Chapter 02: Definite integrals

1/ Definition: Let $f : [a,b] \to \mathbb{R}$ be a continuous function and F is an antiderivative of f in $[a,b]$. The quantity $\int_a^b f(x)\,dx$ is called the definite integral of f from a to b (or of f over $[a,b]$). It's also written by $[F(x)]_a^b$

Remark: The numbers a and b are known as the lower and upper limits of the integral. We simply write the integral as $\int_a^b f(t)\,dt$ or $\int_a^b f(u)\,du$ instead of $\int_a^b f(x)\,dx$

Examples: Evaluate $I_1 = \int_1^4 x^2\,dx$, $I_2 = \int_0^2 e^x\,dx$, $I_3 = \int_0^{\frac{\pi}{2}} \cos x\,dx$

Solution: Note that all the integrations are performed in the normal way.

$$I_1 = \int_1^4 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^4 = \frac{1}{3}\left[x^3\right]_1^4 = \frac{1}{3}\left[4^3 - 1^3\right] = \frac{1}{3}(64-1) = \frac{63}{3} = \boxed{21}$$

$$I_2 = \int_0^2 e^x\,dx = \left.e^x\right]_0^2 = e^2 - e^0 = \boxed{e^2 - 1}$$

$$I_3 = \int_0^{\frac{\pi}{2}} \cos x\,dx = \left.\sin x\right]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = \boxed{1}$$

2/ Properties of definite integrals: Let f and g be two continuous functions defined in $[a,b]$ and k be a constant in $\mathbb{R}$, then we have the following properties

a/ Linearity of the definite integrals:

- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (sum and difference)
- $\int_a^b k\, f(x)\,dx = k\int_a^b f(x)\,dx$ (or constant multiple).

or equivalently:

$$\int_a^b [k\, f(x) + g(x)]\,dx = k\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

b/ Reversing the integration: (or order of integration)

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

c/ Additivity for adjacent integrals (Chasles' relationship)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad \forall c \in [a,b].$$

d/ Zero width interval:

$$\int_a^a f(x)\,dx = 0$$

e/ Max-Min Inequality: If f has maximum value $\max f = M$ and minimum value $\min f = m$, then we have: (i.e., $\forall x \in [a,b], m \leq f(x) \leq M$)

$\therefore \min f \int_a^b dx \leq \int_a^b f(x)\,dx \leq \max f \int_a^b dx$

$\Leftrightarrow \quad m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

$\Leftrightarrow \quad \boxed{m \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq M}$

- The number $\frac{1}{b-a} \int_a^b f(x)\,dx$ is called the average value of f on the interval $[a,b]$ and denoted by f_{ave}.

f/ Domination of the definite integrals:

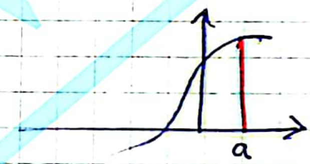
- If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$ (special case)

j/ Absolute value:

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

Remark (1) If f is continuous on $[a,b]$, then

$\int_a^b f(x)\,dx = 0 \Leftrightarrow f(x) = 0$ on $[a,b]$.



(i.e. the integral is just a line and contains no area).

Theorem: (integrability of continuous functions) [existence of definite int

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

* **Other properties of definite integrals:** Let f be continuous function defined in $[a,b]$, then we have:

$y = a+b-x \Rightarrow dy = -dx$.
if $x = a \Rightarrow y = b$, if $x = b \Rightarrow y = a$
$\int_a^b f(x)\,dx = \int_b^a f(y)(-dy) = \int_a^b f(y)\,dy$

1/ $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

2/ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (this property is a particular case of property 1)

3/ $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ $\quad \left[\int_a^{2a} f(x)\,dx = \int_a^0 f(y)(-dy) = \int_0^a f(y)\,dy \right.$ $\quad y = 2a-x$

4/ $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$ (special case of 3/)

5/ $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f \text{ is an even function, i.e. } f(-x) = f(x) \\ 0, & \text{if } f \text{ is an odd function, i.e. } f(-x) = -f(x) \end{cases}$

3/ The fundamental theorem of calculus: (FTC)

If $f(x)$ is a given continuous function, then

$$F(x) = \int_a^x f(t)\,dt$$

is a function of x, where the lower limit of integration, a, is held constant.

In this case:

- $F'(x) = f(x)$, (since $F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$)
- $F(a) = 0$, and (since $F(a) = \int_a^a f(t)\,dt = F(a) - F(a) = 0$)
- $\int_a^b f(x)\,dx = F(b) - F(a)$.

4/ Newton-Leibnitz formula:

Theorem: Let f be a continuous function defined in $[a,b]$. And let $F(x)$ be its primitive or anti-derivative on $[a,b]$, i.e, $\frac{d}{dx}F(x) = f(x)$, then ~~d~~efinite integral of $f(x)$ over $[a,b]$ is denoted by $\int_a^b f(x)dx$ and defined as $[F(b) - F(a)]$.

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Then "a" and "b" are called the limits of integration, where "a" is called the lower limit and "b" is called the upper limit. This is also called Newton-Leibnitz formula.

Remark: The interval $[a,b]$ is called the interval of integration and is also known as range of integration.

5/ Cauchy schwartz inequality for definite integrals:

Theorem: Let a, b be two real numbers with $a < b$, and $f, g : [a,b] \to$ be two continuous functions on $[a,b]$. Suppose neither f nor g is constant zero on $[a,b]$. Then the statements below hold:

(a) The inequality $\left(\int_a^b f(x)\,g(x)\,dx\right)^2 \leq \int_a^b (f(x))^2 dx \int_a^b (g(x))^2\,dx$.

(b) The statements (I) and (II) below, are logically equivalent:

(I) $\left(\int_a^b f(x)\,g(x)\,dx\right)^2 = \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx$

(II) There exist some $p, q \in \mathbb{R}^*$ such that $pf(x) + qg(x) = 0$ for any $x \in [a,b]$.

(the functions f and g are linearly independent over $\mathbb{R}$).

Remark: In the context of the statement of theorem above, if one of the functions f, g is constant zero on $[a,b]$, then the inequality in (a) trivially reduces to the equality in (I) of (b).

proof of the theorem of C.S (Cauchy schwarz):

1st way: We put $p(x) = \int_a^b [f(t) + x\, g(t)]^2\, dt$ —— (I1)

we have from the domination property of definite integrals, we have $\forall x \in \mathbb{R}$: $p(x) \geq 0$ since $[f(t) + x\, g(t)]^2 \geq 0, \forall t$. —— ($\alpha$)

If we suppose: $x_0 = \dfrac{-\int_a^b f(t)\, g(t)\, dt}{\int_a^b g^2(t)\, dt}$, then $p(x_0) \geq 0$, where $\int_a^b g^2(t)\, dt \neq 0$. $\left| p(x_0) = \int_a^b \left[f(t) - \dfrac{\int_a^b f(t) g(t)\, dt}{\int_a^b g^2(t)\, dt} \cdot g(t) \right]^2 dt \geq 0 \right.$ (from (α))

hence (I1) becomes:

$$
\begin{aligned}
0 \leq p(x_0) &= \int_a^b [f(t) + x_0 g(t)]^2\, dt \\
&= \int_a^b [f^2(t) + 2x_0 f(t) g(t) + (x_0 g(t))^2]\, dt \\
&= \int_a^b f^2(t)\, dt - 2\frac{\left(\int_a^b f(t) g(t)\, dt\right)^2}{\int_a^b g^2(t)\, dt} + \frac{\left(\int_a^b f(t) g(t)\, dt\right)^2}{\left(\int_a^b g^2(t)\, dt\right)^2} \int_a^b g^2(t)\, dt \\
&= \int_a^b f^2(t)\, dt - \frac{\left(\int_a^b f(t) g(t)\, dt\right)^2}{\int g^2(t)\, dt} \geq 0 \quad (\text{since } p(x_0) \geq 0)
\end{aligned}
$$

this implies that:

$$\int_a^b f^2(t)\, dt \geq \frac{\left(\int_a^b f(t) g(t)\, dt\right)^2}{\int_a^b g^2(t)\, dt}$$

which gives:

$$\left(\int_a^b f(t) g(t)\, dt\right)^2 \leq \int_a^b f^2(t)\, dt \int_a^b g^2(t)\, dt$$

hence the proof of theorem is completed.

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2nd way of proof of (a).

Define the function $F: \mathbb{R} \to \mathbb{R}$ by $F(x) = \int_a^b (x f(t) + g(t))^2 \, dt$ for any $x \in \mathbb{R}$.

Pick any $x \in \mathbb{R}$. For any $t \in [a,b]$, we have $(x f(t) + g(t))^2 \geq 0$.

Then from property (f) of domination of definite integrals:

$$F(x) = \int_a^b (x f(t) + g(t))^2 \, dt \geq 0.$$

Define: $A = \int_a^b (f(t))^2 dt$, $B = 2\int_a^b f(t) g(t) \, dt$, $C = \int_a^b (g(t))^2 dt$

and $\Delta = B^2 - 4AC$

By definition, for any $x \in \mathbb{R}$, we have:

$$F(x) = \int_a^b (x f(t) + g(t))^2 dt = x^2 \int_a^b f^2(t) \, dt + 2x \int_a^b f(t) g(t) dt + \int_a^b g^2(t) \, dt$$

$$\Rightarrow \boxed{F(x) = x^2 A + Bx + C = Ax^2 + Bx + C} \;(*)$$

since $(f(t))^2 \geq 0$ for any $t \in [a,b]$, and f is not constant zero on $[a,b]$, we have:

$$A = \int_a^b f^2(t) \, dt > 0$$

[since if $\int_a^b f^2(t) \, dt = 0 \Rightarrow F(x)$ is a constant not a polynomial $\Rightarrow$ $\deg(F) = 0 \Rightarrow 0 \leq \left(\int_a^b f(t) g(t) dt\right)^2 \leq \underbrace{\int_a^b f^2(t) dt}_{0} \int_a^b g^2(t) \, dt = 0 \Rightarrow \left(\int_a^b f(t) g(t) dt\right)^2 = 0$

$\Rightarrow \int_a^b f(t) g(t) dt = 0$, hence the inequality is true for $\int_a^b f^2(t) dt$ « trivial »

Then F is a quadratic polynomial function with real coefficients from $(*)$. The discriminant of F is Δ.

Recall that $F(x) \geq 0$ for any $x \in \mathbb{R}$. Then $\Delta \leq 0$. Therefore $\frac{B^2}{4} \leq AC$.

Hence: $$\left(\int_a^b f(t) g(t) \, dt\right)^2 \leq \int_a^b (f(t))^2 dt \int_a^b (g(t))^2 dt.$$

(b): (I) $\Rightarrow$ (II) ?

suppose $\left(\int_a^b f(t) g(t) \, dt\right)^2 = \int_a^b f^2(t) \, dt \int_a^b g^2(t) \, dt$.

Then $\Delta = B^2 - 4AC = 0 \Rightarrow$ the quadratic polynomial function F has a repeated real root. It is $x_0 = -B/2A$.

Then, for the same x_0, we have $F(x_0)=\int_a^b (x_0 f(t)+g(t))^2 dt=0$
Therefore by remark ①, we have $x_0 f(t)+g(t)=0$ for any $t\in[a,b]$.
Note that $x_0\neq 0$; otherwise it would happen that $g(t)=0$ for any $t\in[a,b]$

ii) (II) $\Rightarrow$ (I)?
Suppose there exist some $p,q\in\mathbb{R}-\{0\}$ such that for any $x\in[a,b]$, the equality $pf(x)+qg(x)=0$ holds.
Define $x_0=\frac{p}{q}$. then for each $x\in[a,b]$, we have: $x_0 f(x)+g(x)=0$
Therefore: $F(x_0)=\int_a^b (x_0 f(t)+g(t))^2 dt=0$.
Now the quadratic polynomial $F(t)$ has a real root, namely x_0.
Then $\Delta\geq 0$. Also recall $\Delta\leq 0$. Then $\Delta=0$. Hence:

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^2=\int_a^b f^2(t)\,dt\int_a^b g^2(t)\,dt.$$

3) Subdivisions and Darboux sums:

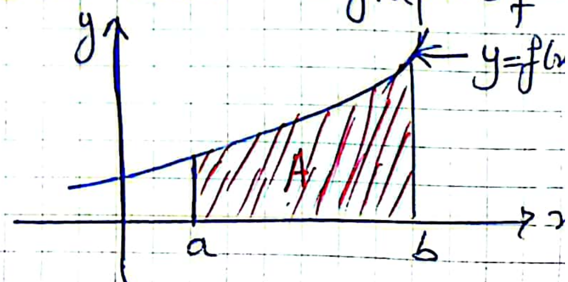
Definition ①: Let $n\in\mathbb{N}^*$ be a fixed number. The finite part $d=\{x_0=a, x_1, \dots, x_n=b\}$ is called a subdivision (or partition) of order n of the interval $[a,b]$ such that:

$$a=x_0<x_1<x_2<\dots<x_{n-1}<x_n=b$$

- The number $h=\max\limits_{i=1}^{n}(x_i-x_{i-1})$ is called the step of the subdivision.
- Let $h_i=x_i-x_{i-1}$ for all $i=\overline{1,n}$, h_i is called the step of the subinterval $[x_{i-1},x_i]=I_i$. If all h_i are equal for $i=\overline{1,n}$, then the subdivision d is called regular and irregular if not.

Remark: If d is a regular subdivision of the interval $[a,b]$, of order n, then the step $h=\frac{b-a}{n}$ and $x_i=a+ih$, $\forall i=\overline{1,n}$.

Definition ②: assume that $f(x)>0$, $\forall x\in[a,b]$, then the area of the region (shown below) over $[a,b]$ but below the graph of $y=f(x)$ is given by the limit:



$$\text{Area}=A=\lim_{n\to+\infty}\sum_{i=1}^{n} f(x_i)(x_i-x_{i-1})$$

which equivalent to : $A = \lim_{n \to +\infty} \sum_{i=1}^{n} h f(x_i)$

where for any n, we have $h = \frac{b-a}{n}$ and $x_i = a + i h$.

- This area formula is a limit of the special sum form

$$\sum_{i=1}^{n} h f(x_i),$$

which we interpreted as a sum of areas of rectangles.
In general, a sum having this form is called a Riemann sum.
That is, a Riemann sum is the sum of areas of n rectangles, which for large n approximates the area A under $f(x)$.
However, Riemann sums don't necessarily approximate just area.
In fact they can be negative. If $f(x)$ is ever negative on $[a,b]$ then some (or all) of the terms $f(x_i)h$ in the Riemann sum can be negative. Note that any rectangles under the x-axis give a negative contribution to the sum $\sum_{i=1}^{n} f(x_i) h$.

Definition ③ : Let f be a function defined on the closed interval $[a,b]$, the definite integral of $f(x)$ from a to b is the number, denoted as $\int_a^b f(x)\,dx$ and defined as

$$\int_a^b f(x)\,dx = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\,\underbrace{(x_i - x_{i-1})}_{h},$$

where for any n, we have: $h = \frac{b-a}{n}$ and $x_i = a + i h$.

- If the limit doesn't exist, then we say the definite integral doesn't exist.

Remark ①: Take note:

- A definite integral is a number, for example $\int_1^2 x\,dx = \frac{3}{2}$.
- An indefinite integral is a set of functions, for example $\int x\,dx = \frac{x^2}{2} + c$

Theorem ①: If $f(x)$ is continuous on $[a,b]$, then $\int_a^b f(x)\,dx$ exists.

- This means that the limit $\lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i) h$ in the definition of $\int_a^b f(x)\,dx$ is guaranteed to exist as long as f is continuous on $[a,b]$

However, even though the limit exists, it can be quite difficult to evaluate it directly.

for Example: consider the definite integral of $f(x)=x^2+\cos x$ over $[-\pi,\pi]$:
following definition (3), we have: $h=\frac{\pi-(-\pi)}{n}=\frac{2\pi}{n}$ and $x_i=-\pi+ih$

$\Rightarrow x_i=-\pi+\frac{2\pi i}{n}$, so

$$\int_{-\pi}^{\pi}(x^2+\cos x)\,dx=\lim_{n\to+\infty}\sum_{i=1}^{n}\left(x_i^2+\cos(x_i)\right)h$$

$$=\lim_{n\to+\infty}\sum_{i=1}^{n}\left[\left(-\pi+\frac{2\pi i}{n}\right)^2+\cos\left(-\pi+\frac{2\pi i}{n}\right)\right]\frac{2\pi}{n}.$$

This is not an easy limit, But by theorem above it does exist.

Definition (4): Let $f:[a,b]\to\mathbb{R}$ be a bounded function, the quantities

$$\bar{S}(f,d):=\sum_{i=1}^{n}(x_i-x_{i-1})\sup_{x\in[x_{i-1},x_i]}f(x)$$

and

$$s(f,d):=\sum_{i=1}^{n}(x_i-x_{i-1})\inf_{x\in[x_{i-1},x_i]}f(x)$$

are called the upper sum of Darboux and the lower sum of Darboux, respectively of the function f depending on the partition (subdivision) $d=\{x_0=a,x_1,\dots,x_n=b\}$.

Remark: the values $\bar{S}(f,d)$ and $s(f,d)$ are finite, since f is bounded on $[a,b]$ and $\sup_{[x_{i-1},x_i]}f(x)$ and $\inf_{[x_{i-1},x_i]}f(x)$ are finite values.

In addition: $\bar{S}(f,d)\geq s(f,d)$. (since $\sup f(x)\geq\inf f(x)$ $\forall x\in[x_{i-1},x_i]$)

Geometric interpretation: (definite integral and sums of Darboux)

Let f be a continuous function on $[a,b]$, we assume that: $\forall x\in[a,b]$: $f(x)\geq 0$ (and $f(x)\nearrow$). Then the definite integral $\int_a^b f(x)\,dx$ equals the area of the region over $[a,b]$ and below the graph of $y=f(x)$ and above the x-axis.

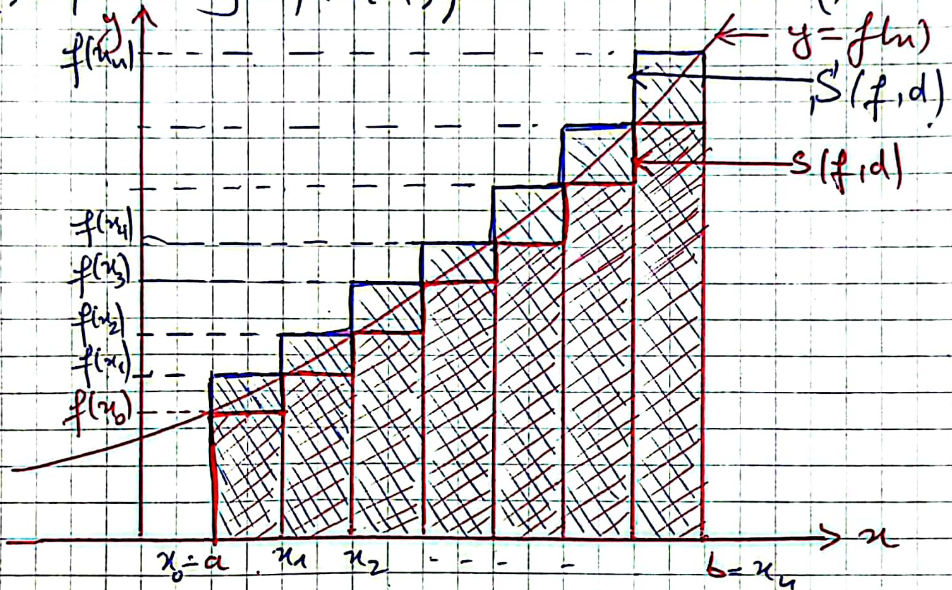
from definition (3), we put $h=\frac{b-a}{n}$ and $x_i=a+ih$, $\forall i=\overline{1,n}$

this devides the interval $[a,b]$ into n subintervals:

$[x_0,x_1]=I_1$, $I_2=[x_1,x_2]$, ... , $I_n=[x_{n-1},x_n]$.

each of length h. Next, in each subinterval $[x_{i-1},x_i]$, we have $\sup_{[x_{i-1},x_i]} f(x)=f(x_i)$ and $\inf_{[x_{i-1},x_i]} f(x)=f(x_{i-1})$ (since $f(x)\nearrow$ on $[a,b] \Rightarrow [x_{i-1}$

Then, the upper (respectively, lower) sum of Darboux equals to the sums of the areas of rectangles above (respectively below) the graph of $y=f(x)$, such that the ith rectangle has height $f(x_i)$ (respectively $f(x_{i-1})$) and base h. (See drawing below).



4) Integrable Reimann functions:

proposition ④: the set of lower sums of Darboux has an upper bound, which we note s_b^a.

• The set of upper sums of Darboux has a lower bound, denoted S_b^a.

Moreover, we have $S_b^a \geqslant s_b^a$.

Definition ⑤ Let f be a bounded function on $[a,b]$, f is called Riemann integrable iff the values S_b^a and s_b^a coincide, that is if and only if: $S_b^a := \inf_{d\in S_{[a,b]}} S(f,d) = s_b^a := \sup_{d\in S_{[a,b]}} s(f,d)$

where $S_{[a,b]}$ is the set of subdivisions (partitions) of $[a,b]$.

in this case $S_b^a = s_b^a$ is denoted by $\int_a^b f(x)\,dx$.

Example ①: Let f be the Direchlet function defined as:

$$f(x) = \begin{cases} 1, & \text{if } x \in [a,b] \cap \mathbb{Q} \\ 0, & \text{if } x \notin [a,b] \cap \mathbb{Q} \end{cases}$$

Solution:

for each subdivision $d \in S_{[a,b]}$, we have:

$S'(f,d) = b-a$ and $s(f,d) = 0$ (since in each subinterval $[x_{i-1}, x_i]$, we have: $\sup\limits_{x \in [x_{i-1},x_i]} f(x) = 1$ ($\mathbb{Q}$ is dense in $\mathbb{R}$) and $\inf\limits_{x \in [x_{i-1},x_i]} f(x) = 0$)

we observe that $S(f,d) = b-a \neq s(f,d) = 0 \Rightarrow f$ isn't Riemann integrable.

Example ②: Let $[a,b]$ be a closed interval from $\mathbb{R}$. and let $f(x) = \lambda$, $\lambda \in \mathbb{R}$ and $g(x) = x$, $\forall x \in [a,b]$

∘ Are the functions f and g Riemann integrable?

Solution: Let d be a regular subdivision of $[a,b]$ such that $d = \{x_0 = a, x_1, \dots, x_n = b\}$ where $h = \frac{b-a}{n}$ and $x_i = a + ih$.

for $f(x)$: we have: $\forall i \in \{1, \dots, n\}$ $\inf f(x) = \sup f(x) = \lambda$, $\forall x \in [x_{i-1}, x_i]$

Hence: $S(f,d) = \sum\limits_{i=1}^{n} (x_i - x_{i-1}) \sup\limits_{x \in [x_{i-1},x_i]} f(x)$

$$= \sum_{i=1}^{n} h\lambda = nh\lambda = (b-a)\lambda \quad \left(\text{since } h = \frac{b-a}{n}\right)$$

in the same way, we get:

$s(f,d) = \sum\limits_{i=1}^{n} (x_i - x_{i-1}) \inf\limits_{x \in [x_{i-1},x_i]} f(x)$

$$= \sum_{i=1}^{n} h\lambda = (b-a)\lambda \Rightarrow \lim_{n \to +\infty} s(f,d) = (b-a)\lambda$$

we obtain $\lim\limits_{n} S(f,d) = \lim\limits_{n} s(f,d)$, that is f is Riemann integrable.

for $g(x)$: we have g is an increasing function (since $\forall x \in [a,b]: g'(x) > 0$)

Hence: $\inf\limits_{[x_{i-1},x_i]} g(x) = g(x_{i-1})$ and $\sup\limits_{[x_{i-1},x_i]} g(x) = g(x_i)$, $\forall i \in \{1, \dots, n\}$.

or, $\inf g(x) = x_{i-1} = a + (i-1)h$ and $\sup g(x) = x_i = a + ih$

Thus: $\bar{S}(g,d) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x\in[x_{i-1},x_i]} g(x_i)$

$$= \sum_{i=1}^{n} h\,(a+ih) = \sum_{i=1}^{n} ah + \sum_{i=1}^{n} i h^2$$

$$= nah + h^2 \sum_{i=1}^{n} i$$

$$= nah + h^2\left(\frac{n}{2}(1+n)\right)$$

$$= a(b-a) + \frac{(b-a)^2(1+n)}{2n} \quad \left(\text{since } h = \frac{b-a}{n}\right) \quad —1$$

in the same way, we get:

$$\underline{S}(g,d) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x\in[x_{i-1},x_i]} g(x_i)$$

$$= \sum_{i=1}^{n} h\,(a+(i-1)h) = \sum_{i=1}^{n} ah + \sum_{i=1}^{n} (i-1)h^2$$

$$= nah + h^2 \sum_{i=1}^{n} (i-1)$$

$$= nah + h^2 \sum_{i=0}^{n-1} i = nah + h^2 \frac{n}{2}(n-1)$$

$$= a(b-a) + \frac{(b-a)^2(n-1)}{2n} \quad \left(\text{since } h = \frac{b-a}{n}\right) \quad —2$$

from 1, we have: $\lim_{n\to+\infty} \bar{S}(g,d) = \lim_{n\to+\infty}\left(a(b-a) + \frac{(b-a)^2(1+n)}{2n}\right)$

$$= a(b-a) + \frac{(b-a)^2}{2}$$

from 2, we have: $\lim_{n\to+\infty} \underline{S}(g,d) = \lim_{n\to+\infty}\left(a(b-a) + \frac{(b-a)^2(n-1)}{2n}\right)$

$$= a(b-a) + \frac{(b-a)^2}{2}$$

Hence $\lim_{n\to+\infty} \bar{S}(g,d) = \lim_{n\to+\infty} \underline{S}(g,d) \Rightarrow g$ is Riemann integrable.

proposition 2: Let f be a Riemann integrable function, then $\forall d \in S_{[a,b]}$:

$$\underline{S}(f,d) \leq \int_a^b f(x)\,dx \leq \bar{S}(f,d) \quad — a$$

$$(b-a) \inf_{x\in[a,b]} f(x) \leq \int_a^b f(x)\,dx \leq (b-a) \sup_{x\in[a,b]} f(x) \quad b$$

proof: (a) is a direct consequence of definition of Darboux sums.
for (b) it is enough to take $d=\{a,b\}$.

proposition ③: The following three assertions are equivalent:

(i) the bounded function f is Riemann integrable on $[a,b]$.

(ii) $\forall \varepsilon>0, \exists d\in S_{[a,b]}: \ S(f,d)-s(f,d)<\varepsilon$

(iii) $\lim\limits_{h\to 0}\left(S(f,d)-s(f,d)\right)=0$.

Theorem ②: All bounded and monotonic function on $[a,b]$ is Riemann integrable on $[a,b]$.

proof: (if d is a regular subdivision, the result is obvious).
Now, suppose that $d=\{x_0=a, x_1, \dots, x_n=b\}$ is not regular subdivision of $[a,b]$ and that f is increasing on $[a,b]$, then we have,

$$0\leq S(f,d)-s(f,d)\leq \sum_{i=1}^{n}(x_i-x_{i-1})\sup_{[x_{i-1},x_i]}f(x)-\sum_{i=1}^{n}(x_i-x_{i-1})\inf_{[x_{i-1},x_i]}f(x)$$

$$=\sum_{i=1}^{n}(x_i-x_{i-1})\left(\sup f(x)-\inf f(x)\right), x\in[x_{i-1},x_i]$$

$$=\sum_{i=1}^{n}h_i\left(f(x_i)-f(x_{i-1})\right).$$

$$\leq \sum_{i=1}^{n}h\left(f(x_i)-f(x_{i-1})\right) \Big/ \text{ where } h=\max_{i=1}^{n} h_i$$

$$=h\left(f(x_n)-f(x_0)\right) \quad (*)$$

hence: $0\leq \lim\limits_{h\to 0} S(f,d)-s(f,d)\leq \lim\limits_{h\to 0} h\left(f(x_n)-f(x_0)\right)=0$
which gives the desired result (i.e., $f\in R_{[a,b]}$).

2nd way: we have gotten from $(*)$ that: $S(f,d)-s(f,d)\leq h\left(f(x_n)-f(x_0)\right)$
we can choose: $h<\dfrac{\varepsilon}{f(b)-f(a)}$, hence we obtain,

$$S(f,d)-s(f,d)\leq h\left(f(b)-f(a)\right)<\frac{\varepsilon}{f(b)-f(a)}\cdot\left(f(b)-f(a)\right)=\varepsilon$$

this gives: $S(f,d)-s(f,d)<\varepsilon$

Therefore from (ii) of proposition ③, f is Riemann integrable.

Remark: from theorem ①, we can deduce that all continuous function on $[a,b]$ is Riemann integrable on $[a,b]$.

Properties of Riemann integrable functions:

① for $f \in R_{[a,b]}$, we have:

(i) $\forall d \in S_{[a,b]}$; $s(f,d) \leq \int_a^b f(x)\,dx \leq S(f,d)$

(ii) $(b-a) \inf_{x \in [a,b]} f(x) \leq \int_a^b f(x)\,dx \leq (b-a) \sup_{x \in [a,b]} f(x)$ (special case)

② $f \in R_{[a,b]} \Leftrightarrow f \in R_{[a,c]}$ and $f \in R_{[c,b]}$, $\forall a \leq c \leq b$

and we have the Chasles' Relationship:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

③ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ and for $a=b$, we have $\int_a^b f(x)\,dx = 0$

④ $\forall f, g \in R_{[a,b]}, \forall \alpha, \beta \in \mathbb{R}$: $\alpha f + \beta g \in R_{[a,b]}$ and we have

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

⑤ for $f, g \in R_{[a,b]}$ and $a < b$, we have:

(i) $f \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$

(ii) $f \geq g \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

(iii) $|f| \in R_{[a,b]}$ and $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

Theorem ③ (Mean value Theorem) Let $f \in C([a,b])$ (f be a continuous function on $[a,b]$), then

$$\exists c \in [a,b];\ \frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

Proof: f is continuous on $[a,b]$, then f reaches it lower and upper values, that is: $\exists x_1, x_2 \in [a,b]$; $\inf_{x \in [a,b]} f(x) = f(x_1)$ and $\sup_{x \in [a,b]} f(x) = f(x_2)$

from (ii) of property ①, we have:

$$(b-a) f(x_1) \leq \int_a^b f(x)\,dx \leq (b-a) f(x_2)$$

$$\Rightarrow f(x_1) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(x_2)$$

from the second theorem of mean values [if f is continuous on $[a,b]$ and $f(x_1) < y < f(x_2)$, then $\exists\, c \in]x_1, x_2[$; $f(c) = y$]

we get :

$$\exists\, c \in]x_1, x_2[: f(c) = y = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

which completes the proof.

Theorem ④ (generalized Mean value theorem).

Let f be a continuous function on the interval $[a,b]$.
and let g be a function defined and continuous on $[a,b]$ such that g have the same sign on all $[a,b]$ (g doesn't change it sign on $[a,b]$)
then :

$$\exists\, c \in]a,b[: \int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx.$$