Analysis 02 (1st level of Mathematic)

Chapter 01: indefinite integrals

indefinite integral, some properties of the indefinite integrals integration methods : integration by change of variable, integration by parts, integration of regular expressions, integration of fractional functions.

Chapter 02: definite integrals

definite integral, some properties of the definite integrals, integral function of its upper bound, Newton-Leibniz formula Cauchy-schwartz integral, Darboux sums. Conditions for the existence of the integral, properties of Darboux sums, integrability of monotonic continuous functions.

Chapter 03: first order differential equations

generalities, classification of first order differential equations, equation with separable variables, homogenous equation, linear equation, Bernoulli method, method of variable of the Lagrange constant, Bernoulli equation, total differential equation, Riccati equation.

Chapter 04: Second order differential equations with constant coefficients.

homogenous second order differential equation with constant coefficients, non-homogenous secondorder differential equation with constant coefficients, methods for solving 2nd order differential equation with constante coefficients.

Chapter 01: indefinite integrals and the primitive

I/ The primitives:

1/ Definition: Let f be a function defined on the (interval) Domain D and F be a differentiable function on D. we say that F is a primitive function of f if and only if: $\forall x \in D$: $F'(x) = f(x)$.

Example: • $F(x) = \frac{1}{3}x^3 + C$ is the primitive of $f(x) = x^2$, with $C \in \mathbb{R}$.

• $F(x) = \ln x$ is the primitive of $f(x) = \frac{1}{x}$, $x \in \mathbb{R}_+^*$.

2/ the set of primitive functions and indefinite integrals:

Definition: we call the undefined or indefinite integral of the function f defined on the interval $[a,b]$, which is denoted by $\int f(x)\,dx$, all (every) primitive function of f, and we have:

$$\int f(x)\,dx = F(x) + C \quad \text{with } F'(x) = f(x)$$

theorem: If the function f admits a primitive function F on the domain D, then the functions $F + C$ (with C is a constant) are all primitive functions of f on D, since.

$\forall x \in D$: $(F+C)'(x) = F'(x) + 0 = f(x)$

. the set of primitive functions of f on D is denoted by $\int_D f(x)\,dx$

where: $\int_D f(x)\,dx = F(x) + C$, C = constant $\in D$.

Example:

$$I_1 = \int_D \frac{1}{x}\,dx, \quad D = \mathbb{R}^* \Rightarrow I_1 = \ln|x| + C$$

$$I_2 = \int_D \cos x\,dx, \quad D = \mathbb{R} \Rightarrow I_2 = \sin x + C$$

3/ properties of the indefinite integrals (or of the primitives)

Let f and g be two functions defined on the Domain D, then

1/ $\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$, $\forall \lambda \in \mathbb{R}$.

2/ $\int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$.

4/ integration methods:

1-4/ integration by parts: If f and g be two functions defined and differentiable on the same domain D, then according to the properties of differentiation, we have

$$(fg)'(x) = f'(x)\, g(x) + g'(x)\, f(x)$$

by integration, we get:

$$(fg)(x) = \int (fg)'(x)\, dx = \int f'(x)\, g(x)\, dx + \int g'(x)\, f(x)\, dx$$

in another meaning:

$$\boxed{\int f'(x)\, g(x)\, dx = (fg)(x) - \int g'(x)\, f(x)\, dx} \quad (*)$$

this expression is called the expression of integration by parts.

Remark: To calculate the integral or the primitive function of the function $u = f'g$ by using the integration by parts, we must choose the functions f' and g well, so that the integral $\int f'(x)\, g(x)\, dx$ is easy to calculate.

Illustrative examples:

1/ $I_1 = \int x^n \ln x\, dx,\ n \in \mathbb{Z} - \{-1\}$

put: $\begin{cases} g(x) = \ln x \\ f'(x) = x^n dx \end{cases} \Rightarrow \begin{cases} g'(x) = \frac{1}{x} dx \\ f(x) = \frac{x^{n+1}}{n+1} \end{cases}$

Using $(*)$, we get:

$$I_1 = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int \frac{x^{n+1}}{x}\, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n\, dx$$

$$\Rightarrow \boxed{I_1 = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C}$$

2/ $I_2 = \int \arcsin x\, dx$

put: $\begin{cases} g(x) = \arcsin x \\ f'(x) = 1\, dx \end{cases} \Rightarrow \begin{cases} g'(x) = \frac{dx}{\sqrt{1-x^2}} \\ f(x) = x. \end{cases}$

Using $(*)$, we obtain

$$I_2 = x \arcsin x - \int \frac{x\, dx}{\sqrt{1-x^2}} = x \arcsin x - \frac{1}{2} \int 2x (1-x^2)^{-\frac{1}{2}}\, dx.$$

$$\Rightarrow I_2 = x \arcsin x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C = \boxed{x \arcsin x + \sqrt{1-x^2} + C}$$

1.1-4: Some integrals that allow us to use the mtd of integration by parts

a/ the product of a polynomial with an exponential function

to calculate the integral $I = \int P_n(x) e^{\alpha x} dx$, where P_n is a polynomial function of the degree n (i.e, $\deg(P_n(x)) = n$) we can use the method of integration by parts, and in this case we always put:

$$\begin{cases} g(x) = P_n(x) \\ f'(x) = e^{\alpha x} dx \end{cases} \Rightarrow \begin{cases} g'(x) = P_{n-1}(x) \\ f(x) = \frac{1}{\alpha} e^{\alpha x}, \ \alpha \neq 0. \end{cases}$$

using (*), I become

$$I = \left[\frac{1}{\alpha} P_n(x) e^{\alpha x} \right] - \frac{1}{\alpha} \underbrace{\int e^{\alpha x} P_{n-1}(x)\, dx}_{I_1}$$

Then, we integrate I_1 by parts again, and we always set g as the polynomial function and f' as the exponential function, and we repeat the process n times (i.e, we integrate by parts n times)

Illustrative example: calculate the integral:

$$I = \int (x^2 - 3x + 2) e^{-x} dx$$

we have: $P_2(x) = x^2 - 3x + 2$, in this integral we integrate by parts two times (twice), where we always put:

$$\begin{cases} g(x) = x^2 - 3x + 2 \\ f'(x) = e^{-x} dx \end{cases} \Rightarrow \begin{cases} g'(x) = (2x - 3)\, dx \\ f(x) = -e^{-x}. \end{cases}$$

Using (*) we get:

$$I = -(x^2 - 3x + 2) e^{-x} + \underbrace{\int (2x-3) e^{-x} dx}_{I_1}$$

Now, we integrate I_1 by parts, where

$$\begin{cases} g(x) = 2x - 3 \\ f'(x) = e^{-x} dx \end{cases} \Rightarrow \begin{cases} g'(x) = 2\, dx \\ f(x) = -e^{-x} \end{cases}$$

using again (*), we get

$$I_1 = -(2x-3) e^{-x} + 2 \int e^{-x} dx = -(2x-3) e^{-x} - 2e^{-x} + c.$$

replacing the value of I_1 in I, we get

$$I = -(x^2-3x+2)e^{-x} - (2x-3)e^{-x} - 2x + C$$

hence :

$$\boxed{I = e^{-x}(-x^2+x-1)+C} \quad — ②$$

2nd mtd of integration:

$$\int (x^2-3x+2)e^{-x}dx = (ax^2+bx+c)e^{-x}+C_1 \quad — ①$$

hence by derivating ① we get :

$$(x^2-3x+2)e^{-x} = \left[(ax^2+bx+c)e^{-x}+C_1\right]' = (-ax^2+(2a-b)x+b-c)e^{-x}$$

$$\Rightarrow \begin{cases} -a = 1 \\ 2a-b = -3 \\ b-c = 2 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 1 \\ c = -1 \end{cases}$$

replacing in ①, we get :

$$\boxed{\int (x^2-3x+2)e^{-x}dx = (-x^2+x-1)e^{-x}+C_1}$$

which is the same as ②.

c/ the product of an exponential function with sin or cos

to calculate the integral $I = \int \cos\beta x \, e^{\alpha x}dx$ or $I = \int \sin\beta x \, e^{\alpha x}dx$

we use the integration by parts only two times.

Remark: Choosing f' and g in the first time is qualitative or arbitrary, but in the second time it is related to what we chose in the first time.

Illustrative example: calculate the following integral

$I = \int \sin 2x \, e^{x} dx.$

$$\begin{cases} f'(x) = e^{x}dx \\ g(x) = \sin 2x \end{cases} \Rightarrow \begin{cases} f(x) = e^{x} \\ g'(x) = 2\cos 2x \, dx \end{cases}$$

using (*), we get :

$$I = \sin 2x \, e^{x} - 2\underbrace{\int \cos 2x \, e^{x}dx}_{I}$$

we integrate I_1 by parts again, there we must put:

$$\begin{cases} f'(x) = e^x\,dx \\ g(x) = \cos 2x \end{cases} \Rightarrow \begin{cases} f(x) = e^x \\ g'(x) = -2\sin 2x\,dx \end{cases}$$

using (*), we get

$$I_1 = e^x \cos 2x + 2\underbrace{\int e^x \sin 2x\,dx}_{I} + C.$$

replacing the value I_1 in I, we obtain

$$I = \sin 2x\, e^x - 2 I_1$$

$$= \sin 2x\, e^x - 2\left[e^x \cos 2x + 2I\right] + C.$$

$$\Rightarrow I = \sin 2x\, e^x - 2e^x \cos 2x - 4I + C.$$

$$\Rightarrow 5I = (\sin 2x - 2\cos 2x)e^x + C.$$

$$\Rightarrow \boxed{I = \frac{1}{5}(\sin 2x - 2\cos 2x)e^x + C}$$

2-4: the integration by change of variable:

theorem: Let h be a function of the class C^1 defined from $[\alpha, \beta]$ to $[a, b]$ and f be a continuous function on $[a, b]$ then for $t = h(x)$, we have: $h: [\alpha, \beta] \to [a, b]$, $x \mapsto h(x) = t$

$$\int_\alpha^\beta f(h(x))\, h'(x)\, dx = \int_{h(\alpha)}^{h(\beta)} f(t)\, dt = \int_a^b f(t)\, dt \quad (1)$$

in addition, if h is a bijection (or bijective function), then for $x = h(t)$ $h: [\alpha, \beta] \to [a, b]$, $t \to h(t) = x$

we have:

$$\int_a^b f(x)\, dx = \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(t))\, h'(t)\, dt = \int_\alpha^\beta f(h(t))\, h'(t)\, dt.$$

therefore, if F is a primitive function of f, then $F(h)$ is a primitive function of $f(h)\cdot h'$, that is,

$$\int (f \circ h)\, h' = F \circ h + C = F(h) + C$$

proof: Since F is a primitive function of f, then $F'(x) = f(x)$

Using the properties of differentiation of the compound/composite function, we obtain:

$(F\circ h)'(t) = F'(h(t))\, h'(t) = f(h(t)) \cdot h'(t) = (f\circ h)(t) \cdot h'(t)$

therefore: $(F\circ h)$ is the primitive function of $(f\circ h)\, h'$.

Hence:

$$\int_\alpha^\beta f(h(x))\, h'(x)\, dx = [F\circ h]_\alpha^\beta = F(h(\beta)) - F(h(\alpha)) = \int_{h(\alpha)}^{h(\beta)} f(t)\, dt.$$

which gives integral (1) of the theorem.

Illustrative examples:

(1) by using the following change of variable $x = a\sin t$, $a > 0$ calculate the integral: $I = \int_0^a \sqrt{a^2 - x^2}\, dx$ —— (*1)

$x = 0 \Rightarrow a\sin t = 0 \Rightarrow \sin t = 0 \overset{\sin \text{ is bijection}}{\Rightarrow} t = \arcsin 0 = 0 \Rightarrow \boxed{t = 0}$

$x = a \Rightarrow a\sin t = a \Rightarrow \sin t = 1 \overset{\sin \text{ is bijection}}{\Rightarrow} t = \arcsin 1 = \arcsin(\sin \frac{\pi}{2}) = \boxed{\frac{\pi}{2}}$

$x = a\sin t \Rightarrow dx = a\cos t\, dt$

integral (*1) becomes:

$$I = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t}\, (a\cos t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2(1 - \sin^2 t)}\; a\cos t\, dt$$

$$= a^2 \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t}\, \cos t\, dt$$

$$= a^2 \int_0^{\frac{\pi}{2}} \cos^2 t\, dt \qquad = a^2 \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2}\, dt$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2t)\, dt$$

$$= \frac{a^2}{2} \left[t + \frac{1}{2}\sin 2t \right]_0^{\frac{\pi}{2}} = \boxed{\frac{a^2\pi}{4}}$$

(2) by using the change of variable $t = \cos x$, calculate

$I = \int_0^\pi \sin^3 x\, dx$ —— (2)

$x = 0 \Rightarrow t = \cos 0 = 1$

$x = \pi \Rightarrow t = \cos \pi = -1$

$t = \cos x \Rightarrow dt = -\sin x\, dx.$

integral (2) becomes: $I = \int_0^\pi \sin^2 x \sin x\, dx$, thus

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$$I = -\int_0^{\pi} \sin^2 x\,(-\sin x\,dx) = -\int_0^{\pi} (1-\cos^2 x)\,(-\sin x)\,dx$$

$$= -\int_1^{-1} (1-t^2)\,dt = \int_{-1}^{1} (1-t^2)\,dt = t - \frac{t^3}{3}\Big]_{-1}^{1} = \boxed{\frac{4}{3}}$$

(3) by using the chang of variable $t = e^x$, calculate:

$$I = \int_1^2 \frac{dx}{e^x + e^{-x}} \quad \text{— (3)}$$

$x = 1 \Rightarrow t = e^1$

$x = 2 \Rightarrow t = e^2$

$t = e^x \Rightarrow dt = e^x\,dx.$

hence integral (3) becomes:

$$I = \int_1^2 \frac{e^x\,dx}{e^{2x}+1} = \int_e^{e^2} \frac{dt}{t^2+1} = \operatorname{arctg} t \Big]_e^{e^2} = \boxed{\operatorname{arctg} e^2 - \operatorname{arctg} e}.$$

Remark) Note that in the first example we stipulated (exig) that the function f ($\sin t$) be bijective, while we didn't stipulate that on them in the last two examples.

II. Integration of regular expressions) In the following table we give the integral of some usual functions.

1/ $\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C, & \text{if } n \neq -1 \\ \ln|x| + C, & \text{if } n = -1 \end{cases}$

2/ $\int e^{\alpha x}\,dx = \dfrac{e^{\alpha x}}{\alpha} + C \quad (\alpha \in \mathbb{R}^*)$

3/ $\int \dfrac{dx}{1+x^2} = \operatorname{arctg} x + C$

4/ $\int \dfrac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

5/ $\int \cos x\,dx = \sin x + C$

6/ $\int \sin x\,dx = -\cos x + C$

7/ $\int \dfrac{dx}{\cos^2 x} = \operatorname{tg} x + C$

8/ $\int \dfrac{dx}{\sin^2 x} = -\operatorname{cotg} x + C$

9/ $\int \frac{dx}{\cos x} = \ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$

10/ $\int \frac{dx}{\sin x} = \ln\left|tg\,\frac{x}{2}\right| + C$

11) $\int tg\,x\,dx = -\ln|\cos x| + C$

12/ $\int cotg\,x\,dx = \ln|\sin x| + C$

13/ $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

14/ $\int \frac{dx}{\sqrt{x^2+\alpha}} = \ln\left|x+\sqrt{x^2+\alpha}\right| + C$, $\alpha \neq 0$

15/ $\int ch\,x\,dx = sh\,x + C$

16/ $\int sh\,x\,dx = ch\,x + C$

17/ $\int \frac{dx}{ch^2 x} = th\,x + C$

18/ $\int \frac{dx}{sh^2 x} = -coth\,x + C$

19/ $\int \frac{dx}{ch\,x} = 2\,arctg\,e^x + C$

20/ $\int \frac{dx}{sh\,x} = \ln\left|th\,\frac{x}{2}\right| + C$

21/ $\int th\,x\,dx = \ln(ch\,x) + C$

22/ $\int coth\,x\,dx = \ln|sh\,x| + C$

*/ particular cases:

1/ $\int f'(x)\,f^n(x)\,dx = \begin{cases} \frac{f^{n+1}(x)}{n+1} + C, & \text{if } n \neq -1 \\ \ln|f(x)| + C, & \text{if } n = -1. \end{cases}$

2/ $\int f'(x)\,e^{f(x)}\,dx = e^{f(x)} + C$

III – integration of rational functions.

1/ Definitions:

Definition 01) we call a rational function all function of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are two polynomials that it

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coefficients in $\mathbb{R}$.

Example: the functions. $\frac{x^4-3x^3+5x^2+x-3}{6x^3+2x-1}$ and $\frac{x+1}{2x^3+x-2}$ are rational functions.

Definition (2): (Simple elements)

a/ we call a simple element of the first type with degree n ($n \in \mathbb{N}^*$) all fractional function that is written in the form:

$\frac{\alpha}{(x-a)^n}$, where α and a are a constant coefficients in $\mathbb{R}$.

its primitive is:

$$\int \frac{\alpha}{(x-a)^n}\,dx = \begin{cases} \alpha \ln|x-a| + C, & \text{if } n=+1 \\ \frac{-1}{n-1}\,\frac{\alpha}{(x-a)^{n-1}} + C, & \text{if } n > +1 \end{cases}, \quad C = cte \in \mathbb{R}.$$

Example: each of the following functions is a simple element of the first type: $\frac{-1}{x+1}$, $\frac{5}{(x-3)^4}$ and $\frac{1}{(x+2)^3}$.

b/ we call a simple element of the second type with the degree $n \in \mathbb{N}^*$, all fractional function, that is written in the form $\frac{\alpha x+\beta}{(ax^2+bx+c)^n}$, where $\alpha, \beta, a, b, c = cte \in \mathbb{R}$

and b^2-4ac is always negative ($b^2-4ac<0$).

example: each of the following functions: $\frac{x+2}{(x^2+x+1)^2}$ and $\frac{x+1}{2x^2+x+1}$ is a simple element of the second type.

Definition (3) (pole of rational function).

Let $f(x) = \frac{p(x)}{(x-a)^\alpha\, q_1(x)}$, where $p(a) \neq 0$ and $q_1(a) \neq 0$

be a fractional function.

then a is called a pole of the degree α of f.

*/ If $\alpha = 1$, then a is called a simple pole.

*/ If $\alpha > 1$, then a is called a multiple pole of degree α.

Example: Let $f(x) = \dfrac{x^2+1}{(x-2)(x-1)(x-3)^2}$ be a fractional functio[n]

then $x=1$ and $x=2$ are a simple poles, while $x=3$ is a multiple pole of degree $n=2$.

2- Factoring a fractional function:

Theorem: Each fractional function $f(x) = \dfrac{p(x)}{q(x)}$ can be decompose[d] (factorized) into the sum of simple fractions of the first or second type.

Example ① Let $f(x) = \dfrac{x^2+2}{(x+1)^3(x-2)}$ and $g(x) = \dfrac{1}{(x-1)(x-2)}$ be two fractional functions, then

$$f(x) = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{(x+1)^3} + \frac{d}{x-2}$$

1st mtd:
To find the value of the constants a, b, c and d, we standardiz[e] the denominators and compare with $f(x)$ to obtain

$$a = -\frac{2}{9},\ b = \frac{1}{3},\ c = -1 \text{ and } d = \frac{2}{9}$$

in the same way, we get the value of a and b of g such that

$$g(x) = \frac{a}{x-1} + \frac{b}{x-2}, \text{ here } a = -1 \text{ and } b = 1$$

2nd mtd:
Or we follow the following method

*/ for the function g, we have:

$$g(x) = \frac{a}{x-1} + \frac{b}{x-2} = \frac{1}{(x-1)(x-2)} \quad --- (*)$$

Step 1: computing the value of a:
- we multiply both sides of $(*)$ by $(x-1)$ to get

$$a + \frac{b(x-1)}{x-2} = \frac{1}{x-2} \quad —— (**)$$

then, we put $x=1$ in $(**)$ to get the value $\boxed{a=-1}$

Step 2: Computing the value of b:
Now, multiplying both the sides of $(*)$ by $(x-2)$ to get

$$\frac{a(x-2)}{x-1} + b = \frac{1}{x-1} \quad —— (**')$$

then, put $x=2$ in $(**')$ to get the value: $\boxed{b=1}$

*/ for function f, we have:

$$f(x) = \frac{x^2+2}{(x+1)^3(x-2)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{(x+1)^3} + \frac{d}{x-2} \quad \text{--- (2)}$$

step 1: computing the value of c:

we multiply both sides of (2) by $(x+1)^3$ and then, we put $x=-1$

we get: $c = -1$

step 2: computing the value of d:

we multiply both sides of (2) by $(x-1)$ and then, we put $x=2$

we get: $d = \frac{2}{9}$

step3: computing the values of a and b.

for computing the values of a and b, we put in (2), $x=0$ and $x=1$

we obtain the following system:

$$\begin{cases} a+b = \frac{1}{9} \\ 4a+2b = -\frac{2}{9} \end{cases} \Rightarrow \begin{cases} a = -\frac{2}{9} \\ b = \frac{1}{3} \end{cases}$$

Example (2): Let $f(x) = \frac{x^3+1}{(x+2)^4}$ be a fractional function.

Decompose the function f into the sum of simple fractions of the first or second type. we have:

$$f(x) = \frac{x^3+1}{(x+2)^4} = \frac{a}{x+2} + \frac{b}{(x+2)^2} + \frac{c}{(x+2)^3} + \frac{d}{(x+2)^4}$$

1st mtd: to compute the values of a, b, c and d, we standardiz the denominators and compare with $f(x)$ to get:

$a=1$, $b=-6$, $c=12$ and $d=-7$.

2nd mtd: to decompose functions of the form $\frac{p(x)}{(x-a)^\alpha}$ where $\boxed{\deg(p) \geqslant}$

we perform the process of successive division of the functio $p(x)$ by $(x-a)$ until a constant quotient is obtained.

for example (2), we have, $f(x) = \frac{x^3+1}{(x+2)^4}$, where $\deg(x^3+1) = 3 \geqslant 1$.

Here, we perform the process of successive division of the functio x^3+1 by $(x+2)$ until a constant quotient is obtained, that is

$$\begin{array}{l|l}
x^3+1 & x+2 \\ \hline
-\left(x^3+2x^2\right) & x^2-2x+4 \\
\quad -2x^2+1 & \\
-\left(-2x^2-4x\right) & \\
\quad 4x+1 & \\
-\left(4x+8\right) & \\
\quad \boxed{-7} &
\end{array}
\qquad
\begin{array}{l|l}
x^2-2x+4 & x+2 \\ \hline
-\left(x^2+2x\right) & x-4 \\
\quad -4x+4 & \\
-\left(-4x-8\right) & \\
\quad \boxed{12} &
\end{array}
\qquad
\begin{array}{l|l}
x-4 & x+2 \\ \hline
-\left(x+2\right) & 1 \\
\quad \boxed{-6} &
\end{array}$$

constant quotient

Hence: $\dfrac{x^3+1}{(x+2)^4} = \dfrac{-7}{(x+2)^4} + \dfrac{12}{(x+2)^3} - \dfrac{6}{(x+2)^2} + \dfrac{1}{x+2}.$

Example 03 Decompose the function f into the sum of simple fractions of the first and/or second type, such that

$$f(x) = \frac{4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)} = \frac{ax+b}{(x^2+2x+3)^2} + \frac{cx+d}{x^2+2x+3} + \frac{e}{x+1}$$

after calculations and simplifications, we find:

$a=1$, $b=-1$, $c=d=0$ and $e=1$

3/ integrating a fractional functions:

to integrate the fractional function $f(x) = \dfrac{p(x)}{q(x)}$ we follow the following steps:

Step 1: if $\deg(p) \geq \deg(q)$ we perform the Euclidean division of the function $p(x)$ by the function $q(x)$ until an irreducible fraction is obtained (i.e, until we obtain a remainder of division $(R(x))$ with a degree less than $p(x)$), that is,

$$\frac{p(x)}{q(x)} = E(x) + \frac{R(x)}{q(x)}, \quad \deg(R) < \deg(q).$$

Example: $f(x) = \dfrac{x^3+x+1}{x^2+1} = x + \dfrac{1}{x^2+1}.$

- If $\deg(p) < \deg(q)$, we go to the next step (step 2):

Step 2: we decompose the function $\dfrac{p(x)}{q(x)}$ $\left(\text{or } \dfrac{R(x)}{q(x)}\right)$ into the sum of simple fractions of the first and/or second type.

step 3: Now, it remains to integrate the obtained simple elements gotten in step 2.

3-1: integrating simple elements of the first type:

we previously saw that: if the function f is written as $f(x)=\frac{\alpha}{(x-a)}$ then

$$\int f(x)\,dx = \int \frac{\alpha\,dx}{(x-a)^n} = \begin{cases} \alpha \ln|x-a| + c, & \text{if } n=1 \\ \frac{\alpha}{1-n}\,\frac{1}{(x-a)^{n-1}} + C, & \text{if } n \neq 1 \end{cases}$$

3-2: integrating simple elements of the second type: $n>0$, $n\neq 1$

If the function f is written in the form: $\frac{\alpha x+\beta}{(ax^2+bx+c)^n}$ with $\Delta = b^2-4ac<0$, then we try to write the derivative of the denominator in the numerator as follows:

$$\frac{\alpha x+\beta}{(ax^2+bx+c)^n} = \frac{\frac{\alpha}{2a}(2ax+b)+\beta-\frac{\alpha}{2a}b}{(ax^2+bx+c)^n}$$

$$= \underbrace{\frac{\alpha}{2a}}_{\text{constant}}\,\frac{2ax+b}{(ax^2+bx+c)^n} + \underbrace{\left(\beta-\frac{\alpha}{2a}b\right)}_{\text{constant}}\frac{1}{(ax^2+bx+c)^n} \quad (*)$$

we remark that $\frac{2ax+b}{(ax^2+bx+c)^n}$ is of the form $\frac{U'(x)}{[U(x)]^n}$ and its primitive is:

$$\int\frac{2ax+b}{(ax^2+bx+c)^n}dx = \begin{cases} \ln|ax^2+bx+c| + k, & \text{if } n=1 \\ \frac{1}{1-n}\,\frac{1}{(ax^2+bx+c)^{n-1}} + k, & \text{if } n\neq 1 \end{cases}$$

Hence, it remains to integrate the following term: $\frac{1}{(ax^2+bx+c)^n}$

for this, we try to write $\frac{1}{ax^2+bx+c}$ in the form $\frac{1}{1+t^2}$ as follows:

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

$$= a\,\frac{(4ac-b^2)}{4a^2}\left[1+\frac{\left(x+\frac{b}{2a}\right)^2}{\frac{4ac-b^2}{4a^2}}\right]$$

$$= \frac{4ac-b^2}{4a}\left[1+\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)^2\right]$$

we put Now, $t=\frac{2ax+b}{\sqrt{4ac-b^2}}$, hence:

$$\int\frac{dx}{(ax^2+bx+c)^n} = \alpha\int\frac{dt}{(1+t^2)^n} = \alpha\, I_n, \text{ with } I_n=\int\frac{dt}{(1+t^2)^n}$$

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• if $n = 1$, then $I_n = I_1 = \int \frac{dt}{1+t^2} = \text{arctg}\, t + C_1 = \text{arctg}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + C$

• if $n > 1$, then I_n is computed by the following reccurence relationship:

$$\boxed{(2n-1)\, I_n = \frac{t}{(1+t^2)^{n-1}} + (2n-3)\, I_{n-1}, \text{ with } I_1 = \text{arctg}\, t + k}$$

example: compute $I = \int \frac{x+1}{(x^2+x+1)^2}\, dx$

$$\frac{x+1}{(x^2+x+1)^2} = \frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{(x^2+x+1)^2} = \frac{1}{2}\frac{2x+1}{(x^2+x+1)^2} + \frac{1}{2}\cdot\frac{1}{(x^2+x+1)^2} \quad (*)$$

hence: $\int \frac{2x+1}{(x^2+x+1)^2}\, dx = \int (2x+1)(x^2+x+1)^{-2}\, dx =$

$$= -\left[(x^2+x+1)^{-1}\right] + C = \boxed{\frac{-1}{x^2+x+1} + C} \quad \text{I}$$

Now, it remains to integrate the term: $\frac{1}{(x^2+x+1)^2}$

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$
$$= \frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1\right)$$
$$= \frac{3}{4}\left(\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2 + 1\right)$$
$$= \frac{3}{4}\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1\right)$$

we put: $t = \frac{2x+1}{\sqrt{3}} \Rightarrow dt = \frac{2}{\sqrt{3}}\, dx \Rightarrow dx = \frac{\sqrt{3}}{2}\, dt$

hence:
$$\int \frac{dx}{(x^2+x+1)^2} = \int \frac{\frac{dt}{2}}{\frac{3}{4}(t^2+1)} = \frac{4}{3}\cdot\frac{1}{2}\int \frac{dt}{(1+t^2)^2} = \frac{2}{3}\int \frac{dt}{(1+t^2)^2}$$

thus: $\int \frac{dx}{(x^2+x+1)^2} = \frac{2}{3}\, I_2$ — II

from the relationship of reccurence, we have:

$$(2n-1)\, I_n = \frac{t}{(1+t^2)^{n-1}} + (2n-3)\, I_{n-1}, \text{ with } I_1 = \text{arctg}\, t + k$$

for $\underline{n=2}$: we get the value of I_2 as follows:

$$(2\times 2-1)\, I_2 = \frac{t}{(1+t^2)^1} + (2\times 2-3)\, I_1$$

$$\Rightarrow I_2 = \frac{1}{3}\left(\frac{t}{1+t^2} + \text{arctg}\, t\right) + k \quad (**)$$

replacing (**) in (II), we get:

$$\int \frac{dx}{(x^2+x+1)^2} = \frac{2}{3}\cdot\frac{1}{3}\left(\frac{t}{1+t^2}+\operatorname{arctg} t\right)+k \quad \text{(III)}$$

Now, replacing (I) and (III) in (*), we obtain the value of I, i.e.

$$\int \frac{x+1}{(x^2+x+1)^2}dx = \frac{-1}{2(x^2+x+1)} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{3}\left[\frac{t}{1+t^2}+\operatorname{arctg} t\right]+\mathcal{C}$$

$$= -\frac{1}{2(x^2+x+1)} + \frac{1}{9}\left(\frac{t}{t^2+1}+\operatorname{arctg} t\right)+\mathcal{C} \;/\; \mathcal{C} = c+k$$

$$\Rightarrow \boxed{\int \frac{x+1}{(x^2+x+1)^2}dx = \frac{-1}{2(x^2+x+1)} + \frac{1}{9}\left[\frac{\frac{2x+1}{\sqrt{3}}}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2} + \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)\right]+\mathcal{C}}$$

4) integrating the functions of the form $\int R(x, x^{\frac{m_1}{n_1}}, x^{\frac{m_2}{n_2}}, \dots, x^{\frac{m_i}{n_i}})$

where R is a fractional function written in terms of $x, x^{\frac{m_1}{n_1}}, \dots, x^{\frac{m_i}{n_i}}$ and $i \in \mathbb{N}^*$ [R is not a polynomial function].

In this case, to integrate the function R. we search the least common multiple of the numbers $n_1, n_2, \dots, n_i$ and let it be k, that is $ppcm(n_1, n_2, \dots, n_i) = k$,

then, we put $x = t^k$, in this case, the function R becomes a fractional function of the polynomials.

illustrative example: compute the following integral $I = \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}+2}dx$

note that: $ppcm(2,4) = 4$, hence $x = t^4 \Rightarrow dx = 4t^3dt$.

therefore: $I = \int \frac{t^2}{t^3+2}.4t^3dt = 4\int \frac{t^5}{t^3+2}dt$

$$= 4\int \left(t^2 - \frac{2t^2}{t^3+2}\right)dt$$

$$= 4\int t^2dt - 8\int \frac{t^2}{t^3+2}dt = 4\int t^2dt - \frac{8}{3}\int \frac{3t^2}{t^3+2}dt$$

$$= 4\frac{t^3}{3} - \frac{8}{3}\ln|t^3+2| + C$$

$$\Rightarrow \boxed{\int \frac{x^{\frac{1}{2}}}{x^{3/4}+2}dx = \frac{4}{3}\left[x^{3/4} - 2\ln\left(x^{\frac{3}{4}}+2\right)\right]+C, \; t = x^{\frac{1}{4}}}$$

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5/ integrating the functions of the form $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$.
where $ad - bc \neq 0$
in this case, we put $t = \sqrt[n]{\frac{ax+b}{cx+d}}$ to obtain a rational function in terms of t that is easy to integrate by following the following steps:

$$t = \sqrt[n]{\frac{ax+b}{cx+d}} \Rightarrow t^n = \frac{ax+b}{cx+d} \Rightarrow x = \frac{dt^n - b}{a - ct^n} \Rightarrow dx = \frac{n(ad-cb)t^{n-1}}{(a-ct^n)^2} dt$$

$$\Big[\text{since } dx = \frac{ndt^{n-1}(a-ct^n) + nct^{n-1}(dt^n - b)}{(a-ct^n)^2} dt$$

$$= \frac{nt^{n-1}(ad - dct^n + dct^n - bc)}{(a-ct^n)^2} dt \Big]$$

therefore, the integration becomes as follows:

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx = \int R\left(\frac{dt^n - b}{a - ct^n}, t\right) \frac{n(ad-cb)t^{n-1}}{(a-ct^n)^2} dt$$

$$= \int f(t)\, dt.$$

where f is a rational function in terms of t. that easy to integrate:

Illustrative example: compute $\int \frac{dx}{1+\sqrt{1+x}}$

note that $a = b = d = n = 1$ and $c = 0$
we put: $t = \sqrt{1+x} \Rightarrow x = t^2 - 1 \Rightarrow dx = 2t\,dt$

$$\int \frac{dx}{1+\sqrt{1+x}} = \int \frac{2t\,dt}{1+t} = 2\int \frac{t}{1+t} dt = 2\left[\int \left(1 - \frac{1}{1+t}\right) dt\right]$$

$$= 2t - 2\ln|1+t| + C.$$

subtituting the value of t, we find:

$$\boxed{\int \frac{dx}{1+\sqrt{1+x}} = 2\sqrt{1+x} - 2\ln\left[1+\sqrt{1+x}\right] + C}$$

6/ integrating the functions of the form: $\int R(x, \sqrt{ax^2+bx+c})\, dx$
where $a, b, c \in \mathbb{R}$ and $a \neq 0$
• If $a = 0$, we get a special case of integration of functions of

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the previous form (form (5) i.e., $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$)

- If $a \neq 0$, the integral $\int R(x, \sqrt{ax^2+bx+c})\, dx$ can be transformes into an integral of fractional function by using a change in the Euler variables

a/ First Euler substitution:

- If $\Delta = b^2 - 4ac > 0$, then the equation $ax^2+bx+c=0$ admits two different real roots, denoted by x_1 and x_2.

in this case, we put:

$$\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)(x-x_2)} = (x-x_1)t$$

hence: $a(x-x_1)(x-x_2) = (x-x_1)^2 t^2 \Leftrightarrow a(x-x_2) = (x-x_1)t^2$

$$\Rightarrow x(a-t^2) = ax_2 - x_1 t^2 \Rightarrow x = \frac{ax_2 - x_1 t^2}{a-t^2} = \boxed{\frac{x_1 t^2 - a x_2}{t^2-a}}$$

therefore: $x = \dfrac{x_1 t^2 - a x_2}{t^2 - a} \Rightarrow dx = \dfrac{2at(x_2 - x_1)}{(t^2-a)^2}$

[since : $dx = \dfrac{2x_1 t(t^2-a) - 2t(x_1 t^2 - a x_2)}{(t^2-a)^2}\, dt$]

in this case, the function R becomes a fractional function in terms of t that easy to integrate.

Illustrative example: compute the value of the following integral

$$I = \int \frac{dx}{\sqrt{x^2-x-2}}$$

we have: $x^2 - x - 2 = (x+1)(x-2)$

then, we put $\sqrt{(x+1)(x-2)} = (x+1)t$, hence:

$$x - 2 = (x+1)t^2 \Rightarrow \boxed{x = \frac{2+t^2}{1-t^2}} \text{ — } (*)$$

$$\Rightarrow dx = \frac{6t}{(1-t^2)^2}\, dt \quad \left[\text{since: } dx = \frac{2t(1-t^2) + 2t(2+t^2)}{(1-t^2)^2} = \frac{6t}{(1-t^2)^2}\, dt \right]$$

Thus, integral I becomes:

$$I = \int \frac{dx}{\sqrt{x^2-2x-2}} = \int \frac{1}{(x+1)t} \cdot \frac{6t}{(1-t^2)^2}\, dt \overset{(*)}{=} \int \frac{1}{\left(\frac{2+t^2}{1-t^2}+1\right)\cancel{t}} \cdot \frac{6\cancel{t}}{(1-t^2)^2}\, dt$$

$$I = \int \frac{6\,dt}{\cancel{(1-t^2)^2}\cdot\left(\frac{2+\cancel{t^2}+1-\cancel{t^2}}{\cancel{1-t^2}}\right)} = \int \frac{6\,dt}{3(1-t^2)} = 2\int \frac{dt}{(1-t^2)} = 2\int \frac{dt}{(1-t)(1+t)}$$

$$\frac{2}{1-t^2} = \frac{a}{1-t} + \frac{b}{1+t} \quad \text{—— (I)}$$

- to obtain "a" we multiply both sides of (I) by $(1-t)$ and then we put $t=1$, we get : $\boxed{a=1}$
- to obtain "b" we multiply both sides of (I) by $(1+t)$ and then we put $t=-1$, we get : $\boxed{b=1}$

hence : $I = \int \frac{2\,dt}{1-t^2} = \int \frac{dt}{1-t} + \int \frac{dt}{1+t} = -\ln|1-t| + \ln|1+t| + C$

$$\boxed{I = \ln\left|\frac{1+t}{1-t}\right| + C}$$

but : $\sqrt{(x+1)(x-2)} = (x+1)t \Rightarrow t = \sqrt{\frac{x-2}{x+1}}$

hence I becomes :

$$I = \ln\left|\frac{1+\sqrt{\frac{x-2}{x+1}}}{1-\sqrt{\frac{x-2}{x+1}}}\right| + C$$

Remark : we can put $\sqrt{ax^2+bx+c} = (x-x_2)t \Rightarrow t = \sqrt{\frac{a(x-x_1)}{x-x_2}}$ and we obtain the same result. [since $\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)(x-x_2)}$]

2nd mtd : we can put in the previous example :

$$\sqrt{(x+1)(x-2)} = (x-2)t \Rightarrow t = \sqrt{\frac{x+1}{x-2}}$$

hence ; $x+1 = (x-2)t^2 \Rightarrow x = \frac{1+2t^2}{t^2-1}$

$$\Rightarrow dx = \frac{-6t}{(t^2-1)^2}\,dt$$

I becomes : $I = \int \frac{dx}{\sqrt{x^2-x-2}} = \int \frac{-6\cancel{t}}{(t^2-1)^2}\cdot\frac{1}{\left(\frac{1+2t^2}{t^2-1}-2\right)\cancel{t}}\,dt$

$$= \int \frac{-6\,dt}{\cancel{(t^2-1)^2}\left(\frac{1+2t^2-2t^2+2}{\cancel{t^2-1}}\right)} = \int \frac{-6\,dt}{3(t^2-1)}$$

$$= -2\int \frac{dt}{(t^2-1)} = \int \frac{2\,dt}{1-t^2}$$

which is the same as the previous integral.

b) Second Euler substitution:

If $\Delta < 0$ and $a > 0$ of the equation ($ax^2+bx+c=0$), then we put

$$\sqrt{ax^2+bx+c} = \pm\sqrt{a}\,x + t \quad \text{--- (1)}$$

Taking the sign + before $\sqrt{a}$, we have:

$$ax^2+bx+c = (\sqrt{a}\,x+t)^2 = ax^2+t^2+2\sqrt{a}\,xt$$

$$\Rightarrow bx+c = t^2+2\sqrt{a}\,xt \Rightarrow \boxed{x = \frac{t^2-c}{b-2\sqrt{a}\,t}} \quad \text{--- (2)}$$

replacing (2) in (1), we obtain

$$\sqrt{ax^2+bx+c} = \sqrt{a}\,x+t = \sqrt{a}\,\frac{t^2-c}{b-2\sqrt{a}\,t}+t$$

from (2), we have:

$$dx = \frac{2t(b-2\sqrt{a}\,t)+2\sqrt{a}\,t(t^2-c)}{(b-2\sqrt{a}\,t)^2}\,dt = \frac{2bt-2\sqrt{a}\,t^2-2\sqrt{a}\,c}{(b-2\sqrt{a}\,t)^2}\,dt$$

Hence, the function R becomes a fractional function in terms of t that easy to integrate it.

Example: compute $I = \int \frac{dx}{\sqrt{x^2+5}}$

we have: $\Delta < 0$ and $a = 1 > 0$, then we put

$$\sqrt{x^2+5} = x+t \; (*) \Rightarrow \boxed{t = \sqrt{x^2+5}-x} \quad \text{--- } (\alpha)$$

$$\Rightarrow x^2+5 = x^2+2xt+t^2 \Rightarrow \boxed{x = \frac{5-t^2}{2t}} \; (*') \Rightarrow \boxed{dx = \frac{-5-t^2}{2t^2}\,dt} \; (*'')$$

replacing $(*')$ in $(*)$, we get:

$$\sqrt{x^2+5} = x+t = \frac{5-t^2}{2t}+t = \boxed{\frac{5+t^2}{2t}} \; (3*)$$

Now, replacing $(*'')$ and $(3*)$ in I, we get:

$$I = \int \frac{\frac{-5-t^2}{2t^2}\,dt}{\frac{5+t^2}{2t}} = \int \frac{\cancel{2t}}{5+t^2}\cdot\frac{-5-t^2}{\cancel{2t^2}}\,dt = -\int \frac{5+t^2}{5+t^2}\,dt$$

$$= -\int \frac{dt}{t} = -\ln|t| + C \overset{(\alpha)}{=} -\ln\left|\sqrt{x^2+5}-x\right| + C$$

hence:

$$I = \int \frac{dx}{\sqrt{x^2+5}} = -\ln\left|\sqrt{x^2+5}-x\right| + C$$

c/ ~~third Euler substitutions~~

If $\Delta<0$ and $c>0$ of the equation $ax^2+bx+c=0$, then we put

$\sqrt{ax^2+bx+c} = xt \pm\sqrt{c} \quad (*) \Rightarrow \boxed{x = \frac{2\sqrt{c}\,t-b}{a-t^2}}$ (by taking the sign + before $\sqrt{c}$).

hence : $dx = \frac{2\sqrt{c}(a-t^2)+2t(2\sqrt{c}\,t-b)}{(a-t^2)^2}dt = \frac{2\sqrt{c}\,a-2bt}{(a-t^2)^2}dt$

$\Rightarrow \boxed{dx = \frac{2(a\sqrt{c}-bt)}{(a-t^2)^2}dt}$

from (*), we have: $\sqrt{ax^2+bx+c} = xt+\sqrt{c} \Rightarrow$

$\sqrt{ax^2+bx+c} = \frac{(2\sqrt{c}\,t-b)}{a-t^2}t+\sqrt{c}$

Therefore, the function R becomes a fractional function in terms of t that easy to integrate.

Example: compute the integral: $I = \int \frac{1-\sqrt{x^2+x+1}}{x^2\sqrt{x^2+x+1}}dx.$

Here, we have: $\Delta<0$ and $c>0$, then we put:

$\underbrace{\sqrt{x^2+x+1} = xt+1}_{2} \Leftrightarrow x^2+x+1 = x^2t^2+2xt+1$

$\Leftrightarrow x^2(1-t^2)+x(1-2t) = 0$

$\Leftrightarrow x\left((1-t^2)x+(1-2t)\right) = 0$

$\Rightarrow \begin{cases} x = 0 & (\text{rejected}, \Delta<0, x\notin\mathbb{R}, x\in\mathbb{C}) \\ \vee \\ x = \frac{2t-1}{1-t^2} & (\text{accepted}) \end{cases}$

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$\Rightarrow \boxed{x = \frac{2t-1}{1-t^2}} \Rightarrow dx = \frac{2t^2-2t+2}{(1-t^2)^2}dt$

hence the integral becomes:

$I = \int \frac{1-\sqrt{x^2+x+1}}{x^2\sqrt{x^2+x+1}}dx \overset{1\,2}{=} \int \frac{\cancel{1}-\frac{2t-1}{1-t^2}t+\cancel{1}}{\left(\frac{2t-1}{1-t^2}\right)^2\left(\frac{2t-1}{1-t^2}t+1\right)} \cdot \frac{2t^2-2t+2}{(1-t^2)^2}dt$

$= \int \frac{(2t-1)t}{1-t^2} \cdot \frac{2t^2-2t+2}{(1-t^2)^2} \cdot \frac{1}{\left(\frac{2t-1}{1-t^2}\right)^2\left(\frac{2t^2-t+1-t^2}{1-t^2}\right)}dt$

$= \int \frac{\cancel{(2t-1)}t(2t^2-2t+2)\cancel{(1-t^2)^3}}{\cancel{(1-t^2)^3}(2t-1)^{\cancel{2}}(t^2-t+1)}dt = \int \frac{t(2t^2-2t+2)}{(2t-1)(t^2-t+1)}dt$

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$$\Rightarrow I = \int \frac{2t}{2t-1}\,dt = \int \frac{2t-1+1}{2t-1}\,dt = \int dt + \int \frac{dt}{2t-1}$$

$$\Rightarrow I = t + \frac{1}{2}\ln|2t-1| + C.$$

Hence : $I = \int \frac{1-\sqrt{x^2+x+1}}{x^2\sqrt{x^2+x+1}}\,dx = \boxed{\frac{\sqrt{x^2+x+1}-1}{x} + \ln\left|\frac{2\sqrt{x^2+x+1}-2-x}{x}\right| + C}$

[since : $\sqrt{x^2+x+1} = xt + 1 \Rightarrow t = \frac{\sqrt{x^2+x+1}-1}{x}$]

7/ Some other forms of integration:

a/ integration of the form $\int \cos^n x \sin^m x\,dx$, with $n, m \in \mathbb{N}^*$

a-1/ If n is odd and m is even:

in this case, we put $t = \sin x \Rightarrow dt = \cos x\,dx$.

Example: $I = \int \sin^2 x \cos^3 x\,dx$.

we put $t = \sin x \Rightarrow dt = \cos x\,dx$, hence I becomes

$$I = \int \underbrace{\sin^2 x}_{t^2}\,\underbrace{\cos^2 x}_{1-t^2}\,\underbrace{\cos x\,dx}_{dt} = \int t^2(1-t^2)\,dt = \int (t^2 - t^4)\,dt$$

$$\Rightarrow I = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

a-2/ If n is even and m is odd:

In this case, we put: $t = \cos x \Rightarrow dt = -\sin x\,dx$.

Example: compute : $I = \int \cos^2 x \sin^3 x\,dx$.

we put : $t = \cos x \Rightarrow dt = -\sin x\,dx$, then I becomes

$$I = \int \underbrace{\cos^2 x}_{t^2}\,\underbrace{\sin^2 x}_{1-t^2}\,\underbrace{\sin x\,dx}_{-dt} = -\int t^2(1-t^2)\,dt = -\int (t^2 - t^4)\,dt$$

$$= -\frac{t^3}{3} + \frac{t^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

a-3/ If n and m are odd together:

In this case, we can put $t = \sin x$ or $t = \cos x$ and follow the same previous steps.

Example: Compute $I = \int \cos^3 x \sin^3 x\,dx$

we put : $t = \sin x \Rightarrow dt = \cos x\, dx$, hence I becomes

$$I = \int \underbrace{\cos^2 x}_{1-t^2}\ \underbrace{\sin^3 x}_{t^3}\ \underbrace{\cos x\, dx}_{dt} = \int (1-t^2)\, t^3\, dt = \int (t^3 - t^5)\, dt$$

$$\Rightarrow I = \frac{t^4}{4} - \frac{t^6}{6} + C = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C.$$

a-4/ If n and m are even together :

In this case, we use the following trigonometric equalities: $\cos^2 x = \frac{1+\cos 2x}{2} \wedge \sin^2 x = \frac{1-\cos 2x}{2}$.

Example: compute : $I = \int \sin^2 x \cos^2 x\, dx$

using the above equations, we get:

$$I = \int \sin^2 x \cos^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \qquad (*)$$

$$I = \frac{1}{4}\int (1-\cos^2 2x)\, dx = \frac{1}{4}\int dx - \frac{1}{4}\underbrace{\int \cos^2 2x\, dx}_{I'} = \boxed{\frac{1}{4}x - \frac{1}{4}I'}$$

using again the relation: $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$, hence, we get

$$\int \cos^2 2x\, dx = \int \left(\frac{1+\cos 4x}{2}\right) dx = \frac{1}{2}\left[\int dx + \int \cos 4x\, dx\right.$$

$$\Rightarrow I' = \frac{1}{2}x + \frac{1}{8}\sin 4x + C$$

replacing I' in $(*)$, we get:

$$I = \frac{1}{4}x - \frac{1}{4}\left[\frac{1}{2}x + \frac{1}{8}\sin 4x\right] + C$$

$$\Rightarrow \boxed{I = \frac{1}{8}x - \frac{1}{32}\sin 4x + C}$$

b/ integration of the form: $\int \cos \alpha x \sin \beta x\, dx$, $\alpha, \beta \in \mathbb{R}^*$

in this case, we use the following relationship

$$\cos \alpha x \sin \beta x = \frac{1}{2}\left[\sin(\alpha+\beta)x - \sin(\alpha-\beta)x\right] \quad \text{— (I)}$$

Example: compute the integral : $I = \int \cos 3x \sin 5x\, dx$.

from (I), we have:

$\cos 3x \sin 5x = \frac{1}{2}\left[\sin 8x - \sin(-2x)\right]$, hence I becomes

$$I = \frac{1}{2}\int (\sin 8x + \sin 2x)\, dx = \left(\frac{-1}{16}\right)\cos 8x + \left(\frac{-1}{4}\right)\cos 2x + C$$

$$\boxed{I = -\frac{1}{4}\left[\frac{1}{4}\cos 8x + \cos 2x\right] + C}$$

C/ integration of the form $\int f(\cos x, \sin x)\,dx$.
where f is a fractional function.
in this case, we put generally $t = tg\frac{x}{2}$

$$\Rightarrow dt = \frac{1}{2}\frac{1}{\cos^2\frac{x}{2}}\,dx = \frac{1}{2}\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\,dx = \frac{1}{2}\left(1+tg^2\frac{x}{2}\right)dx$$

$$\Rightarrow dt = \frac{1+t^2}{2}\,dx \Rightarrow \boxed{dx = \frac{2\,dt}{1+t^2}}$$

in the same way, we get:

$$\cos x = \cos\left(\frac{x}{2}+\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}$$

$$\Rightarrow \cos x = \frac{\cos^2\frac{x}{2}\left(1-tg^2\frac{x}{2}\right)}{\cos^2\frac{x}{2}\left(1+tg^2\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2} \Rightarrow \boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

and $\sin x = \sin\left(\frac{x}{2}+\frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,tg\frac{x}{2}\cdot\cos^2\frac{x}{2}$

$$\Rightarrow \sin x = \frac{2\,tg\frac{x}{2}}{\frac{1}{\cos^2\frac{x}{2}}} = \frac{2\,tg\frac{x}{2}}{\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\Rightarrow \boxed{\sin x = \frac{2t}{1+t^2}}$$

Example: compute the following integrals: $I_1 = \int\frac{dx}{\sin x}$ $\wedge$ $I_2 = \int\frac{dx}{\cos x}$
we put $t = tg\frac{x}{2} \Rightarrow dx = \frac{2\,dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

Hence: $I_1 = \int\frac{\cancel{2}\,dt}{\cancel{1+t^2}}\cdot\frac{\cancel{1+t^2}}{\cancel{2}t} = \int\frac{dt}{t} = \ln|t| + C = \boxed{\ln\left|tg\frac{x}{2}\right| + C}$

$I_2 = \int\frac{2\,dt}{\cancel{1+t^2}}\cdot\frac{\cancel{1+t^2}}{1-t^2} = \int\frac{2\,dt}{1-t^2} = \int\frac{dt}{1-t} + \int\frac{dt}{1+t} = \ln\left|\frac{1+t}{1-t}\right| + C$

Hence: $\boxed{I_2 = \ln\left|\frac{1+tg\frac{x}{2}}{1-tg\frac{x}{2}}\right| + C}$