Chapter 04: 2^{nd}-order ordinary differential equations with constant coefficients (2^{nd}-order ODE)

Definition: we call second-order differential equation with constant coefficients any equation of the form:

$$ay'' + by' + cy = f(x), \quad a \neq 0 \quad \text{---} (E)$$

where a, b and c are real constants and f is a continuous function.

Remark: If $f(x) = 0$, then (E) is called homogenuous equation. else (E) is a non-homogenuous equation.

Theorem: Let y_p is a particular solution of non-homogenuous equation (E) and y_h is a general solution of homogenuous equation of (E), then $y = y_p + y_h$ is the general solution of non-homogenuous equation (E).

Method of solution of equation: $ay'' + by' + cy = f(x), \ a \neq 0$

1/ **homogenuous solution (y_h) or (solution without second member)**

Let $ay'' + by' + cy = 0$ --- (H)

we assume that $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$, $r \in \mathbb{R}$

(H) becomes:

$(ar^2 + br + c)e^{rx} = 0 \Rightarrow ar^2 + br + c = 0$ (since $e^{rx} > 0, \forall r \in \mathbb{R}$)

Now, finding a solution of (H) is equivalent to find a solution of characteristic equation $ar^2 + br + c = 0$ — (1)

by computing the value of $\Delta = b^2 - 4ac$, we distinguishe three cases:

a/ **If $\Delta > 0$**, hence equation (1) has two real solutions r_1 and r_2 where $r_1 \neq r_2$. In this case the homogenuous solution of equation (H) is written as follows:

$$\boxed{y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \text{ with } C_1, C_2 \text{ are two real constants}}.$$

b/ **If $\Delta = 0$**, hence equation (1) has a real multiple solution

r_0. In this case the homogeneous solution of equation (H) is written as follows:

$$\boxed{y_h = (c_1 x + c_2)e^{rx}}$$, where c_1 and c_2 are two real constants.

c/ If $\Delta < 0$, then equation (1) has two complex solutions r_1 and r_2 where $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R}$.

In this case the homogeneous solution of equation (H) is written as follows:

$$\boxed{y_h = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x), \ c_1, c_2 \in \mathbb{R} \text{ are constants.}}$$

2/ particular solution (y_p) (or nonhomogeneous solution or solution with second member)

The solution to differential equations with a second side (member) depends on the nature of its second side (member) $f(x)$.

a/ The second side is the product of a polynomial with an exponential function:

Theorem1: Let (E) be the following differential equation

$$ay'' + by' + cy = P_n(x)\, e^{\alpha x} \quad \text{----}(E).$$

The solution y_p to the equation (E) with the second side is as follows:

$y_p = P_m(x)\, e^{\alpha x}$, where P_m is a polynomial of degree:

a/ $\deg(P_m) = \deg(P_n)$, if α isn't a root of the characteristic equation.

b/ $\deg(P_m) = \deg(P_n) + 1$, if α is a simple root of the characteristic equation

c/ $\deg(P_m) = \deg(P_n) + 2$, if α is a double root of the characteristic equation

Example: $y'' - y' - 2y = 3x e^{2x} \ldots (E)$

1/ homogeneous solution (y_h):

Firstly, we solve the homogeneous equation: $y'' - y' - 2y = 0$ --(H)

Assume that: $y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2 e^{rx}$

Hence, (H) becomes:

$(r^2 - r - 2)e^{rx} = 0 \Leftrightarrow r^2 - r - 2 = 0$.(1) (since $e^{rx} > 0, \forall r \in \mathbb{R}$).

Hence Δ of (1) equals to: $\Delta = 1+8 = 9 > 0$,

then equation (1) has two real solutions:

$r_1 = \frac{1-3}{2} = -1$ and $r_2 = \frac{1+3}{2} = 2$

Therefore, the general solution of homogeneous equation (H) is

$$\boxed{y_h = c_1 e^{-x} + c_2 e^{2x} \quad \text{, where } c_1 \text{ and } c_2 \text{ are real constants}}$$

2/ particular solution (y_p): here, we have: $P_n(x) = 3x \wedge e^{\alpha x} = e^{2x}$

$\Rightarrow y_p = P_m(x) e^{2x}$, where P_m is a polynomial of degree:

- $\deg(P_m) = \deg(P_n) + 1 = 1+1 = 2$ (since $\alpha = 2$ is a simple root of characteristic equation (1)), hence:

$P_m(x) = ax^2 + bx + c$, this implies

$y_p = P_m(x) e^{2x} = (ax^2+bx+c) e^{2x}$

$\Rightarrow y_p' = ((2a+2b)x + 2ax^2 + 2c + b) e^{2x}$

$\Rightarrow y_p'' = [4ax^2 + (8a+4b)x + (2a+4b+4c)] e^{2x}$.

Replacing y_p, y_p' and y_p'' in (E), we obtain:

$$[4ax^2 + (8a+4b)x + (2a+4b+4c)]e^{2x} - [(2a+2b)x + 2ax^2 + 2c + b]e^{2x}$$
$$- 2[ax^2+bx+c]e^{2x} = 3x e^{2x}$$

by simplifying, we get:

$6ax + 2a + 3b = 3x$

In comparison, we find: $\begin{cases} 6a = 3 \Rightarrow a = \frac{1}{2} \\ 2a+3b = 0 \Rightarrow b = -\frac{1}{3} \end{cases}$, we can take $c = 0$,

Hence: $P_m(x) = \frac{1}{2}x^2 - \frac{1}{3}x$

and then, $\boxed{y_p = \left(\frac{1}{2}x^2 - \frac{1}{3}x\right) e^{2x}}$.

Hence, the general solution of the equation (E) is:

$$\boxed{y = y_p + y_h = c_1 e^{-x} + c_2 e^{2x} + \left(\frac{1}{2}x^2 - \frac{1}{3}x\right) e^{2x}, \quad c_1, c_2 \in \mathbb{R}}$$

b/ the second side is the product of a polynomial with sinus and/or cosinus func

Theorem 2: Let (E) be the following differential equation:

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$$ay'' + by' + cy = A_n(x)\cos\gamma x + B_m(x)\sin\gamma x \quad \text{---} (E)$$

where A_n and B_m are polynomials of degree n and m, respectively.

The solution (y_p) to the equation (E) is as follows:

$$y_P = P_k(x)\cos\gamma x + q_k(x)\sin\gamma x.$$

where P_k and q_k are polynomials of degree k such that:

$k = \max(n,m)$, if $i\beta$ isn't a root of the characteristic equation.

$k = \max(n,m)+1$, if $i\beta$ is a simple root of the characteristic equation

Example: $y'' + y = \cos x \quad \text{---} (E) \Leftrightarrow y'' + y = 1\cos x + 0\sin x.$

Solution:

1/ y_h: we solve the homogeneous equation: $y'' + y = 0 \quad \text{—} (H)$

Assume that: $y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2 e^{rx}$.

Hence (H) becomes: $r^2 + 1 = 0 \; (1) \Leftrightarrow r = \pm i$

$$\Rightarrow \boxed{y_h = (c_1\cos x + c_2\sin x)e^{0x}, \quad c_1, c_2 \in \mathbb{R}.}$$

2/ y_P: $y_p = P_k(x)\cos x + q_k(x)\sin x$, where P_k and q_k are polynomials of degree k such that:

$k = \max(0,0) + 1$ (since (i) is a simple root of equation (1).

Hence: $y_p = (a_1x + b_1)\cos x + (a_2x + b_2)\sin x.$

$$\Rightarrow y_p' = (a_1 + a_2x + b_2)\cos x + (-a_1x + b_1 + a_2)\sin x.$$

$$\Rightarrow y_p'' = (-a_1x - b_1 + 2a_2)\cos x + (-a_2x - 2a_1 - b_2)\sin x.$$

Replacing by y_p, y_p' and y_p'' in (E), we obtain:

$$(-a_1x - b_1 + 2a_2)\cos x + (-a_2x - 2a_1 - b_2)\sin x + (a_1x + b_1)\cos x + (a_2x + b_2)\sin x = \cos x + 0\sin x$$

$$\Rightarrow 2a_2\cos x - 2a_1\sin x = \cos x + 0\sin x.$$

in comparaison, we find:

$$\begin{cases} 2a_2 = 1 \Rightarrow a_2 = 1/2 \\ -2a_1 = 0 \Rightarrow a_1 = 0 \end{cases}$$

we can take $b_1 = b_2 = 0$, Hence:

$y_p = \frac{1}{2} x \sin x$ is the partialar solution of (E).

As conclusion: the general solution of equation (E) is as follows:

$$\boxed{y = y_p + y_h = c_1 \cos x + c_2 \sin x + \frac{1}{2} x . \sin x, \quad c_1, c_2 \in \mathbb{R}}$$

c/ The second side is the product of a polynomial with an exponential and sinus and/or cosinus functions:

Theorem 3: Let (E) be the following equation:

$$ay'' + by' + cy = e^{\alpha x}\left(P_n(x) \cos \beta x + P_m(x) \sin \beta x\right) \text{ — (E)}$$

where P_n and P_m are polynomials of degree n and m, respectively

The solution (y_p) of (E) is as follows:

$y_p = e^{\alpha x}\left(P_k(x) \cos \beta x + q_k(x) \sin \beta x\right)$, where P_k and q_k are polynomials of degree k such that:

- $k = \max(n, m)$, if $\alpha \pm i\beta$ isn't a root of characteristic equation
- $k = \max(n, m) + 1$, if $\alpha \pm i\beta$ is a simple root of characteristic equation

Example: Let: $y'' + y' + y = 2x \cos \sqrt{3} x + e^{x} \sin \sqrt{3} x$ --(E)

1/ y_h: (homogeneous solution): for finding y_h, we solve the equation: $y'' + y' + y = 0$. — (H)

assume that: $y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2 e^{rx}$

Replacing in (H), we get:

$(r^2 + r + 1) e^{rx} = 0 \Leftrightarrow r^2 + r + 1 = 0$ --(1) (since $e^{rx} > 0, \forall x \in \mathbb{R}$)

$\Delta = 1 - 4 \times 1 = -3 < 0 \Rightarrow$ (1) has two complex solutions:

$r_1 = \frac{-1 - \sqrt{\Delta}}{2a} = \frac{-1 - i\sqrt{3}}{2} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ and $r_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Hence $$\boxed{y_h = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right), \quad c_1, c_2 \in \mathbb{R}}$$

2/ y_p: (particular solution) $y_p = y_{p_1} + y_{p_2}$, where y_{p_1} and y_{p_2} are particular solutions of equations:

$y'' + y' + y = (2x \cos \sqrt{3} x + 0 \sin \sqrt{3} x) e^{0x}$ — (E_1) ($: y_{p_1}$)

and $y'' + y' + y = (\sin \sqrt{3} . x + 0 \cos \sqrt{3} x) e^{x}$. — (E_2) ($: y_{p_2}$)

respectively.

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•) finding (y_{p_1}) of (E_1) : we have:

$$y_{p_1} = e^{0x}\left(p_k(x)\cos\sqrt{3}x + q_k(x)\sin\sqrt{3}x\right)$$

where p_k and q_k are polynomials of degree :

$k = \max(\deg(p_n), \deg(p_m)) = \max(n, m) = \max(1, 0) = 1$ (since $i\sqrt{3}$ isn't a root of equation characteristic. (1))

hence : $p_k(x) = a_1 x + b_1$ and $q_k(x) = a_2 x + b_2$, thus:

$$y_{p_1} = (a_1 x + b_1)\cos\sqrt{3}x + (a_2 x + b_2)\sin\sqrt{3}x .$$

$$\Rightarrow y'_{p_1} = (\sqrt{3}a_2 x + a_1 + \sqrt{3}b_2)\cos\sqrt{3}x + (-\sqrt{3}a_1 x + a_2 + \sqrt{3}b_1)\sin\sqrt{3}x .$$

$$\Rightarrow y''_{p_2} = (-3a_1 x + 2\sqrt{3}a_2 - 3b_1)\cos\sqrt{3}x - (3a_2 x + 2\sqrt{3}a_1 + 3b_2)\sin\sqrt{3}x .$$

Replacing by y_{p_1}, y'_{p_1} and y''_{p_2} in (E_1), we obtain:

$$\left[(-2a_1 + \sqrt{3}a_2)x + 2\sqrt{3}a_2 - 2b_1 + \sqrt{3}b_2\right]\cos\sqrt{3}x +$$
$$+ \left[(-2a_2 + \sqrt{3}a_1)x + a_2 - 2\sqrt{3}a_2 - 2b_2 - \sqrt{3}b_1\right]\sin\sqrt{3}x = 2x\cos\sqrt{3}x$$

by matching :

$$\begin{cases} -2a_1 + \sqrt{3}a_2 = 2 \\ 2\sqrt{3}a_2 - 2b_1 + \sqrt{3}b_2 = 0 \\ -2a_2 + \sqrt{3}a_1 = 0 \\ -2\sqrt{3}a_1 - 2b_2 + a_2 - \sqrt{3}b_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = \frac{-4}{7} \\ a_2 = \frac{2\sqrt{3}}{7} \\ b_1 = \frac{46}{49} \\ b_2 = \frac{12\sqrt{3}}{49} \end{cases}$$

hence : $$\boxed{y_{p_1}(x) = \left(\frac{-4}{7}x + \frac{46}{49}\right)\cos\sqrt{3}x + \left(-\frac{2\sqrt{3}}{7}x + \frac{12\sqrt{3}}{49}\right)\sin\sqrt{3}x}$$

•) finding (y_{p_2}) of (E_2), we have

$y_{p_2} = e^{x}\left(p_k(x)\cos\sqrt{3}x + q_k(x)\sin\sqrt{3}x\right)$, where p_k and q_k are polynomials of degree :

$k = \max(\deg(p_n), \deg(p_m)) = \max(0, 0) = 0$ (since $1 \pm i\sqrt{3}$ isn't a root of characteristic equation (1)).

Hence : $p_k(x) = a$ and $q_k(x) = b$, thus :

$$y_{p_2}(x) = e^x\left(a\cos\sqrt{3}\,x + b\sin\sqrt{3}\,x\right)$$

$$\Rightarrow y'_{p_2}(x) = \left[(a+b\sqrt{3})\cos\sqrt{3}\,x + (b-a\sqrt{3})\sin\sqrt{3}\,x\right]e^x.$$

$$\Rightarrow y''_{p_2}(x) = \left[(-2a+2b\sqrt{3})\cos\sqrt{3}\,x + (-2b-2a\sqrt{3})\sin\sqrt{3}\,x\right]e^x.$$

Replacing by this values in (E_2), we obtain :

$$\Big[(a+b\sqrt{3})\cos\sqrt{3}\,x + (b-a\sqrt{3})\sin\sqrt{3}\,x + (-2a+2b\sqrt{3})\cos\sqrt{3}\,x + (-2b-2a\sqrt{3})\sin\sqrt{3}\,x$$
$$+ (a\cos\sqrt{3}\,x + b\sin\sqrt{3}\,x)\Big]e^x = \sin\sqrt{3}\,x\,e^x$$

in comparaison, we get :

$$\begin{cases} 3b\sqrt{3} = 0 \\ -3a\sqrt{3} = 1 \end{cases} \Rightarrow \begin{cases} b = 0 \\ a = -\dfrac{1}{3\sqrt{3}} \end{cases}$$

hence $\boxed{y_{p_2} = e^x\left(-\frac{1}{3\sqrt{3}}\cos\sqrt{3}\,x\right)} \Rightarrow y_p = y_{p_1} + y_{p_2}$

Therefore, the general solution of equation (E) is :

$$y = y_p + y_h = y_{p_1} + y_{p_2} + y_h$$

$$\Rightarrow \boxed{y = \left(\frac{-4}{7}x + \frac{46}{49}\right)\cos\sqrt{3}\,x + \left(\frac{-2\sqrt{3}}{7}x + \frac{12\sqrt{3}}{49}\right)\sin\sqrt{3}\,x + \left(\frac{-1}{3\sqrt{3}}\cos\sqrt{3}\,x\right)e^x + e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right),\ c_1, c_2 \in \mathbb{R}.}$$

Examples : Solve the following equations :

1/ $y'' + y' - 2y = 0 \Rightarrow y = y_h = c_1 e^{-2x} + c_2 e^x$, $c_1, c_2 \in \mathbb{R}$.

2/ $y'' + 9y = 0 \Rightarrow y = y_h = c_1\cos 3x + c_2\sin 3x$, $c_1, c_2 \in \mathbb{R}$

3/ $y'' - 4y' + 4y = 0 \Rightarrow y = y_h = (c_1 + c_2 x)e^{2x}$, $c_1, c_2 \in \mathbb{R}$.

4/ $y'' - 5y' + 6y = (2x+1)e^x \Rightarrow y = y_h + y_p = c_1 e^{2x} + c_2 e^{3x} + (x+2)e^x$, c_1, c_2

5/ $y'' - 5y' + 6y = 3e^{2x} + e^{3x} \Rightarrow y = ?$

6/ $y'' - 2y' + y = e^x \Rightarrow y = ?$

7/ $y'' + y' + y = \cos x + \sin 3x \Rightarrow y = ?$

8/ $y'' + y' + y = e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x \Rightarrow y = ?$

9/ $y'' + y' + y = e^{\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + \sin x$.

10/ $y'' + y' = (x-1) \sin x$

11/ $y'' + y = \cos x + 2$.

We can summarize the particular solutions of 2^{nd}-order differential equation with constant coefficients (E) in the following table: where $ay'' + by' + cy = f(x)$ --- (E), with $a \neq 0$

form of the function $f(x)$	root of the characteristic equ	form of (y_p) of (E)
$P_m(x)$	"0" isn't a root of characteristic equation $[ar^2 + br + c = 0]$	$q_m(x)$
	"0" is a root of degree "S" of characteristic equation.	$x^S q_m(x)$
$P_m(x) e^{\alpha x}$	"α" isn't a root of characteristic equation.	$q_m(x) e^{\alpha x}$
	"α" is a root of degree "S" of characteristic equation	$x^S q_m(x) e^{\alpha x}$
$P_m(x) \cos \beta x + q_m(x) \sin \beta x$	"$\pm i\beta$" isn't a root of characteristic equation	$P_k(x) \cos \beta x + q_k(x) \sin \beta x$
	"$\pm i\beta$" is a simple root of characteristic equation	$x \left(P_k(x) \cos \beta x + q_k(x) \sin \beta x \right)$
$e^{\alpha x}\left(P_m(x) \cos \beta x + q_m(x) \sin \beta x\right)$	"$\alpha \pm i\beta$" isn't a root of characteristic equation	$e^{\alpha x}\left(P_k(x) \cos \beta x + q_k(x) \sin \beta x\right)$
	"$\alpha \pm i\beta$" is a simple root of characteristic equation	$x e^{\alpha x}\left(P_k(x) \cos \beta x + q_k(x) \sin \beta x\right)$

where P_n and q_m are a polynomials of degree n and m, respectively. and P_k and q_k are a polynomials of degree $k = \max(n, m)$.