\Big(\delta_{\varphi(n+1)}(x)\Big) = o\Big(\delta_{\varphi(n)}(x)\Big)$ near x_0. Because φ is increasing $n+1 > n \Rightarrow \varphi(n+1) > \varphi(n)$. □

Theorem 1.2.2. *Let $(\delta_n(x))_{n\in\mathbb{N}}$ be an asymptotic sequence near x_0. If the functions $\delta_n(x)$ are integrable, then the sequence*

$$\beta_n(x) = \left(\int_{x_0}^{x} |\delta_n(t)|\, dt\right)_{n\in\mathbb{N}}$$

is also asymptotic near x_0.

Proof. Let

$$g_n(x) = \int_{x_0}^{x} |f_n(t)|\, dt.$$

We will show that $(g_n(x))_{n\in\mathbb{N}}$ is an asymptotic sequence, i.e.,

$$\forall n \in \mathbb{N}: \quad g_{n+1}(x) = o(g_n(x)).$$

□

$$g_{n+1}(x) = \int_{x_0}^{x} f_{n+1}(t)\,dt = \int_{x_0}^{x} o(f_n(t))\,dt = o\Big(\int_{x_0}^{x} |f_n(t)|\,dt\Big) = o(g_n(x)).$$

Theorem 1.2.3. *Let $(\delta_n(x,\xi))_{n\in\mathbb{N}}$ be a uniformly asymptotic sequence compared to the parameter $\xi \in]a,b[$, in a neighborhood of x_0.*
If the functions $\delta_n(x,\xi)$ are integrable with respect to ξ, then the sequence

$$\beta_n(x) = \left(\int_a^b |f_n(x,\xi)|\,d\xi\right)_{n\in\mathbb{N}}$$

is asymptotic in a neighborhood of x_0.

Theorem 1.2.4. 1. *If $(\delta_n(x))_{n\in\mathbb{N}}$ be an asymptotic sequence near x_0, and $\alpha > 0$, then $|\ \delta_n(x)\ |^{\alpha}$ is also asymptotic near x_0.*

2. *If $(\delta_n(x))_{n\in\mathbb{N}}$ and $(\beta_n(x))_{n\in\mathbb{N}}$ are two equivalent sequences .*
   $\forall n, \delta_n(x) = O(\beta_n(x))$ and $\beta_n(x) = O(\delta_n(x)), \forall x \in V(x_0)$.
   if one is asymptotic, the other is also asymptotic near x_0 near x_0.

Remark 1.2.3. *The differentiations are not allowed.*
$\delta_n(x)$ is asymptotic near $x_0 \nRightarrow \delta_n'(x)$ is asymptotic near x_0. ***Exemple:***

Definition 1.2.2 (1.17)**.** *Two sequences $(f_n(x))_{n\in\mathbb{N}}$ and $(g_n(x))_{n\in\mathbb{N}}$ are said to be* equivalent *in a neighborhood of x_0 if*

$$\forall n \in \mathbb{N}: \quad f_n(x) \sim g_n(x) \quad \textit{near } x_0.$$

Theorem 1.2.5. *Let $(f_n(x))_{n\in\mathbb{N}}$ and $(g_n(x))_{n\in\mathbb{N}}$ be two equivalent sequences in a neighborhood of x_0. Then $(f_n(x))_{n\in\mathbb{N}}$ is asymptotic near x_0 if and only if $(g_n(x))_{n\in\mathbb{N}}$ is asymptotic near x_0.*

Proof. Assume that $(f_n(x))_{n\in\mathbb{N}}$ is an asymptotic sequence near x_0, i.e.

$$\forall n \in \mathbb{N}: \quad f_{n+1}(x) = o(f_n(x)).$$

We show that $(g_n(x))_{n\in\mathbb{N}}$ is also asymptotic near x_0, i.e.

$$\forall n \in \mathbb{N}: \quad g_{n+1}(x) = o(g_n(x)).$$

Indeed,

$$\forall n \in \mathbb{N}: \quad g_{n+1}(x) = O(f_{n+1}(x)) = O(o(f_n(x))) = o(f_n(x)) = o(O(g_n(x))) = o(g_n(x)).$$

Hence, $(g_n(x))_{n\in\mathbb{N}}$ is asymptotic near x_0. The converse implication is proved in the same way. □

1.3 Asymptotic Series

Definition 1.3.1 (1.19)**.** *The series*

$$\sum_{n>0} a_n f_n(x)$$

is said to be asymptotic *in a neighborhood of* x_0 *if the sequence* $(f_n(x))_{n\in\mathbb{N}}$ *is asymptotic near* x_0.

In this case, we have

$$\sum_{n>0} a_n f_n(x) = \sum_{k=0}^{n} a_k f_k(x) + o(f_n(x)), \quad \forall n \in \mathbb{N}.$$

Or equivalently,

$$\sum_{n>0} a_n f_n(x) = \sum_{k=0}^{n} a_k f_k(x) + O(f_{n+1}(x)), \quad \forall n \in \mathbb{N}.$$

Example 1.3.1 (1.20)**.**

1. *Power series are asymptotic series.*
2. *The series*

$$\sum_{n>0} 3^{2n+1} \sin^n x$$

is an asymptotic series near 0.

Chaptere 2

Asymptotic Expansion of Functions

2.1 Asymptotic Expansion

Generally, an asymptotic expansion has in commun with a TAylor expansion the fact that it provides an approximation of a function, which is expressed as a sum of functions, arranged from the "largest" to the "smalles", together with a remainder term that is negligible compared with all the other trems in the sum. This is what is called a comparison scale.
These functions can be of any nature, whereas a Taylor expansion contains only polynominal terms.

Definition 2.1.1. *Let f be a function defined on a subset $D \subset \mathbb{R}$ and let $x_0 \in \overline{D}$. Let $(\delta_i(x))_{i\in\mathbb{N}}$ be an asymptotic sequence in a neighborhood of x_0.*

We say that f admits an asymptotic expansion *near x_0 with respect to the sequence $(\delta_i(x))_{i\in\mathbb{N}}$* ***of order*** *$N$ if there exists a numerical sequence $(a_i)_{i\in\mathbb{N}}$ such that*

$$f(x) - \sum_{i=0}^{n} a_i\delta_i(x) = o(\delta_n(x)). \tag{2.1}$$
$$= O(f_{n+1}(x)).$$

If relation (2.1) holds for all $n > 0$, we say that f is expandable in asymptotic series with respect to the sequence $(\delta_i(x))_{i\in\mathbb{N}}$ in a neighborhood of x_0, and we write

$$f(x) \underset{x_0}{\sim} \sum_{i\geq 0} a_i\delta_i(x).$$

Remark 2.1.1. *The* Taylor expansion *near x_0 is an asymptotic expansion; it suffices to take $\delta_n(x) = (x - x_0)^n$.*

Example 2.1.1. *Near 0, we have*

$$\frac{1}{1+t} = \sum_{n>0} (-1)^n t^n.$$

If we take $t = \sin x$*, then*

$$\frac{1}{1+\sin x} = \sum_{n>0}(-1)^n(\sin x)^n.$$

Since the sequence $(\delta_n(x))_{n\in\mathbb{N}} = (\sin^n x)_{n\in\mathbb{N}}$ *is asymptotic near* 0*, Indeed,* $\lim\limits_{x\to 0}\frac{\delta_{n+1}(x)}{\delta_n(x)} = \lim\limits_{x\to 0}\frac{sin^{n+1}}{sin^n} = \lim\limits_{x\to 0} sinx = 0.$
It follows that

$$g(x) = \frac{1}{1+\sin x}$$

is expandable near 0 *in asymptotic series with respect to the sequence* $(\delta_n(x))_{n\in\mathbb{N}} = (\sin^n x)_{n\in\mathbb{N}}$.

Theorem 2.1.1. *The asymptotic expansion with respect to an asymptotic sequence* $(\delta_i(x))_{n\in\mathbb{N}}$ *of a given function* $f(x)$ *if it exists it is unique.*
In other words: The coefficients of the asymptotic expansion of a function $f(x)$ *with respect to an asymptotic sequence* $(\delta_i(x))_{i\in\mathbb{N}}$ *are unique.*

Proof. Assume that

$$f(x) = \sum_{i=0}^{n} a_i\delta_i(x) + o(\delta_n(x)) = \sum_{i=0}^{n} b_i\delta_i(x) + o(\delta_\delta n(x)).$$

For $n = 0$ we have $a_0 = \lim\limits_{x\to x_0}\frac{f(x)}{\delta_0(x)}$ and $b_0 = \lim\limits_{x\to x_0}\frac{f(x)}{\delta_0(x)}$. From the uniqueness of the limit we find $a_0 = b_0$.
Assume that the property is true up to order n, i.e. $a_i = b_i, \forall i = \overline{0,n}$. We have for $n+1$

$$a_{n+1} = \lim_{x\to x_0}\frac{f(x) - \sum_{i=0}^{N} a_i\delta_i(x)}{\delta_{n+1}(x)} = \lim_{x\to x_0}\frac{f(x) - \sum_{i=0}^{n} b_i\delta_i(x)}{\delta_{n+1}(x)} = b_{n+1}.$$

So $a_i = b_i \forall i \geq 0$, hence the uniqueness of the coefficients. □

Definition 2.1.2. *we say that* $f(x)$ *and* $g(x)$ *are asymptotically equal (equivalent) in the neighborhood of* x_0 *with respect to the asymptotic sequence* $(\delta_i(x))_{i\in\mathbb{N}}$ *if* $f(x)$ *and* $g(x)$ *admit the same asymptotic expansion in the neighborhood of* x_0 *with respect to the asymptotic sequence* $(\delta_i(x))_{i\in\mathbb{N}}$*. That is to say*

$$f(x) - g(x) = o(\delta_i(x)),\ x \to x_0, \forall i = 1, 2, ...$$

Proposition 2.1.1 (Operations on the asymptotic expansion). *Let* $(\delta_i(x))_{i\in\mathbb{N}}$ *be an asymptotic sequence in a neighborhood of* x_0*, and let* f *and* g *be two functions defined on* D *such that*

$$f(x) \underset{x_0}{\sim} \sum_{i\geq 0} a_i\delta_i(x) \quad and \quad g(x) \underset{x_0}{\sim} \sum_{i\geq 0} b_i\delta_i(x).$$

Then the following properties hold:

1. *If* $\alpha, \beta \in \mathbb{R}$ *such that* $(\alpha, \beta) \neq (0,0)$, *then*

$$\alpha f(x) + \beta g(x) \underset{x_0}{\sim} \sum_{i\geq 0} (\alpha a_i + \beta b_i)\delta_i(x).$$

2.

$$\begin{aligned} f(x).g(x) &\underset{x_0}{\sim} \Big(\sum_{i\geq 0} a_i\delta_i(x)\Big)\Big(\sum_{i\geq 0} b_i\delta_i(x)\Big). \\ &\underset{x_0}{\sim} a_1 b_1 \delta_i^2(x) + ... \end{aligned}$$

3. *If the functions* $f(x)$ *and* $\delta_i(x)$ *are integrable, then*

$$\int_{x_0}^{x} f(t)\,dt \underset{x_0}{\sim} \sum_{i\geq 0} a_i \int_{x_0}^{x} \delta_i(t)\,dt.$$

Proof. .
(1) We have

$$f(x) \sim_{x_0} \sum_{i\geq 0} a_i\delta_i(x) \iff f(x) = \sum_{k=0}^{n} a_k\delta_k(x) + o(\delta_n(x)),$$

and

$$g(x) \sim_{x_0} \sum_{n>0} b_n\delta_n(x) \iff g(x) = \sum_{k=0}^{n} b_k\delta_k(x) + o(\delta_n(x)).$$

Hence,

$$\begin{aligned} \alpha f(x) + \beta g(x) &= \sum_{k=0}^{n} a_k\delta_k(x) + o(\delta_n(x)) + \sum_{k=0}^{n} b_k\delta_k(x) + o(\delta_n(x)) \\ &= \sum_{k=0}^{n} (a_k + b_k)\delta_k(x) + o(\delta_n(x)). \end{aligned}$$

Thus,

$$f(x) + g(x) \sim_{x_0} \sum_{n>0} (a_n + b_n)\delta_n(x).$$

(2) From

$$f(x) \sim_{x_0} \sum_{n>0} a_n\delta_n(x) \iff f(x) = \sum_{k=0}^{n} a_k\delta_k(x) + o(\delta_n(x)),$$

we have

$$\int_{x_0}^{x} f(t)\,dt = \int_{x_0}^{x}\Big(\sum_{k=0}^{n} a_k\delta_k(t) + o(\delta_n(t))\Big)\,dt = \sum_{k=0}^{n} a_k \int_{x_0}^{x}\delta_k(t)\,dt + o\Big(\int_{x_0}^{x}\delta_n(t)\,dt\Big).$$

Hence,

$$\int_{x_0}^{x} f(t)\,dt \sim_{x_0} \sum_{n>0} a_n \int_{x_0}^{x}\delta_n(t)\,dt.$$

□

Proposition 2.1.2. *Let*

$$f(x,t) \sim_{t\to 0} \sum_{n>0} a_i(x)\delta_i(t)$$

uniformly for $t \in [a,b]$.

•*If the functions $f(x,t)$ and $\delta_i(t)$ are integrable with respect to t, then*

$$\int_0^t f(\xi,t)\,d\xi \sim_{x_0} \sum_{i>0} a_i(t)\int_0^t \delta_i(\xi)\,d\xi.$$

•*likewise, if the functions $f(x,t)$ and $a_i(x)$ are integrable with respect to x, then*

$$\int_a^b f(x,t)\,dx \sim_{x_0} \sum_{i>0}\int_a^b a_i(x)\,dx\delta_i(x).$$

Remark 2.1.2. *Differentiation is not allowed:*

$$f(x,t) \sim_{x_0} \sum_{i>0} a_i(t)\delta_i(x) \nRightarrow \frac{\partial f(x,t)}{\partial t} \sim_{x_0} \sum_{i>0} a_i'(t)\delta_i(x)$$

Theorem 2.1.2. *If $f(x,t) \sim_0 \sum_{i>0} a_i(x)\delta_i(t)$ and if*

$$\frac{\partial f(x,t)}{\partial x} \sim_0 \sum_{i>0} b_i(x)\delta_i(t).$$

Then,

$$b_i(x) = a_i'(x) = \frac{da_i(x)}{dx}$$

.

2.2 Calculation of the coefficients of an asymptotic expansion

Let $(\delta_i(x))_{i\in\mathbb{N}}$ be a sequence asymptotique, let $f(x)$ be a function defined on D, $x_0 \in \overline{D}$. let's suppose that $f(x)$ admits an asymptotic expansion x_0 with respect to the $\delta_i(x)$,

$x \to x_0$.

$$f(x) = \sum_{i=0}^{n} a_i \delta_i(x) + o(\delta_n(x)), \quad \forall n \in \mathbb{N}.$$

We divide both sides by $\delta_0(x)$. (We always assume that the limits exist).
we find

$$\frac{f(x)}{\delta_0(x)} = a_0 + a_1 \frac{\delta_1(x)}{\delta_0(x)} + ... + \frac{o(f_n(x))}{\delta_0(x)}. \Rightarrow a_0 = \lim_{x \to x_0} \left(\frac{f(x)}{\delta_0(x)} - a_1 \frac{\delta_1(x)}{\delta_0(x)} + ... + \frac{o(f_n(x))}{\delta_0(x)} \right).$$

$$\Rightarrow a_0 = \lim_{x \to x_0} \frac{f(x)}{\delta_0(x)},$$

since $\lim_{x \to x_0} \frac{\delta_i(x)}{\delta_0(x)} = 0,\ i \geq 1$ and $\lim_{x \to x_0} \frac{o(f_n(x))}{\delta_0(x)} = 0$
Likewise,

$$\frac{f(x) - a_0 \delta_0(x)}{\delta_1(x)} = a_1 + a_2 \frac{\delta_2(x)}{\delta_1(x)} + ... + \frac{o(f_n(x))}{\delta_1(x)} \Rightarrow a_1 = \lim_{x \to x_0} \left(\frac{f(x) - a_0 \delta_0(x)}{\delta_1(x)} \right).$$

and by recurrence,

$$a_0 = \lim_{x \to x_0} \frac{f(x)}{\delta_0(x)}, \text{ and } a_n = \lim_{x \to x_0} \left(\frac{f(x) - \sum_{i=0}^{n-1} a_i \delta_i(x)}{\delta_n(x)} \right), \ \forall n \geq 1$$

.

By induction, we can show that

$$a_n = \lim_{x \to x_0} \frac{f(x) - \sum_{k=0}^{n-1} a_k f_k(x)}{f_n(x)}, \quad \forall n \in \mathbb{N}.$$

Example 2.2.1. *Let*

$$f(x) = \frac{1}{\sin x} \left(1 - \frac{x}{e^x - 1} \right).$$

We will develop the asymptotic expansion of $f(x)$ with respect to the sequence (x^n) near 0.

- $(\delta_n)_n = (x^n)_n$ *is an asymptotique sequence:*

$$\lim_{x \to x_0} \frac{\delta_{n+1}}{\delta_n} = \lim_{x \to x_0} \frac{x^{n+1}}{x^n} = \lim_{x \to x_0} x = 0.$$

Calculation of a_0:

$$\begin{aligned}
a_0 &= \lim_{x\to 0}\frac{f(x)}{x^0} = \lim_{x\to 0}\frac{1}{\sin x}\left(1-\frac{x}{e^x-1}\right)\\
&= \lim_{x\to 0}\frac{x}{\sin x}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)\\
&= \lim_{x\to 0}\frac{e^x-x-1}{x(e^x-1)} \xrightarrow[\text{(L'Hopital's rule)}]{} \lim_{x\to 0}\frac{e^x-1}{(x+1)e^x-1} \xrightarrow[\text{(L'Hopital's rule)}]{} \lim_{x\to 0}\frac{e^x}{(x+2)e^x} = \frac{1}{2}.
\end{aligned}$$

Hence, $a_0 = \frac{1}{2}$.

Calculation of a_1:

$$\begin{aligned}
a_1 &= \lim_{x\to 0}\frac{f(x)-\frac{1}{2}}{x} = \lim_{x\to 0}\left(\frac{1}{x\sin x}\left(1-\frac{x}{e^x-1}\right)-\frac{1}{2x}\right)\\
&= \lim_{x\to 0}\left(\frac{x}{\sin x}\left(\frac{1}{x^2}-\frac{1}{x(e^x-1)}\right)-\frac{1}{2x}\right) = \lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x(e^x-1)}-\frac{1}{2x}\right)\\
&= \lim_{x\to 0}\frac{(2-x)e^x-2-x}{2x^2e^x-2x^2} \xrightarrow[\text{(L'Hopital's rule)}]{} \lim_{x\to 0}\frac{(1-x)e^x-1}{(2x^2+4x)e^x-4x}\\
&\xrightarrow[\text{(L'Hopital's rule)}]{} \lim_{x\to 0}\frac{-xe^x}{(2x^2+8x+4)e^x-4} \xrightarrow[\text{(L'Hopital's rule)}]{} \lim_{x\to 0}\frac{(-1-x)e^x}{(2x^2+12x+12)e^x} = -\frac{1}{12}.
\end{aligned}$$

Therefore,

$$\frac{1}{\sin x}\left(1-\frac{x}{e^x-1}\right) = \frac{1}{2}-\frac{x}{12}+o(x).$$

2.3 The method of integration by Parts to obtain an asymptotic expansion

Let u and v be two functions of class C^1 on $[a,b]\subset\mathbb{R}$. The formula of integration by parts is given by

$$\int_a^b u'(t)v(t)\,dt = \Big[u(t)v(t)\Big]_a^b - \int_a^b u(t)v'(t)\,dt. \tag{2.2}$$

Let f be a function defined on $[a,b]$ by an integral of the form

$$f(x) = \int_{a(x)}^{b(x)} g(x,t)\,dt.$$

We shall apply formula (2.2) to f in order to obtain an asymptotic expansion of f.

Example 2.3.1. *Let the function f be defined on $]0,+\infty[$ by*

$$f(x) = \int_0^{+\infty}\frac{e^{-t}}{x+t}\,dt.$$

We shall use the method of integration by parts *to obtain an asymptotic expansion of f*

near $+\infty$ with respect to the sequence $(\delta_n(x))_n = (x^{-n})_n$.

Let $u'(t) = e^{-t}$ and $v(t) = \dfrac{1}{x+t}$. Then

$$f(x) = \left[-\frac{e^{-t}}{x+t}\right]_0^{+\infty} - \int_0^{+\infty} \frac{e^{-t}}{(x+t)^2}\,dt = \frac{1}{x} - \int_0^{+\infty} \frac{e^{-t}}{(x+t)^2}\,dt.$$

By repeating integration by parts three times, we obtain

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - 2\cdot 3\int_0^{+\infty} \frac{e^{-t}}{(x+t)^4}\,dt.$$

By induction, one shows that

$$f(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{x^k} + (-1)^n n! \int_0^{+\infty} \frac{e^{-t}}{(x+t)^{n+1}}\,dt.$$

It is clear that the sequence $(\delta_n(x))_n = (x^{-n})_n$ is asymptotic near $+\infty$. Indeed,

$$\lim_{x\to\infty} \frac{\delta_{n+1}(x)}{\delta_n(x)} = \lim_{x\to\infty} \frac{x^n}{x^{n+1}} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

It remains to verify that

$$R_n(x) = \int_0^{+\infty} \frac{e^{-t}}{(x+t)^{n+1}}\,dt = o\left(\frac{1}{x^n}\right).$$

We have

$$|R_n(x)| = \left|\int_0^{+\infty} \frac{e^{-t}}{(x+t)^{n+1}}\,dt\right| \le \frac{1}{x^{n+1}} \int_0^{+\infty} e^{-t}\,dt = \frac{1}{x^{n+1}}.$$

Hence,

$$\left|\frac{R_n(x)}{\delta_n(x)}\right| = |x^n R_n(x)| < \frac{x^n}{x^{n+1}} = \frac{1}{x} \xrightarrow[x\to\infty]{} 0 \quad \Rightarrow \quad R_n(x) = o\left(\frac{1}{x^n}\right).$$

Therefore,

$$f(x) = \int_0^{+\infty} \frac{e^{-t}}{x+t}\,dt \sim \sum_{n\ge 1} \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

2.4 Expansion of an Inverse Function

Recall that the inverse of a function f is a function f^{-1} such that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x,$$

at least on part of the domain.

For a function such as $f(x) = x^3 + x$, it is difficult to find an explicit formula for $f^{-1}(x)$. However, we can find the *asymptotic series* of $f^{-1}(x)$ as $x \to \infty$.

Example 2.4.1. *Find the first three terms of the inverse of the function*

$$f(x) = x^3 + x$$

as $x \to \infty$.

Solution: *Since $x^3 + x \sim x^3$ when $x \to \infty$, it is natural to assume that the inverse function behaves like $\sqrt[3]{x}$ as $x \to \infty$. But what will be the next term of the series?*

The idea is to extract this leading term by writing

$$f^{-1}(x) = \sqrt[3]{x} + g(x),$$

with $g(x) = o(\sqrt[3]{x})$ and to seek an asymptotic approximation of $g(x)$.
Since we know that $f(f^{-1}(x)) = x$, we have

$$(\sqrt[3]{x} + g(x))^3 + \sqrt[3]{x} + g(x) = x.$$

That is,

$$x + 3g(x)\sqrt[3]{x^2} + 3g^2(x)\sqrt[3]{x} + g^3(x) + \sqrt[3]{x} + g(x) = x.$$

Using the fact that $g(x) = o(\sqrt[3]{x})$, it is not necessary to keep the terms containing the square (or higher powers) of the unknown function. Hence,

$$x + 3g(x)\sqrt[3]{x^2} + o\Big(g(x)\sqrt[3]{x^2}\Big) + \sqrt[3]{x} + g(x) = x.$$

By eliminating the terms that are known to be smaller, we obtain

$$3g(x)\sqrt[3]{x^2} \sim -\sqrt[3]{x} \quad \Rightarrow \quad g(x) \sim -\frac{1}{3}\frac{1}{\sqrt[3]{x}}.$$

We now have the first two terms of the asymptotic series:

$$f^{-1}(x) \sim \sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}} \quad \text{as } x \to \infty.$$

To find the next term of the series, we repeat the process, assuming that

$$f^{-1}(x) = \sqrt[3]{x} - \frac{1}{3}\sqrt[3]{\frac{1}{x}} + h(x),$$

with $h(x) = o\left(\frac{1}{\sqrt[3]{x}}\right)$.

Since $f(f^{-1}(x)) = x$, we have

$$\left(\sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}} + h(x)\right)^3 + \sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}} + h(x) = x.$$

Expanding, we obtain

$$\left(\sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}}\right)^3 + 3\left(\sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}}\right)^2 h(x) + 3\left(\sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}}\right) h^2(x) + h^3(x) + \sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}} + h(x) = x.$$

Any term involving $h^2(x)$ is negligibly small, so we can write

$$\left(\sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}}\right)^3 + 3\left(\sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}}\right)^2 h(x) + o\left(\sqrt[3]{x^2}h(x)\right) + \sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}} + h(x) = x.$$

It is clear that the largest term containing $h(x)$ is $3\sqrt[3]{x^2}h(x)$. Thus,

$$\left(x - \sqrt[3]{x} + \frac{1}{3}\frac{1}{\sqrt[3]{x}} - \frac{1}{27x}\right) + 3\sqrt[3]{x^2}h(x) + \sqrt[3]{x} - \frac{1}{3}\sqrt[3]{\frac{1}{x}} + o\left(\sqrt[3]{x^2}h(x)\right) = x.$$

Simplifying,

$$\frac{1}{27x} = 3\sqrt[3]{x^2}h(x) + o\left(\sqrt[3]{x^2}h(x)\right),$$

which means

$$3\sqrt[3]{x^2}h(x) \sim \frac{1}{27x}.$$

Hence,

$$h(x) \sim \frac{1}{81\sqrt[3]{x^5}}.$$

Therefore, we obtain

$$f^{-1}(x) \sim \sqrt[3]{x} - \frac{1}{3}\frac{1}{\sqrt[3]{x}} + \frac{1}{81\sqrt[3]{x^5}} \quad \text{as } x \to \infty.$$

Let's recap the steps that were used

1. Determine the first term of the asymptotic series. This can often be done using simple approximations.

2. Add an unknown function to the series obtained so far. Assume this function is smaller than the previous term.

3. Substitute this series into the equation that the function must satisfy.

4. Expand this equation carefully in an asymptotic manner, cancelling terms as much as possible.

5. The remaining terms should yield an equation for the unknown function, which is now easy to solve. This provides the next term of the series.

6. Repeat steps (2–5) to obtain additional terms in the series.

2.5 Expansion of an Implicit Function

Sometimes, it is easy to determine the asymptotic series for the solution of an equation $y = f(x)$, but it generally becomes a problem when the equation is of the form $f(x, y) = 0$.

If there are three or more terms in an equation $f(x, y) = 0$, usually two of the terms dominate the others. Therefore, we can form an asymptotic equation using only the two dominant terms. Such equations are usually very easy to solve.

The problem, of course, lies in determining which two terms are dominant. This can only be established through trial and error. In each case, we must verify whether the other terms are indeed small compared to those assumed to be dominant.

Example 2.5.1. *Find the behavior of the function defined implicitly by*

$$x^2 + xy - y^3 = 0 \quad \text{as} \quad x \to +\infty.$$

solution: Since there are three nonzero terms, there are three possible pairs of terms.

First choice: Suppose y^3 is smaller when $x \to +\infty$, i.e. $y^3 = o(xy)$ and $y^3 = o(x^2)$. Then we have

$$x^2 + xy + o(xy) = 0 \quad \Longleftrightarrow \quad x^2 \sim -xy \quad \Rightarrow \quad y \sim -x.$$

But then $y^3 \sim -x^3 \neq o(x^2)$. Contradiction.

Second choice: Suppose x^2 is smaller. Then we have

$$xy - y^3 + o(xy) = 0 \quad \Longleftrightarrow \quad y^3 \sim xy \quad \Rightarrow \quad y \sim \pm\sqrt{x}.$$

But then $xy \sim \pm x^{3/2} = o(x^2) \Rightarrow xy = o(x^2)$. Contradiction.

Third choice: Suppose xy is smaller. Then we have

$$x^2 - y^3 + o(x^2) = 0 \quad \Longleftrightarrow \quad y^3 \sim x^2 \quad \Rightarrow \quad y \sim x^{2/3}.$$

In this case, $xy \sim x^{5/3} = o(x^2) \Rightarrow xy = o(x^2)$, which proves that the choice is valid.

• To find the next term in the series, we set $y = x^{2/3} + g(x)$ with $g(x) = o(x^{2/3})$. Then we have

$$x^2 + x(x^{2/3} + g(x)) = (x^{2/3} + g(x))^3$$

$$\iff x^2 + x^{5/3} + xg(x) = x^2 + 3x^{4/3}g(x) + o(x^{4/3}g(x))$$

$$\iff x^{5/3} + xg(x) = 3x^{4/3}g(x) + o(x^{4/3}g(x))$$

$$\iff x^{5/3} = 3x^{4/3}g(x) + o(x^{4/3}g(x))$$

$$\iff x^{5/3} \sim 3x^{4/3}g(x)$$

$$\iff g(x) \sim \frac{x^{1/3}}{3} = o(x^{3/2}).$$

Thus,

$$y \sim x^{2/3} + \frac{x^{1/3}}{3} \quad \text{as} \quad x \to +\infty.$$

Let's recap the steps that were used in this method.

1. Guess which terms may be negligible.
2. Eliminate those terms to form a simpler equation and solve it exactly.
3. Check that the solution is consistent with step 1. If not, try eliminating different terms.
4. Determine the next term to verify that the leading behavior is correct.
5. Verify all other possible pairs of dominant terms, since more than one solution may exist.

Chaptere 3

Asymptotic study of functions defined by integrals

3.1 Introduction

In this chapter, we study the asymptotic behavior of certain parametric integrals.

We have already seen that integration by parts is a way to find the asymptotic approximations of integrals, but its application is limited.

An important class of integrals that, under certain conditions, lends itself to this method of integration by parts belongs to the class of **Laplace integrals** of the forme

$$I(x) = \int_0^\infty f(t)\, e^{-xt}\, dt. \tag{3.1}$$

3.1.1 Watson's lemma

We begin with **Watson's lemma**, which provides an asymptotic expansion for Laplace type inegrals of the first kind.

Theorem 3.1.1 (Watson's lemma)**.** .
Let f be a fonction of a real variable t with complex values, satisfying

1. *f is continues on the interval* $[0, \infty]$.

2. *f admits an asymptotic expansion*

$$f(t) \sim \sum_{k=0}^{\infty} a_k\, t^{\lambda_k},\ t \to 0^+$$

3. *For a certain fixed $c > 0$*

$$f(t) = O(e^{ct}),\ t \to +\infty,$$

Then,

$$\int_0^\infty f(t)\, e^{-xt}\, dt \underset{n\to\infty}{\sim} \sum_{k=0}^{\infty} a_k \frac{\Gamma(\lambda_k + 1)}{x^{\lambda_k + 1}}.$$

Proof. By conditions (1, 2 and 3), the integral converges for $x > c$. (we can integrate term by term).
According to (3)

Recall

$$\int_0^\infty e^{-y}\, y^{\lambda_n}\, dy = \Gamma(\lambda_n + 1).$$

□

Special case

$$I(x) = \int_0^\infty e^{-xt}\, t^\alpha\, g(t)\, dt,$$

and $g(t)$ has a Taylor expansion in the neighborhood of 0

$$g(t) = \sum_{n=0}^\infty \frac{g^{(n)}(0)}{n!} t^n,$$

we apply Watson's theorem

$$\begin{aligned} f(t) &= t^\alpha \sum_{n=0}^\infty \frac{g^{(n)}(0)}{n!} t^n \\ &= \sum_{n=0}^\infty \frac{g^{(n)}(0)}{n!} t^{n+\alpha}. \end{aligned}$$

So, $I(x) =\underset{n\to\infty}{\sim} \sum_{n=0}^\infty \frac{g^{(n)}(0)}{n!} \frac{\Gamma(n+\alpha+1)}{x^{n+\alpha+1}}$.

Example 3.1.1. *Consider the function I defined on $\mathbb{R}^+$ by*

$$I(x) = \int_0^{\pi/2} e^{-x\tan^2\theta}\, d\theta.$$

With the change of variable $t = \tan^2\theta$ we have $\theta = \arctan\sqrt{t}$, $d\theta = \dfrac{1}{2\sqrt{t}(1+t)}\, dt$ and

$$I(x) = \int_0^\infty \frac{e^{-xt}}{2\sqrt{t}(1+t)}\, dt.$$

Let's pose $f(t) = \dfrac{1}{2\sqrt{t}(1+t)}$
f is continues on $]0, \infty[$, and we have in the neighborhood of 0,
$f(t) = \dfrac{1}{2\sqrt{t}} \dfrac{1}{(1+t)} = \dfrac{1}{2\sqrt{t}} \sum_{k=0}^\infty (-1)^k t^k = \sum_{k=0}^\infty \dfrac{(-1)^k}{2} t^{k-\frac{1}{2}}$.
Moreover, $\int_0^{+\infty} |f(t)| dt = \int_0^{+\infty} \dfrac{1}{2\sqrt{t}(1+t)}\, dt = \arctan(\sqrt{t})|_0^{+\infty} = \dfrac{\pi}{2} < +\infty$.
So, $f(t) = \sum_{k\geq 0} a_k t^{k-\frac{1}{2}}$ *with* $a_k = \frac{1}{2}(-1)^k$.

Applying Watson's lemma gives

$$I(x) \underset{x\to\infty}{\sim} \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+\frac{1}{2})}{x^{k+\frac{1}{2}}} = \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(k+\frac{1}{2})}{2x^{k+\frac{1}{2}}} = \frac{\sqrt{\pi}}{2}x^{-1/2} - \frac{\sqrt{\pi}}{4}x^{-3/2} + \cdots .$$

Recall
$\Gamma(n) = (n-1)!,\ n \in \mathbb{N}*$.
$\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$

If $a \in \mathbb{R}^+$, $\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.

3.2 Laplace method

Let $I = [a, b]$ be a real interval (finite or infinite), x a large positive parameter and f and g be two functions defined and continuous on I.
Let us consider the integral

$$\int_a^b f(t)e^{x\varphi(t)}\,dt. \tag{3.2}$$

To give an asymptotic expansion of the integral $I(x)$, we can apply Watson's lemma, we make the change of variable: $u = -\varphi(t)$, so, $t = \varphi^{-1}(-u)$, and $du = \varphi'(t)dt$. if $\varphi' \neq 0$, $dt = -\dfrac{du}{\varphi'(\varphi^{-1}(-u))}$. Then,

$$I(x) = -\int_{-\varphi(a)}^{-\varphi(b)} \frac{f(\varphi^{-1}(-u))}{\varphi'(\varphi^{-1}(-u))}e^{-xu}\,dt.$$

But if φ' is zero, we cannot apply Watson's lemma and we need Laplace's method

Theorem 3.2.1 (Laplace's Theorem)**.** *Let $I = [a, b]$ be a real interval (finite or infinite), (or $(-\infty, +\infty)$) be an interval of $\mathbb{R}$. Let f and φ be two functions defined on I such that f is continuous on I and $\varphi C^2(I, \mathbb{R})$. Assume that:*

1. $\int_a^b e^{xg(t)}|f(t)|\,dt < \infty$ *for all* $x > 0$*;*
2. g' *vanishes at a single point* $t_0 \in I$ $(\varphi'(t_0)) = 0$ *and* $g''(t_0) < 0$*;*
   t *a strict absolute maximum point of* φ
3. $f(t_0) \neq 0$.

Then, as $x \to +\infty$*,*

$$I(x) := \int_a^b f(t)e^{x\varphi(t)}\,dt \sim f(t_0)e^{x\varphi(t_0)}\sqrt{\frac{2\pi}{x\,|g''(t_0)|}}.$$

Proof. By Taylor's integral formula applied to g about t_0 we have

$$\begin{aligned} g(t) &= g(t_0) + \frac{1}{2}g''(t_0)(t-t_0)^2 + \frac{(t-t_0)^3}{2!}\int_0^1 (1-u)^2 g^{(3)}(t_0+u(t-t_0))\,du \\ &= g(t_0) + (t-t_0)^2\,\theta(t), \end{aligned} \tag{3.3}$$

where

$$\theta(t) = \tfrac{1}{2}g''(t_0) + \tfrac{1}{2}(t-t_0)\int_0^1 (1-u)^2 g^{(3)}(t_0+u(t-t_0))\,du.$$

Let $J = [t_0 - a, t_0 + a] \subset I$ be a small neighborhood of t_0 and set $K = I \setminus J$. Then

$$L(x) = \int_J f(t)e^{xg(t)}\,dt + \int_K f(t)e^{xg(t)}\,dt =: L_J(x) + L_K(x).$$

We study L_J first. From (3.3) we get

$$L_J(x) = e^{xg(t_0)}\int_J f(t)\,e^{x(t-t_0)^2\theta(t)}\,dt.$$

Perform the change of variable $v = (t-t_0)\sqrt{x}$, i.e.

$$t = t_0 + \frac{v}{\sqrt{x}},$$

to obtain

$$L_J(x) = \frac{e^{xg(t_0)}}{\sqrt{x}}\int_{-\beta}^{\beta} h(v,x)\,dv,$$

where $\beta = a\sqrt{x}$ and

$$h(v,x) = f\Big(t_0 + \frac{v}{\sqrt{x}}\Big)\exp\Big(v^2\,\theta\Big(t_0 + \frac{v}{\sqrt{x}}\Big)\Big).$$

Since $\theta(t) \to \frac{1}{2}g''(t_0)$ as $t \to t_0$, for fixed v and large x we have

$$h(v,x) \to f(t_0)\,\exp\Big(\tfrac{1}{2}g''(t_0)\,v^2\Big).$$

Hence, by dominated convergence (justified by the integrability hypothesis and choice of J),

$$\int_{-\beta}^{\beta} h(v,x)\,dv \sim f(t_0)\int_{-\infty}^{\infty}\exp\Big(\tfrac{1}{2}g''(t_0)\,v^2\Big)\,dv = f(t_0)\sqrt{\frac{2\pi}{|g''(t_0)|}}.$$

Therefore

$$L_J(x) \sim f(t_0)e^{xg(t_0)}\sqrt{\frac{2\pi}{x\,|g''(t_0)|}}.$$

The contribution $L_K(x)$ from the complement is exponentially smaller and does not affect the leading term; this concludes the proof. □

$$|h(t_0,x)| \leq e^{(x\theta(t_0)+\eta)^2} \sup_{|v|\leq\beta|t_0|} \left[\frac{t_0+\frac{v}{\sqrt{x}}}{t_0}\right] \leq 1^2 g''_\beta(t_0)+\eta, \quad \forall\, v\in[-\beta,\beta].$$

Donc

$$|h(t_0,x)| \leq e^{(\theta(t_0)+\eta)^2} \sup_{|v|\leq\beta|t_0|} \left[\frac{t_0+\frac{v}{\sqrt{x}}}{t_0}\right] = Ce^{(\theta(t_0)+\eta)^2}.$$

h est dominÃľe par une fonction positive, intÃľgrable et indÃľpendante de x. DâĂŹaprÃĺs le thÃľorÃĺme de continuitÃľ sous le signe intÃľgral, on obtient :

$$\lim_{x\to+\infty}\int_{-\beta}^{\beta} h(t_0,x)\,dv = \int_{-\infty}^{+\infty} h(t_0)\,dv = f(t_0)\int_{-\infty}^{+\infty} e^{-\frac{v^2}{2g''(t_0)}}\,dv.$$

Et comme $\theta(t)=\dfrac{g(t)-g(t_0)}{t-t_0}<0$ car $g(t_0)$ est le maximum de g, donc

$$\int_{-\beta}^{\beta} h(t_0,x)\,dv \overset{x\to+\infty}{\longrightarrow} f(t_0)\int_{-\infty}^{+\infty} e^{-\frac{v^2}{-g''(t_0)}}\,dv = f(t_0)\sqrt{\frac{2\pi}{-g''(t_0)}}.$$

DâĂŹoÃź

$$L_1(x) \sim_{x\to+\infty} f(t_0)e^{xg(t_0)}\sqrt{\frac{2\pi}{x\,g''(t_0)}}.$$

ÃĽtude de L_k

D'aprÃĺs ce qui prÃľcÃĺde,

$$\frac{L_k(x)}{L_1(x)} \sim_{x\to+\infty} e^{-xg(t_0)}\sqrt{\frac{xg''(t_0)}{-2\pi}}\int_K f(t)e^{xg(t)}\,dt. \tag{3.2}$$

Donc, pour prouver que $L_k(x)=o(L_1(x))$, il suffit de prouver que le cÃťtÃľ droit de (3.2) tend vers 0.

On a :

$$g(t)<g(t_0) \quad\Rightarrow\quad \exists\varepsilon>0 : g(t)<g(t_0)-\varepsilon.$$

Donc

$$xg(t)=g(t)+(x-1)g(t)<g(t_0)+(x-1)(g(t_0)-\varepsilon).$$

Alors

$$\left|e^{-xg(t_0)}\sqrt{x}\int_K f(t)e^{xg(t)}dt\right| \leq e^{-xg(t_0)}\sqrt{x}\int_K |f(t)|e^{g(t_0)}e^{(x-1)(g(t_0)-\varepsilon)}dt.$$

$$=\sqrt{x}e^{-\varepsilon(x-1)}\int_K |f(t)|e^{g(t_0)}dt.$$

$$\leq M\sqrt{x}e^{-\varepsilon x} \longrightarrow 0.$$

Donc

$$f \sim g, \quad h_i = o(g) \Rightarrow f + h_i \sim g.$$

3.3 Method of stationary phase(Fourier method)

This method is a derivative of Laplace's method; it allows us to study the behavior of integrals of the following form:

$$I(x) = \int_a^b f(t)e^{ix\varphi(t)},\, dt \quad x \to +\infty(or - \infty) \tag{3.4}$$

we will assume that f has compact support in $]a, b[$. ($\text{supp}(f) = \{x \in X : f(x) \neq 0\}$)
The function $f(t)$ is said the amplitude.
$\varphi(t)$: is referred to as the phase, and if its derivative is zero, it is said to be stationary.
Note: The behavior of the integral I is approximated by its contributions near the endpoints of integration and near the points where the phase $\varphi(t)$ is stationary, that is, points for which the first derivative of φ is zero, or more generally, points where the first $k-1$ derivatives are non-zero and the k^{th} derivative is zero.
We first note that I is finite.
Indeed,

$$\begin{aligned} |\int f(t)e^{ix\varphi(t)}\, dt| &\leq \int_a^b |f(t)|\, |e^{ix\varphi(t)}|\, dt \\ &\leq \int_a^b |f(t)|\, dt < \infty \end{aligned}$$

(because f has a compact support).
the asymptotic behavior of I determined by the points satisfying $\varphi'(t) = 0$. two cases are distinguished:
φ' not equal to zero on the supp(f): **stationary phase phase**.
φ' equal to zero on the supp(f): **unstationary phase phase**.

Theorem 3.3.1 (Stationary phase, compact support case)**.** *Let f and g be C^∞ functions on the interval $[a, b]$. Assume that f has compact support with* supp$(f) \subset [a, b]$, *and that g has no critical points on* supp(f) *(i.e. $g'(t) \neq 0$ for all $t \in$ supp(f)). Define*

$$F(x) = \int_a^b f(t)\, e^{ixg(t)}\, dt.$$

Then for every integer $n \geq 0$,

$$F(x) = O(x^{-n}) \qquad (x \to \infty).$$

Proof. Since $g'(t) \neq 0$ on $\operatorname{supp}(f)$ we may write

$$e^{ixg(t)} = \frac{1}{ixg'(t)} \frac{d}{dt}\Big(e^{ixg(t)}\Big).$$

Hence

$$F(x) = \frac{1}{ix} \int_a^b \frac{f(t)}{g'(t)} \frac{d}{dt}\Big(e^{ixg(t)}\Big)\, dt.$$

Integration by parts gives

$$F(x) = \frac{1}{ix} \left[\frac{f(t)}{g'(t)} e^{ixg(t)}\right]_a^b - \frac{1}{ix} \int_a^b \left(\frac{f(t)}{g'(t)}\right)' e^{ixg(t)}\, dt.$$

The boundary term vanishes because f has compact support, and therefore

$$F(x) = \frac{1}{x} \int_a^b f_1(t)\, e^{ixg(t)}\, dt,$$

where we set

$$f_1(t) = -\frac{1}{i} \left(\frac{f(t)}{g'(t)}\right)'.$$

Since f_1 is again C^∞ with compact support we may repeat the integration by parts. By induction one obtains for every $n \in \mathbb{N}$

$$F(x) = \frac{1}{x^n} \int_a^b f_n(t)\, e^{ixg(t)}\, dt,$$

where f_n is a C^∞ function with compact support depending on f and g. Consequently there exists a constant M (depending on n) such that

$$\Big|x^n F(x)\Big| = \left|\int_a^b f_n(t)\, e^{ixg(t)}\, dt\right| \leq \int_a^b |f_n(t)|\, dt = M,$$

which yields $F(x) = O(x^{-n})$ as $x \to \infty$ for every n. $\square$

Remark 3.3.1. *In the previous theorem, if one of the integration endpoints is infinite (that is, $a = -\infty$ or $b = +\infty$), the condition "f has compact support included in $[a, b]$" should be replaced as appropriate, by*

1. *if $a = -\infty$: Supp$(f) =]-\infty, c[\subset]-\infty, b[$ and $\lim_{t \to -\infty} f(t) = 0$.*
2. *if $b = +\infty$: Supp$(f) =]c, +\infty[\subset]a, +\infty[$ and $\lim_{t \to +\infty} = 0$.*
3. *if $a = -\infty$, and $b = +\infty$: $\lim_{t \to -\infty} f(t) = \lim_{t \to +\infty} f(t) = 0$.*

Remark 3.3.2. *If $\varphi'(t) \neq 0$ near the endpoints of a, that is, if there are no critical points, then the resultis modified. The terms from the bracket of integration by parts must be added to it.*

Example 3.3.1.

$$I(x) = \int_a^b e^{ixt},\, dt \quad x \to +\infty$$

$f(t) = 1$, $\varphi(t) = t$, $\varphi'(t) = 1 \neq 0$.

$$\int_a^b e^{ixt},\, dt = \left[\frac{e^{ixt}}{ix}\right]_a^b = \frac{e^{ixb} - e^{ixa}}{ix} = O(x^{-n}) \qquad (x \to \infty).$$

3.4 Stationary Phase Method

Theorem 3.4.1. *Let $f \in C_c^\infty([a, b])$, $\varphi \in C_c^\infty([a, b])$.*
Assume that f has compact support contained in $[a, b]$, and that φ has a unique critical point $t_0 \in \operatorname{supp}(f)$, and that this point is not degenerate. (i.e $\exists t_c \in \operatorname{supp}(f)$, $\varphi'(t_c) = 0$, and $\varphi''(t_c) \neq 0$).
Then, as $x \to +\infty$,

$$I(x) = \int_a^b f(t) e^{ix\varphi(t)}\, dt \sim f(t_c) e^{ix\varphi(t_0)} e^{i\pi/4(Sgn\varphi''(t_c))} \sqrt{\frac{2\pi}{x\,|\varphi''(t_c)|}},$$

where $s = \operatorname{sgn}(\varphi''(t_c)) \in \{\pm 1\}$.

Proof. Using Taylor's formula around t_0, we write

$$g(t) = g(t_0) + (t - t_0)^2 \theta(t), \qquad \theta(t_0) = \frac{1}{2} g''(t_0).$$

Let $I = [t_0 - a, t_0 + a]$ be a neighbourhood of t_0, and set $K = [a, b] \setminus I$. Then,

$$F(x) = \int_I f(t) e^{ixg(t)}\, dt + \int_K f(t) e^{ixg(t)}\, dt = F_1(x) + F_K(x).$$

From Theorem 3.7, one has $F_K(x) = O(x^{-n})$ for every n, so it suffices to analyse $F_1(x)$.

Make the change of variables $v = (t - t_0)\sqrt{x}$. Then

$$F_1(x) = e^{ixg(t_0)} \frac{1}{\sqrt{x}} \int_{-\beta}^{\beta} h(v,x)\, e^{iv^2\theta(t_0+v/\sqrt{x})}\, dv,$$

where $\beta = a\sqrt{x}$ and

$$h(v,x) = f\left(t_0 + \frac{v}{\sqrt{x}}\right) \exp\Big(iv^2(\theta(t_0 + v/\sqrt{x}) - \theta(t_0))\Big)\,.$$

Since $h(v,x)$ is uniformly bounded, by continuity under the integral sign we obtain

$$\lim_{x\to+\infty} F_1(x)e^{-ixg(t_0)}\sqrt{x} = f(t_0)\, e^{is\pi/4}\sqrt{\frac{\pi}{|\theta(t_0)|}}.$$

Substituting $\theta(t_0) = \frac{1}{2}g''(t_0)$ yields the stated asymptotic formula. □

Remark 3.4.1. *If the stationary point t_0 lies at an endpoint a or b, the result must be divided by 2. If several stationary points occur, their contributions add.*

Example: the Airy function

The Airy function is defined by

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(xt+t^3/3)}\, dt.$$

As $x \to +\infty$, make the substitution $t = u\sqrt{x}$, giving

$$\mathrm{Ai}(x) = \frac{\sqrt{x}}{2\pi} \int_{-\infty}^{+\infty} e^{ix^{3/2}(u+u^3/3)}\, du.$$

Let $g(u) = u + \frac{u^3}{3}$. Then $g'(u) = 1 + u^2 \neq 0$, and

$$\mathrm{Ai}(x) = \frac{1}{2\pi i x} \int_{-\infty}^{+\infty} \frac{1}{g'(u)} \left(e^{ix^{3/2}(u+u^3/3)}\right)'\, du.$$

This yields the classical asymptotic behaviour:

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}}\, x^{-1/4}\, e^{-\frac{2}{3}x^{3/2}}, \qquad x \to +\infty.$$

Asymptotic Analysis of the Airy Integral

We note that

$$f_1(u) = \frac{1}{g'(u)} = \frac{1}{1+u^2}.$$

By integration by parts, we obtain

$$\mathrm{Ai}(x) = \frac{1}{2\pi i x}\left[\int_{-\infty}^{+\infty} \frac{2u}{(1+u^2)^2} e^{ix^{3/2}(u+u^3/3)}\, du\right].$$

Observe that

$$\lim_{u\to\pm\infty} \frac{2u}{(1+u^2)^2} = 0.$$

By Theorem 3.7,

$$\int_{-\infty}^{+\infty} \frac{2u}{(1+u^2)^2} e^{ix^{3/2}(u+u^3/3)}\, du = O\left(\frac{1}{x^{3n/2}}\right), \qquad \forall n \in \mathbb{N}.$$

Hence,

$$\mathrm{Ai}(x) = \frac{1}{x} O\left(\frac{1}{x^{3n/2}}\right) = O\left(\frac{1}{x^{3n/2+1}}\right), \qquad \forall n \in \mathbb{N}.$$

Asymptotics as $y \to -\infty$

Let $y = -x$, and perform the substitution $t = u\sqrt{y}$. Then

$$\mathrm{Ai}(y) = \frac{\sqrt{y}}{2\pi} \int_{-\infty}^{+\infty} e^{iy^{3/2}(-u+u^3/3)}\, du.$$

Set $g(u) = -u + u^3/3$. Then

$$g'(u) = u^2 - 1, \qquad g''(u) = 2u,$$

so g has two critical points at $u = \pm 1$.

Let $\varepsilon > 0$ sufficiently small. Decompose the integral as

$$\int_{-\infty}^{+\infty} = \int_{-\infty}^{-1-\varepsilon} + \int_{-1-\varepsilon}^{-1+\varepsilon} + \int_{-1+\varepsilon}^{1-\varepsilon} + \int_{1-\varepsilon}^{1+\varepsilon} + \int_{1+\varepsilon}^{+\infty}.$$

Thus,

$$\mathrm{Ai}(y) = I_1(y) + I_2(y) + I_3(y) + I_4(y) + I_5(y).$$

By integration by parts (see the asymptotics as $x \to +\infty$), we obtain

$$I_k(y) = O\left(\frac{1}{y}\right), \qquad k = 1, 3, 5.$$

Asymptotics of $I_2(y)$

We apply Theorem 3.10. Let $f(u) = 1$ and $g(u) = -u + u^3/3$. Then

$$g'(u) = u^2 - 1, \qquad g''(u) = 2u.$$

On the interval $[-1-\varepsilon, -1+\varepsilon]$, we have

$$g'(u) = 0 \iff u = u_0 = -1.$$

We set $f(u) = 1$ and $g(u) = -u + \dfrac{u^3}{3}$, so that

$$g'(u) = u^2 - 1, \qquad g''(u) = 2u.$$

On the interval $[-1-\varepsilon,\, -1+\varepsilon]$, the equation $g'(u) = 0$ is equivalent to

$$u = u_0 = -1.$$

We note that

$$f_1(u) = \frac{1}{g'(u)} = \frac{1}{1+u^2}.$$

By integrating by parts, we obtain

$$\mathrm{Ai}(x) = \frac{1}{2\pi i x}\int_{-\infty}^{+\infty} \frac{1}{(1+u^2)^2} e^{x\varphi(u)}\,du - \frac{1}{2\pi i x}\int_{-\infty}^{+\infty} \frac{2u}{(1+u^2)^2} e^{x\varphi(u)}\,du.$$

We observe that

$$\lim_{u\to\pm\infty} \frac{2u}{(1+u^2)^2} = 0,$$

hence, by Theorem 3.7,

$$\int_{-\infty}^{+\infty} \frac{2u}{(1+u^2)^2} e^{x^{3/2}(-u+u^3/3)}\,du = O\left(\frac{1}{x^{3n}}\right), \qquad \forall n \in \mathbb{N}.$$

Thus,

$$\mathrm{Ai}(x) = \frac{1}{x}\,O\left(\frac{1}{x^{3n}}\right) = O\left(\frac{1}{x^{3n+1}}\right), \qquad \forall n \in \mathbb{N}.$$

2) Asymptotic equivalence as $x \to -\infty$

Setting $y = -x$ and applying the change of variables $t = u\sqrt{y}$, we obtain

$$\mathrm{Ai}(y) = \frac{\sqrt{y}}{2\pi}\int_{-\infty}^{+\infty} e^{y^{3/2}(-u+u^3/3)}\,du.$$

Let $g(u) = -u + \frac{u^3}{3}$. Then $g'(u) = u^2 - 1$, and g admits two critical points at $u = \pm 1$. Let $\varepsilon > 0$ sufficiently small. We decompose the integral as

$$\int_{-\infty}^{+\infty} = \int_{-\infty}^{-1-\varepsilon} + \int_{-1-\varepsilon}^{-1+\varepsilon} + \int_{-1+\varepsilon}^{1-\varepsilon} + \int_{1-\varepsilon}^{1+\varepsilon} + \int_{1+\varepsilon}^{+\infty}.$$

Hence,

$$\mathrm{Ai}(y) = I_1(y) + I_2(y) + I_3(y) + I_4(y) + I_5(y).$$

By an integration by parts argument (see the equivalence near $+\infty$),

$$I_k(y) = O\left(\frac{1}{y}\right), \qquad k = 1, 3, 5.$$

Asymptotic form of $I_2(y)$

We apply Theorem 3.10.
Let $f(u) = 1$ and $g(u) = -u + \frac{u^3}{3}$, so that

$$g'(u) = u^2 - 1, \qquad g''(u) = 2u.$$

On the interval $[-1-\varepsilon, -1+\varepsilon]$, the equation $g'(u) = 0$ is equivalent to

$$u = u_0 = -1.$$

Chaptere 4

Disturbance problem

Conclusion General

Ce cours a couvert les aspects fondamentaux de l'analyse mathÃľmatique avancÃľe. Les concepts prÃľsentÃľs constituent la base pour des Ãľtudes plus approfondies en analyse fonctionnelle, Ãľquations aux dÃľrivÃľes partielles, et autres domaines des mathÃľmatiques pures et appliquÃľes.

Perspectives Futures

Les Ãľtudiants intÃľressÃľs sont encouragÃľs Ãă explorer les domaines suivants:

- Analyse ....
- ThÃľorie des opÃľrateurs
- ÃĽquations aux ....
- Analyse ....

Bibliography