Ex. 5 :

1/

f linéaire $\iff \forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3, \forall \alpha, \beta \in \mathbb{R}$

$$f(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)) \overset{??}{=} \alpha f(x_1, y_1, z_1) + \beta f(x_2, y_2, z_2)$$

Soient $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

$$f(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)) = f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$= (-(\alpha x_1 + \beta x_2) + 2(\alpha y_1 + \beta y_2) + 2(\alpha z_1 + \beta z_2),$$

$$-8(\alpha x_1 + \beta x_2) + 7(\alpha y_1 + \beta y_2) + 4(\alpha z_1 + \beta z_2),$$

$$-13(\alpha x_1 + \beta x_2) + 5(\alpha y_1 + \beta y_2) + 8(\alpha z_1 + \beta z_2))$$

$$= \alpha(-x_1 + 2y_1 + 2z_1, -8x_1 + 7y_1 + 4z_1, -13x_1 + 5y_1 + 8z_1)$$

$$+ \beta(-x_2 + 2y_2 + 2z_2, -8x_2 + 7y_2 + 4z_2, -13x_2 + 5y_2, 8z_2)$$

$$= \alpha f(x_1, y_1, z_1) + \beta f(x_2, y_2, z_2)$$

donc f est une application linéaire

2/

$B_c = \{e_1, e_2, e_3\}$

$f(e_1) = f(1, 0, 0) = (-1, -8, -13)$

$f(e_2) = f(0, 1, 0) = (2, 7, 5)$

$f(e_3) = (0, 0, 1) = (2, 4, 8)$

$$f(2e_1+e_2-e_3) = f(2,1,-1)$$
$$= 2f(e_1)+f(e_2)-f(e_3)$$
$$= 2(-1,-8,-13)+(2,7,5)-(2,4,8)$$
$$= (2,-13,-29)$$

3/

$\mathrm{Ker}\, f = ??$

$\mathrm{Ker}\, f = \{(x,y,z)\in\mathbb{R}^3 \,/\, f(x,y,z)=0_{\mathbb{R}^3}\}$

On a $f(x,y,z)=(0,0,0)$

$$\iff \begin{cases} -x+2y+2z=0 & \dots (1) \\ -8x+7y+4z=0 & \dots (2) \\ -13x+5y+8z=0 & \dots (3) \end{cases}$$

$$\xrightarrow{(2)-2(1)} \quad y=2x$$

$$\Rightarrow \begin{cases} 3x+2z=0 & \dots (4) \\ 6x+4z=0 & \dots (5) \\ -3x+8z=0 & \dots (6) \end{cases} \quad \xrightarrow{(4)+(6)} \quad z=0 \Rightarrow x=y=z=0$$

$\Rightarrow \mathrm{Ker}\, f = \{0_{\mathbb{R}^3}\}$

L'unique base de $\mathrm{Ker}\, f$ est $\emptyset$,
et sa dimension est nulle

4/

Puisque $\mathrm{Ker}\, f=\{0_{\mathbb{R}^3}\} \Rightarrow$ l'application
f est injective

On sait que d'après le théorème du rang $\dim E = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

$$\dim \mathbb{R}^3 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$$

$$\Longrightarrow \dim \mathbb{R}^3 = 3 = \dim \operatorname{Im} f$$

alors f est surjective

ce qui implique que f est bijective

5)

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\qquad f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$f \circ g : \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3$$

$$(x,y) \longmapsto (f \circ g)(x,y)$$

$$(f \circ g)(x,y) = f(g(x,y)) = f(x-y,\ x+y,\ x+2y)$$

$$= \big(-(x-y) + 2(x+y) + 2(x+2y),$$
$$-8(x-y) + 7(x+y) + 4(x+2y),$$
$$-13(x-y) + 5(x+y) + 8(x+2y)\big)$$

$$(f \circ g)(x,y) = (3x + 7y,\ 3x + 23y,\ 34y)$$