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Functions of a complex variable

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Introduction

Complex-valued functions of a complex variable are harder "to visualise" than real-valued functions of a real variable. This course focuses on the study of functions of a complex variable, as well as the fundamental theorems associated with this type of functions. It is prepared for second-year undergraduate students in Physics. The course contains five chapters.

In chapter one, we start by giving the fundamental structure of complex numbers and their basic properties, then we introduce the notion of complex-valued functions of a complex variable. The third section of the chapter is devoted to holomorphic and harmonic functions. We end the chapter by giving some exercises.

Chapter two is devoted to some elementary complex-valued functions: Complex exponential function, complex trigonometric functions, complex hyperbolic functions, complex logarithmic function, complex power function, inverse trigonometric and hyperbolic functions. We end the chapter by giving some exercises.

Chapter three deals with fundamental theorems about complex-valued functions. We start by introducing the notion of complex integration. After that, we present the Cauchy integral formula, Taylor and Laurent series and zeros and singularities of holomorphic functions. We end the chapter by giving some exercises.

In chapter four, we start by introducing the notion of residue of a complex-valued function at a singular point z_0, then we present Cauchy's residue theorem and show how we can apply this important theorem to compute some real integrals. We end the chapter by giving some exercises.

In chapter five, we present Liouville's theorem, the maximum modulus principle, Rouché's theorem and we end the chapter by giving some examples about calculus of integrals by using the residue theorem.

Notations

The following notations will be used in this course.

$\mathbb{R}$ the set of real numbers.

$\mathbb{R}_+$ the set of positive real numbers.

$\mathbb{R}_+^*$ the set of nonzero real numbers.

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

$\mathbb{N}$ the set of naturel numbers.

$\mathbb{Z}$ the set of integers.

i the imaginary unit, $i^2 = -1$.

$\mathbb{C}$ the set of complex numbers.

$\mathbb{C}^\star$ the set of nonzero complex numbers.

$\Re(z)$ the real part of a complex number z.

$\Im(z)$ the imaginary part of a complex number z.

$[a, b] = \{x \in \mathbb{R},\ a \leq x \leq b\},\ a, b \in \mathbb{R},\ a < b$.

$]a, b[= \{x \in \mathbb{R},\ a < x < b\},\ a, b \in \mathbb{R},\ a < b$.

$I(\gamma)$ the interior of a closed curve γ.

$\lim\limits_{t \overset{<}{\to} t_0}$ the left limit.

$\lim\limits_{t \overset{>}{\to} t_0}$ the right limit.

$L(\gamma)$ the length of γ.

$\{z_n\}_{n=0}^{+\infty}$ a complex sequence.

$\sum\limits_{n=0}^{+\infty} z_n$ a series of complex numbers.

Chapter 1

Holomorphic functions

In this chapter, we start by giving the fundamental structure of complex numbers and their basic properties, then we introduce the notion of complex-valued functions of a complex variable. The third section of this chapter is devoted to holomorphic and harmonic functions. We end the chapter by giving some exercises.

1.1 Complex plane

Definition 1.1.1 *A complex number is a number z of the form $z = x + iy$ where $x, y \in \mathbb{R}$. If $z = x + iy$, $x, y \in \mathbb{R}$, then $x = \Re(z)$ is the real part of z and $y = \Im(z)$ is the imaginary part of z.*

Remark 1.1.1 **1)** $\mathbb{R} \subset \mathbb{C}$.

2) *Two complex numbers are equal when their real and imaginary parts are equal, that is $z_1 = x_1 + iy_1 = z_2 = x_2 + iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.*

3) *Every complex number $z = x + iy$ can be considered as a point $M(x, y)$ on the cartesian plane. See Figure 1.1*

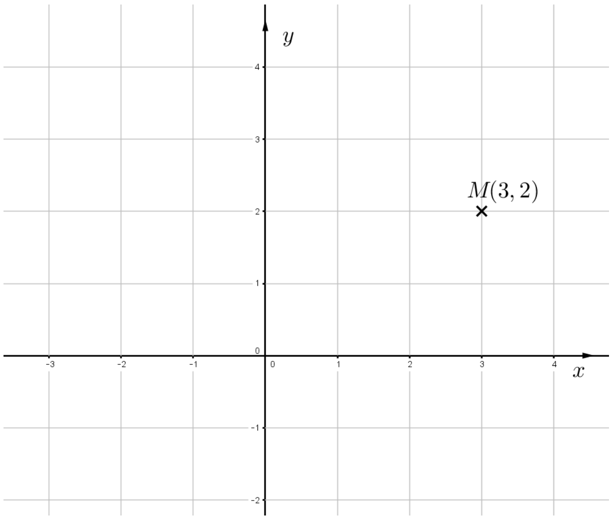


Figure 1.1: The complex number $z = 3 + 2i$ is considered as the point $M(3, 2)$.

Definition 1.1.2 *Let z_1 and z_2 be two complex numbers such that $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2$.*

1) *The addition of z_1 and z_2 is the complex number z given by*

$$z = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

2) *The subtraction of z_1 and z_2 is the complex number z given by*

$$z = z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

3) *The multiplication of z_1 and z_2 is the complex number z given by*

$$z = z_1.z_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2).$$

4) *The division of z_1 by z_2 where $z_2 \neq 0$ is the complex number z given by*

$$z = \frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}.$$

Proposition 1.1.1 *Let $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2, 3$ be complex numbers. Then*

a) *$z_1 + z_2 = z_2 + z_1$ and $z_1.z_2 = z_2.z_1$, that is the addition and the multiplication are commutative.*

b) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ *and* $(z_1.z_2).z_3 = z_1.(z_2.z_3)$, *that is the addition and the multiplication are associative.*

c) $z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3$, *distributivity of multiplication over addition.*

Definition 1.1.3 *Let* $z = x + iy$, $x, y \in \mathbb{R}$ *be a complex number. The conjugate of* z *is the complex number given by* $\overline{z} = x - iy$.

Proposition 1.1.2 *Let* z, z_1 *and* z_2 *be complex numbers. Then*

1) $z = \overline{z}$ *if and only if* $z \in \mathbb{R}$.

2) $z = -\overline{z}$ *if and only if* $z = iy$, $y \in \mathbb{R}$.

3) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ *and* $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

4) $\overline{z_1.z_2} = \overline{z_1}.\overline{z_2}$ *and* $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ *if* $z_2 \neq 0$.

5) $\overline{\overline{z}} = z$.

Proposition 1.1.3 *Let* $z = x + iy$, $x, y \in \mathbb{R}$ *be a complex number and let* $\overline{z}$ *be its conjugate. Then* $z + \overline{z} = 2\Re(z)$ *and* $z - \overline{z} = 2i\Im(z)$. *That is*

$$\Re(z) = \frac{z + \overline{z}}{2} \text{ and } \Im(z) = \frac{z - \overline{z}}{2i}.$$

Remark 1.1.2 *A complex number* $z = x + iy$, $x, y \in \mathbb{R}$ *can be considered as a vector from the origin* $O(0, 0)$ *to the point* $M(x, y)$. *See Figure 1.2*

Definition 1.1.4 *Let* $z = x + iy$, $x, y \in \mathbb{R}$ *be a complex number. The modulus of* z *is the real number* $|z| = \sqrt{x^2 + y^2}$.

Proposition 1.1.4 *Let* z, z_1, z_2 *be complex numbers and let* $\overline{z}$ *be the conjugate of* z. *Then*

1) $|z| \geq 0$ *and* $|z| = 0$ *if and only if* $z = 0$.

2) $|z| = \sqrt{z.\overline{z}}$.

3) $|z| = |\overline{z}|$.

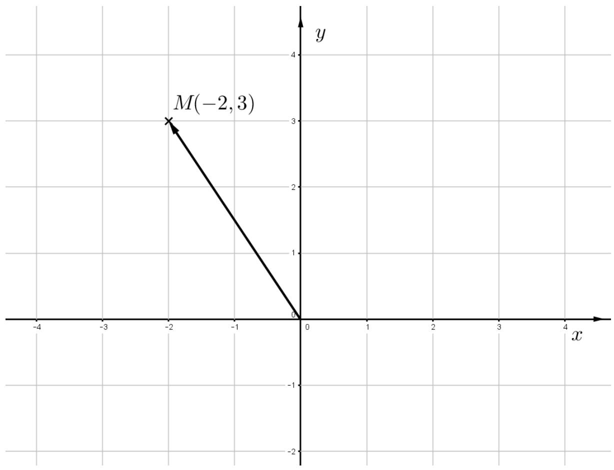


Figure 1.2: The complex number $z = -2 + 3i$ is considered as the vector from the origin $O(0, 0)$ to the point $M(-2, 3)$.

4) $|z_1.z_2| = |z_1||z_2|$ *and* $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$.

5) $|z_1 + z_2| \leq |z_1| + |z_2|$, *(the triangle inequality).*

6) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

Definition 1.1.5 *Let $z = x + iy \in \mathbb{C}^\star$. The number z can be expressed in trigonometric (or polar) form as*

$$z = r(\cos\theta + \sin\theta),$$

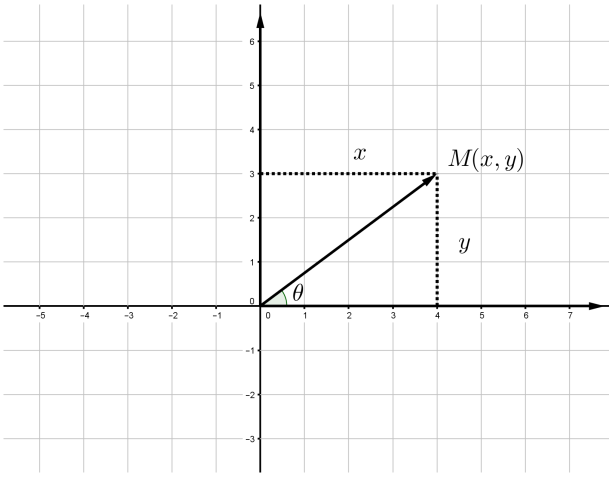
where $r = |z| = \sqrt{x^2 + y^2}$ and θ is a real number such that

$$\cos\theta = \frac{x}{r},\ \sin\theta = \frac{y}{r},\ \text{that is } \theta = \arctan\left(\frac{y}{x}\right).$$

- *The number θ is called an argument of z and we write $\theta = \arg(z)$.*
- *We denote by $Arg(z)$ the principal value of $\arg(z)$ and it is defined as the unique value of $\arg(z)$ such that $-\pi < \arg(z) \leq \pi$.*

Remark 1.1.3 **a)** *$\arg(z)$ is the angle measured in radians that the vector corresponds to z makes with the positive real axis.*

b) *The argument of $z = 0$ is not defined.*



Proposition 1.1.5 *Let z_1 and z_2 be two nonzero complex numbers. Then*

1) $\arg(z_1.z_2) = \arg(z_1) + \arg(z_2)$, *(as a set equality).*

2) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$, *(as a set equality).*

Definition 1.1.6 *Let $z = x + iy \in \mathbb{C}$. The exponential of z is defined by*

$$e^z = e^x(\cos y + i \sin y).$$

Proposition 1.1.6 *Let $z_1, z_2 \in \mathbb{C}$. Then*

1) $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

2) $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$.

Remark 1.1.4 **1)** *If $z = iy,\ y \in \mathbb{R}$, then we have the formulas of Euler*

$$e^{iy} = \cos y + i \sin y.$$

2) *For every $y \in \mathbb{R}$, we have*

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and } \sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$

Definition 1.1.7 *Let $z = x + iy \in \mathbb{C}^\star$. The number z can be expressed in exponential form as $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$.*

Proposition 1.1.7 *Let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ be two nonzero complex numbers. Then*

(i) $z_1.z_2 = r_1 e^{i\theta_1}.r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

(ii) $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$.

(iii) *The conjugate of* $z = re^{i\theta} \in \mathbb{C}^\star$ *is given by* $\overline{z} = \overline{(re^{i\theta})} = re^{-i\theta}$.

Proposition 1.1.8 (De Moivre's formula) *Let* $\theta \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *then*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Corollary 1.1.1 *Let* $z = re^{i\theta}$ *be a nonzero complex number where* $r = |z|$ *and* $\theta = Arg(z)$. *Then the n distinct roots of z are given by*

$$z_k = (r)^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}} = (r)^{\frac{1}{n}} \left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right),\ k = 0, 1, \ldots, n-1,\ n \in \mathbb{N}^\star.$$

Definition 1.1.8 *Let* $z_0 \in \mathbb{C}$ *and* $r \in \mathbb{R}_+^\star$.

1) *The set* $B(z_0, r) = \{z \in \mathbb{C},\ |z - z_0| < r\}$ *is called the open disk centered at* z_0 *with radius r.*

2) *The set* $\overline{B}(z_0, r) = \{z \in \mathbb{C},\ |z - z_0| \leq r\}$ *is called the closed disk centered at* z_0 *with radius r.*

3) *The set* $C(z_0, r) = \{z \in \mathbb{C},\ |z - z_0| = r\}$ *is called the circle centered at* z_0 *with radius r.*

4) *The set* $B(0, 1) = \{z \in \mathbb{C},\ |z| < 1\}$ *is called the open unit disk.*

5) *The set* $C(0, 1) = \{z \in \mathbb{C},\ |z| = 1\}$ *is called the unit circle.*

Remark 1.1.5 $B(z_0, r)$ *is also called the* $r-$*neighborhood of* z_0 *or a neighborhood of* z_0.

Definition 1.1.9 *Let A be a subset of* $\mathbb{C}$ *and let* $z_0 \in A$. *We say that* z_0 *is an interior point of A if there is a neighborhood of* z_0 *that is completely contained in A.*

Example 1.1.1 **1)** *Let A be the right half-plane, that is* $A = \{z \in \mathbb{C},\ \Re(z) > 0\}$, *then* $z_0 = \frac{1}{2}$ *is an interior point of A.*

2) *Every point z in the open disk* $B(z_0, r)$ *is an interior point of* $B(z_0, r)$.

3) *Let* $A = \overline{B}(0, 1)$, *then every complex number z such that* $|z| = 1$ *is not an interior point of A and every complex number z such that* $|z| < 1$ *is an interior point of A.*

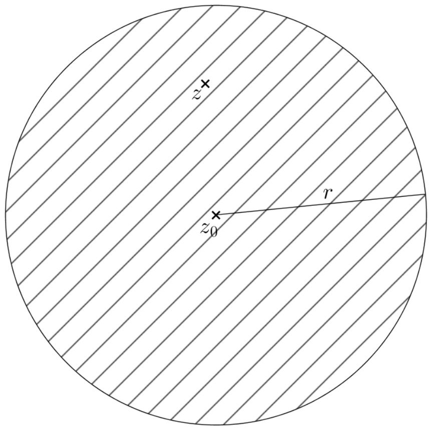


Figure 1.3: The closed disk $\overline{B}(z_0, r)$

Definition 1.1.10 *Let $A \subset \mathbb{C}$ be a subset. We say that A is an open set if every point of A is an interior point of A.*

Example 1.1.2 *$\mathbb{C}$ and $\emptyset$ are open.*

Definition 1.1.11 *Let $A \subset \mathbb{C}$ be a subset and $z_0 \in \mathbb{C}$.*

- *We say that z_0 is an exterior point of A if there is some neighborhood of z_0 that does not contain any points of A.*
- *We say that z_0 is a boundary point of A if every neighborhood of z_0 contains at least one point of A and at least one point not in A.*
- *We denote by ∂A the set of all boundary points of A and it is called the boundary (or frontier) of A.*

Definition 1.1.12 *Let $A \subset \mathbb{C}$ be a subset. We say that A is closed if it contains all of its boundary points, that is $\partial A \subseteq A$.*
The set $A \cup \partial A = \overline{A}$ is called the closure of A.

Example 1.1.3 **1)** *$\mathbb{C}$ and $\emptyset$ are closed.*

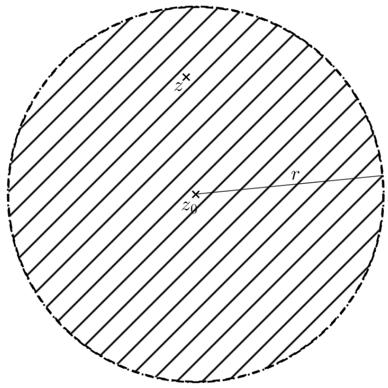


Figure 1.4: The open disk $B(z_0, r)$

2) *The closed disk* $\overline{B}(z_0, r)$ *is closed and it is the closure of the open disk* $B(z_0, r)$.

3) *The circular annulus* $A = \{z \in \mathbb{C},\ r_1 < |z| \leq r_2\}$ *where* $r_1, r_2 \in \mathbb{R}_+^\star$ *is neither open nor closed and its boundary is given by* $\partial A = \{z \in \mathbb{C},\ |z| = r_2\} \cup \{z \in \mathbb{C},\ |z| = r_1\}$.

Remark 1.1.6 *Let* A *be a subset of* $\mathbb{C}$. *Then,* A *is open if and only if its complement* $A^c_{\mathbb{C}} = \{z,\ z \in \mathbb{C}$ *and* $z \notin A\}$ *is closed.*

Definition 1.1.13 *Let* $A \subset \mathbb{C}$ *be a subset and* $z_0 \in \mathbb{C}$. *We say that* z_0 *is an accumulation point (or limit point) of* A *if every neighborhood of* z_0 *contains infinitely many points of* A.

Remark 1.1.7 *Let* A *be a subset of* $\mathbb{C}$. *Then,* A *is closed if it contains all of its accumulation points.*

Definition 1.1.14 *Let* A *be a subset of* $\mathbb{C}$. *We say that* A *is bounded if there exists a positive real number* M *such that* $|z| \leq M,\ \forall z \in A$.
If A *is not bounded, we say that* A *is unbounded.*

Definition 1.1.15 *Let* A *be a subset of* $\mathbb{C}$. *We define the diameter of* A *by*

$$diam\, A = \sup_{z,w \in A} |z - w|\,.$$

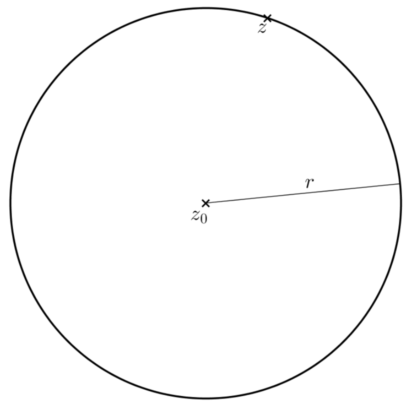


Figure 1.5: The circle $C(z_0, r)$

Remark 1.1.8 *It is clear that a subset A is bounded if and only if diam $A < +\infty$.*

Theorem 1.1.1 (Bolzano-Weierstrass) *Let A be an infinite bounded set of complex numbers, then A has at least one accumulation point.*

Definition 1.1.16 *Let A be a subset of $\mathbb{C}$. we say that A is compact if it is closed and bounded.*

Example 1.1.4 *The closed disk $\overline{B}(z_0, r)$ is compact but the open disk $B(z_0, r)$ is not compact.*

Definition 1.1.17 1) *Let $z_1, z_2 \in \mathbb{C}$. The set $[z_1, z_2] = \{z \in \mathbb{C},\ z = (1-t)z_1 + tz_2,\ t \in [0, 1]\}$ is called the line segment joining z_1 and z_2. See Figure 1.6*

2) *Let $z_1, z_2, \ldots, z_{n+1} \in \mathbb{C}$ and let l_k be the line segment joining z_k and z_{k+1}, $k \in \{1, 2, \ldots, n\}$. The successive line segments $l_1, l_2, \ldots, l_n$ from a continuous chain is called a polygonal path joining z_1 to z_{n+1}. See Figure 1.7*

Definition 1.1.18 *1) Let A be an open subset of $\mathbb{C}$. we say that A is connected if every pair of points $z_1, z_2 \in A$ can be joined by a polygonal path that lies entirely in A.*
2) A domain is an open connected set.

Example 1.1.5 *See Figures 1.8, 1.9 and 1.10.*

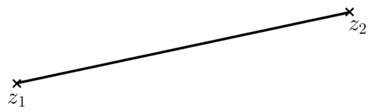


Figure 1.6: The line segment $[z_1, z_2]$

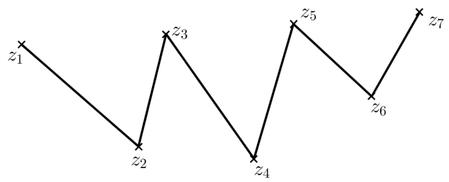


Figure 1.7: A polygonal path joining z_1 to z_7

Example 1.1.6 *The open disk $B(z_0, r)$ is a domain.*

Definition 1.1.19 *A region is a domain together with some, none, or all of its boundary points.*

Definition 1.1.20 *Let A be a subset of $\mathbb{C}$. We say that A is convex if each pair of points $z_1, z_2 \in A$ can be joined by a line segment $[z_1, z_2]$ such that every point in $[z_1, z_2]$ lies in A.*

Example 1.1.7 *Open disks and closed disks are convex.*

Remark 1.1.9 **1)** *If A and B are two convex sets, then $A \cap B$ is also convex.*

2) *If A is a convex set, then it is connected.*

1.2 Complex-valued functions

Definition 1.2.1 *Let E be a subset of $\mathbb{C}$. A complex-valued (or complex-valued of a complex variable) function f from E to $\mathbb{C}$ is a rule that assigns to each $z \in E$ a complex number $w \in \mathbb{C}$*

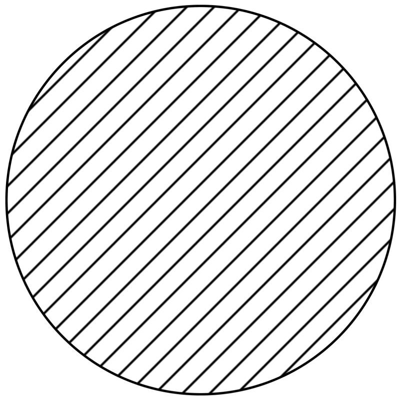


Figure 1.8: Connected

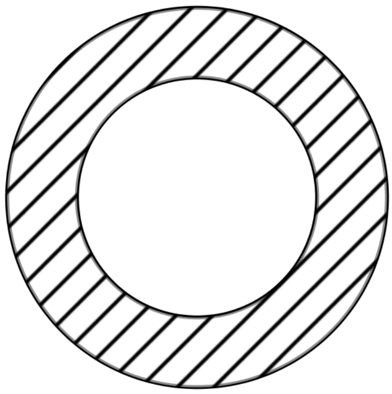


Figure 1.9: Connected

and this w is called the value of f at z.

The set E is called the domain of f and the set $\{w \in \mathbb{C}, \exists z \in E, w = f(z)\}$ is called the rang of f.

Definition 1.2.2 *Let E be a subset of $\mathbb{C}$ and let $f : E \longrightarrow \mathbb{C}$ be a complex-valued function.*

1) *We say that f is single-valued if for each element $z \in E$, f assigns one and only one value $w = f(z)$.*

2) *We say that f is multi-valued if for each element $z \in E$, f assigns a finite or infinite non-empty subset of $\mathbb{C}$.*

3) *We say that f is periodic in E if there exists a constant $T \in \mathbb{C}$ such that $f(z+T) = f(z)$, $\forall z \in E$ and in this case, T is a period of f.*

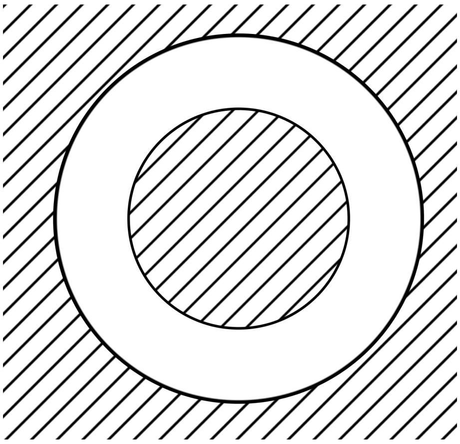


Figure 1.10: Not connected

Example 1.2.1 **a)** *The function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by* $f(z) = iz$, $\forall z \in \mathbb{C}$ *is single-valued.*

b) *The function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by* $f(z) = \arg(z)$, $\forall z \in \mathbb{C}$ *is multi-valued.*

Definition 1.2.3 *Let* $f : E \longrightarrow \mathbb{C}$ *be a multi-valued function. A branch of* f *is any single-valued function* $h : D \subseteq E \longrightarrow \mathbb{C}$*, that is continuous on* D *and at each point* $z \in D$*, it assigns one of the values of* $f(z)$.

Definition 1.2.4 *Let* $E \subset \mathbb{C}$, $f : E \longrightarrow \mathbb{C}$ *be a complex-valued function and* $z = x + iy \in E$*, then the value of* f *at* z *can be written as* $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$ *and* u *is called the real part of* f *and* v *is called the imaginary part of* f.

Example 1.2.2 **(i)** *The domain of the function* f *defined by* $w = f(z) = z^2 + 5z$ *is* $\mathbb{C}$ *and for all* $z = x+iy$, $x, y \in \mathbb{R}$ *we have* $w = f(x+iy) = x^2-y^2+5x+i(2xy+5y)$*, then the real part of* f *is the function* $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $u(x, y) = x^2-y^2+5x$, $\forall (x, y) \in \mathbb{R}^2$*. The imaginary part of* f *is the function* $v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $v(x, y) = 2xy + 5y$, $\forall (x, y) \in \mathbb{R}^2$.

(ii) *The domain of the function* f *defined by* $w = f(z) = |z|$ *is* $\mathbb{C}$ *and it is clear that* f *is a real-valued function, then the real part of* f *is the function* $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $u(x, y) = \sqrt{x^2 + y^2}$, $\forall (x, y) \in \mathbb{R}^2$ *and the imaginary part of* f *is* $v \equiv 0$.

Remark 1.2.1 *From now on, We use the term "complex-valued function" to refer to single-valued complex function.*

Definition 1.2.5 *Let $z_0 \in \mathbb{C}$ and let f be a complex-valued function defined in some neighborhood of z_0. We say that f has a limit w_0 as z approaches z_0 if we have*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ |z - z_0| \leq \delta \Longrightarrow \left|f(z) - w_0\right| < \varepsilon.$$

In this case we write $\lim\limits_{z \to z_0} f(z) = w_0$.

Example 1.2.3 **1)** *Let f be the function defined by $f(z) = 8z + 1$, $\forall z \in \mathbb{C}$. We will prove that $\lim\limits_{z \to i-1} f(z) = -7 + 8i$. Let $z_0 = -1 + i$ and $w_0 = -7 + 8i$. We have*

$$\begin{aligned} \left|f(z) - w_0\right| &= |8z + 1 - (-7 + 8i)| \\ &= 8\,|z - (-1 + i)|, \end{aligned}$$

hence,

$$\forall \varepsilon > 0,\ \exists \delta = \frac{\varepsilon}{8} > 0,\ |z - z_0| \leq \delta \Longrightarrow \left|f(z) - w_0\right| < \varepsilon.$$

2) *Let f be the function defined by $f(z) = \frac{z^2+4}{z-2i}$, $\forall z \in \mathbb{C} \setminus \{2i\}$. We will prove that $\lim\limits_{z \to 2i} f(z) = 4i$. Let $z_0 = 2i$ and $w_0 = 4i$. We have*

$$\begin{aligned} \left|f(z) - w_0\right| &= \left|\frac{z^2 + 4}{z - 2i} - 4i\right| \\ &= |z - 2i|, \end{aligned}$$

hence,

$$\forall \varepsilon > 0,\ \exists \delta = \varepsilon > 0,\ |z - z_0| \leq \delta \Longrightarrow \left|f(z) - w_0\right| < \varepsilon.$$

Theorem 1.2.1 *Let $f : E \longrightarrow \mathbb{C}$ be a function defined by*

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),\ \forall z = x + iy \in E$$

and let $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$, $x_0, y_0, u_0, v_0 \in \mathbb{R}$. Then

$$\lim_{z \to z_0} f(z) = w_0 \Longleftrightarrow \begin{cases} \lim\limits_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \\ \text{and} \\ \lim\limits_{(x,y) \to (x_0,y_0)} v(x, y) = v_0. \end{cases}$$

Proposition 1.2.1 *Let $E \subset \mathbb{C}$, $z_0 \in E$ and let $f, g : E \longrightarrow \mathbb{C}$ be two complex-valued functions. Suppose that each function has a limit as z approaches z_0. Then*

1) $\lim\limits_{z \to z_0}(f(z) + g(z)) = \lim\limits_{z \to z_0} f(z) + \lim\limits_{z \to z_0} g(z)$.

2) $\lim\limits_{z \to z_0}(f(z)g(z)) = (\lim\limits_{z \to z_0} f(z))(\lim\limits_{z \to z_0} g(z))$.

2) *If* $\lim\limits_{z \to z_0} g(z) \neq 0$, *then* $\lim\limits_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim\limits_{z \to z_0} f(z)}{\lim\limits_{z \to z_0} g(z)}$.

Remark 1.2.2 *We use the notation* $z \to \infty$ *if we have* $|z| \to +\infty$.

Definition 1.2.6 *Let* $z_0, w_0 \in \mathbb{C}$ *and let* f *be a complex-valued function defined in some neighborhood of* z_0.

1) *We say that* $\lim\limits_{z \to z_0} f(z) = \infty$ *if we have*

$$\forall \alpha > 0,\ \exists \delta > 0,\ |z - z_0| \leq \delta \Longrightarrow |f(z)| > \alpha.$$

2) *We say that* $\lim\limits_{z \to \infty} f(z) = w_0$ *if we have*

$$\forall \varepsilon > 0,\ \exists \beta > 0,\ |z| > \beta \Longrightarrow |f(z) - w_0| < \varepsilon.$$

3) *We say that* $\lim\limits_{z \to \infty} f(z) = \infty$ *if we have*

$$\forall \alpha > 0,\ \exists \beta > 0,\ |z| > \beta \Longrightarrow |f(z)| > \alpha.$$

Example 1.2.4 **1)** *Let* $f(z) = \frac{z+1}{5z-2}$, *then* $\lim\limits_{z \to \infty} f(z) = \frac{1}{5}$.

2) *Let* $f(z) = \frac{z-i}{2z^2+1}$, *then* $\lim\limits_{z \to \infty} f(z) = 0$.

Definition 1.2.7 *Let* $E \subset \mathbb{C}$, $z_0 \in E$ *and* $f : E \longrightarrow \mathbb{C}$ *be a complex-valued function.*

- *We say that* f *is continuous at* z_0 *if* $\lim\limits_{z \to \infty} f(z) = f(z_0)$.
- *We say that* f *is continuous on* E *if it is continuous at each point of* E.

Example 1.2.5 *The following complex-valued functions are continuous on* $\mathbb{C}$.

♣ $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \Re(z)$.

♣ $g : \mathbb{C} \longrightarrow \mathbb{C}$, $g(z) = z^2 + z - 3$.

Remark 1.2.3 *Let $E \subset \mathbb{C}$, $z_0 = x_0 + iy_0 \in E$ and let $f : E \longrightarrow C$ be a complex-valued function such that $f(z) = u(x, y) + iv(x, y)$, $\forall z \in E$. Then f is continuous at z_0 if and only if u and v are continuous at (x_0, y_0).*

Proposition 1.2.2 *Let $E \subset \mathbb{C}$, $z_0 \in E$ and let $f, g : E \longrightarrow C$ be two complex-valued functions. If f and g are continuous at z_0, then $f \pm g$, fg and $\frac{f}{g}$, $g(z_0) \neq 0$ are continuous at z_0.*

Remark 1.2.4 **1)** *Polynomial functions defined by $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $\forall z \in \mathbb{C}$, $a_j \in \mathbb{R}$, $j \in \{0, 1, \ldots, n\}$ are continuous on the whole plane $\mathbb{C}$.*

2) *Rational functions defined by $R(z) = \frac{P(z)}{Q(z)}$ where P and Q are two polynomial functions, are continuous at each point z such that $Q(z) \neq 0$.*

Example 1.2.6 *Let us consider the functions f, g, h defined respectively by $f(z) = z^3 - 2z^2 + 6$, $g(z) = \frac{z+2i}{z}$ and $h(z) = \frac{z^2+4}{z(z-3i)}$. Then f is continuous on $\mathbb{C}$, g is continuous on $\mathbb{C}^\star$ and h is continuous on $\mathbb{C} \setminus \{0, 3i\}$.*

1.3 Holomorphic and harmonic functions

Definition 1.3.1 *Let $E \subset \mathbb{C}$, $z_0 \in E$ and let $f : E \longrightarrow \mathbb{C}$ be a complex-valued function. We say that f is differentiable at z_0 if the following limit exists and belongs to $\mathbb{C}$.*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{1.1}$$

In this case, we denote by $f'(z_0)$ (or $\frac{df}{dz}(z_0)$) the derivative of f at z_0 and

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If f is differentiable at every point $z \in E$, we say that f is differentiable on E.

Remark 1.3.1 *If we put $\Delta z = z - z_0$, then the limit given in* (1.1) *becomes*

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{1.2}$$

Example 1.3.1 **1)** *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be the function defined by* $f(z) = z^2$, $\forall z \in \mathbb{C}$ *and let* $z_0 \in \mathbb{C}$. *Then*

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \frac{z^2 - z_0^2}{z - z_0} \\ &= \frac{(z - z_0)(z + z_0)}{z - z_0} \\ &= z + z_0. \end{aligned}$$

Hence

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0,$$

then f *is differentiable at* z_0 *and* $f'(z_0) = 2z_0$.

Similarly, we have

$$\frac{d}{dz}(z^n) = nz^{n-1}, \ \forall n \in \mathbb{N}^\star, \ \forall z \in \mathbb{C}.$$

2) *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be the function defined by* $f(z) = \overline{z}$, $\forall z \in \mathbb{C}$ *and let* $z_0 \in \mathbb{C} \setminus \mathbb{R}$. *Then*

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{\overline{z_0 + \Delta z} - \overline{z_0}}{\Delta z} \\ &= \frac{\overline{z_0} + \overline{\Delta z} - \overline{z_0}}{\Delta z} \\ &= \frac{\overline{\Delta z}}{\Delta z}. \end{aligned}$$

Let $z = x + iy$, $x, y \in \mathbb{R}$, *then* $\Delta z = \Delta x + i\Delta y$ *and*

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \begin{cases} 1, & \text{if } \Delta y = 0 \\ -1, & \text{if } \Delta x = 0, \end{cases}$$

therefore, the limit given by (1.2) *does not exist, and* f *is not differentiable at any* $z_0 \in \mathbb{C} \setminus \mathbb{R}$.

Remark 1.3.2 *All rules known for real-valued functions are also satisfied for complex-valued functions.*

Proposition 1.3.1 *Let* $E \subset \mathbb{C}$, $z_0 \in E$, *c be a real number and let* $f, g : E \longrightarrow \mathbb{C}$ *be complex-valued functions. If* f *and* g *are differentiable at* z_0, *then*

1) $f \mp g$ *is differentiable at* z_0 *and* $(f \mp g)'(z_0) = f'(z_0) \mp g'(z_0)$.

2) *cf is differentiable at z_0 and $(cf)'(z_0) = cf'(z_0)$.*

3) *fg is differentiable at z_0 and $(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0)$.*

4) *$\frac{f}{g}$ is differentiable at z_0 and $\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0)-f(z_0)g'(z_0)}{(g(z_0))^2}$, if $g(z_0) \neq 0$.*

5) *If f is differentiable at $g(z_0)$, then $f \circ g$ is differentiable at z_0 and $(f \circ g)'(z_0) = f'(g(z_0))\, g'(z_0)$.*

Theorem 1.3.1 *Let $E \subset \mathbb{C}$, $z_0 \in E$ and $f : E \longrightarrow \mathbb{C}$ be a complex-valued function. If f is differentiable at z_0, then it is continuous at z_0.*
The converse is not true.

Example 1.3.2 *The function f defined by $f(z) = |z|^2$, $\forall z \in \mathbb{C}$ is continuous on $\mathbb{C}$, but it is differentiable only at $z = 0$.*

Theorem 1.3.2 (Cauchy-Riemann equations) *Let $E \subset \mathbb{C}$, $z_0 = x_0 + iy_0 \in E$ and let $f : E \longrightarrow \mathbb{C}$ be a complex-valued function such that $f(z) = u(x, y) + iv(x, y)$, $\forall z = x + iy \in E$. If f is differentiable at z_0, then the first order partial derivatives of u and v satisfy, at (x_0, y_0), the following equations*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{1.3}$$

The equations given in (1.3) are called the Cauchy-Riemann equations.

Example 1.3.3 *Let $f(z) = x^2 + y + i(y^2 - x) = u(x, y) + iv(x, y)$, $\forall z = x + iy \in \mathbb{C}$. Then*

$$u(x, y) = x^2 + y \text{ and } v(x, y) = y^2 - x,$$

therefore

$$\frac{\partial u}{\partial x}(x, y) = 2x,\ \frac{\partial v}{\partial y}(x, y) = 2y \text{ and } \frac{\partial u}{\partial y}(x, y) = 1,\ \frac{\partial v}{\partial x}(x, y) = -1.$$

Hence f is not differentiable at any point $z \in \mathbb{C} \setminus \{z \in \mathbb{C},\ z = x + ix,\ x \in \mathbb{R}\}$.

Theorem 1.3.3 *Let $E \subset \mathbb{C}$, $z_0 = x_0 + iy_0 \in E$ and let $f : E \longrightarrow \mathbb{C}$ be a complex-valued function such that $f(z) = u(x, y) + iv(x, y)$, $\forall z = x + iy \in E$. If u and v have continuous* *partial derivatives* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *in a neighborhood of* z_0 *and if they satisfied, at* (x_0, y_0)*, the Cauchy-Riemann equations given in* (1.3)*, then* f *is differentiable at* z_0*. Furthermore,*

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \\ &= \frac{\partial u}{\partial x}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \end{aligned}$$

Example 1.3.4 *Let* $f(z) = z^3$, $\forall z \in \mathbb{C}$, *then*

$$f(z) = f(x+iy) = x^3 - 3xy^2 + i(3x^2y - y^3) = u(x,y) + iv(x,y)$$

where

$$u(x,y) = x^3 - 3xy^2 \text{ and } v(x,y) = 3x^2y - y^3.$$

Then for all $(x,y) \in \mathbb{R}^2$*, we have*

$$\frac{\partial u}{\partial x}(x,y) = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}(x,y) \text{ and } \frac{\partial u}{\partial y}(x,y) = -6xy = -\frac{\partial v}{\partial x}(x,y).$$

Therefore, u *and* v *have continuous partial derivatives and they satisfy* (1.3)*, hence* f *is differentiable on* $\mathbb{C}$ *and for all* $z \in \mathbb{C}$*, we have*

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \\ &= 3x^2 - 3y^2 + i6xy \\ &= 3(x^2 - y^2 + 2ixy) = 3z^2. \end{aligned}$$

Definition 1.3.2 *Let* $E \subset \mathbb{C}$, $z_0 \in E$ *and let* $f : E \longrightarrow \mathbb{C}$ *be a complex-valued function. We say that* f *is holomorphic at* z_0 *if it is differentiable at* z_0 *and in a neighborhood of* z_0*.*

Remark 1.3.3 *A function can be differentiable at a point but not holomorphic at the same point.*

Example 1.3.5 *The function* $z \longmapsto |z|^2$ *is differentiable at* $z = 0$ *but it is not holomorphic at* $z = 0$*.*

Definition 1.3.3 *Let $E \subset \mathbb{C}$ be a domain and let $f : E \longrightarrow \mathbb{C}$ be a complex-valued function. We say that f is holomorphic on E if it is holomorphic at each point $z_0 \in E$.*

Definition 1.3.4 *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex-valued function. We say that f is entire if it is holomorphic on $\mathbb{C}$.*

Definition 1.3.5 *We say that a complex-valued function f is holomorphic at $z = \infty$ if $z \mapsto f\left(\frac{1}{z}\right)$ is holomorphic at $z = 0$.*

Example 1.3.6 *Polynomial complex functions are entire functions.*

Proposition 1.3.2 **(L'Hôpital)** *Let $E \subset \mathbb{C}$, $z_0 \in E$ and let $f, g : E \longrightarrow \mathbb{C}$ be two complex-valued functions such that f and g are holomorphic at z_0 with $f(z_0) = g(z_0) = 0$, but $g'(z_0) \neq 0$, then*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Example 1.3.7 *Let $z_0 = i$ and*

$$h(z) = \frac{z^3 + i}{z^2 + 1} = \frac{f(z)}{g(z)}$$

where $f(z) = z^3 + i$ and $g(z) = z^2 + 1$, then f and g are holomorphic at z_0. In addition, $f(i) = g(i) = 0$ and $g'(z_0) = 2i \neq 0$, hence by applying the Proposition 1.3.2, we get

$$\lim_{z \to z_0} h(z) = \lim_{z \to i} \frac{z^3 + i}{z^2 + 1} = \lim_{z \to i} \frac{f'(z)}{g'(z)} = \lim_{z \to i} \frac{3z^2}{2z} = \frac{3}{2} i.$$

Theorem 1.3.4 *Let $E \subset \mathbb{C}$ be a domain and let $f : E \longrightarrow \mathbb{C}$ be a complex single-valued function. If f is holomorphic on E and $f'(z) = 0$, $\forall z \in E$, then $f(z) = k \in \mathbb{C}$, $\forall z \in E$. That is f is constant on E.*

Example 1.3.8 *Let $E = \{z \in \mathbb{C}, |z| < 1\} \cup \{z \in \mathbb{C}, |z| > 2\}$ and let $f : E \longrightarrow \mathbb{C}$ be the function defined by*

$$f(z) = \begin{cases} 1, & \text{if } |z| < 1 \\ 2, & \text{if } |z| > 2. \end{cases}$$

It is clear that E is not a domain, in addition f is holomorphic on E and $f'(z) = 0$ for all $z \in E$, but f is not constant on E.

Theorem 1.3.5 *Let $E \subset \mathbb{C}$ be a domain and let $f : E \longrightarrow \mathbb{C}$ be a complex-valued function. If f is holomorphic on E and if $|f|$ is constant on E, then f is constant on E.*

Definition 1.3.6 *Let $D \subset \mathbb{R}^2$, $(x_0, y_0) \in D$ and let $h : D \longrightarrow \mathbb{R}$ be a real function. We say that h is harmonic at (x_0, y_0) if it has continuous partial derivatives of the first order and of the second order and these partial derivatives satisfy, at (x_0, y_0), the following Laplace equation*

$$\triangle h(x_0, y_0) = \frac{\partial^2 h}{\partial x^2}(x_0, y_0) + \frac{\partial^2 h}{\partial y^2}(x_0, y_0) = 0.$$

We say that h is harmonic on D if it is harmonic at each point (x, y) of D.

Theorem 1.3.6 *Let $E \subset \mathbb{C}$ be a domain and let $f : E \longrightarrow \mathbb{C}$ be a complex-valued function such that $f(z) = u + iv$, $\forall z \in E$. If f is holomorphic on E, then u and v are harmonic on E.*

Definition 1.3.7 *Let $D \subset \mathbb{R}^2$ and let $u, v : D \longrightarrow \mathbb{R}$ be two real functions. If u and v are harmonic on D and their first order partial derivatives satisfy "at any point of D" the Cauchy-Riemann equations* (1.3), *then we say that v is a harmonic conjugate of u.*

Example 1.3.9 *Let $u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be two real functions defined by*

$$u(x, y) = x^2 - y^2,\ v(x, y) = 2xy,\ \forall (x, y) \in \mathbb{R}^2,$$

then

$$\frac{\partial u}{\partial x}(x, y) = 2x,\ \frac{\partial u}{\partial y}(x, y) = -2y,\ \forall (x, y) \in \mathbb{R}^2,$$

hence

$$\frac{\partial^2 u}{\partial x^2}(x, y) = 2,\ \frac{\partial^2 u}{\partial y^2}(x, y) = -2,\ \forall (x, y) \in \mathbb{R}^2$$

and

$$\triangle u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0,$$

then u is harmonic on $\mathbb{R}^2$.
Similarly, we can prove that v is harmonic on $\mathbb{R}^2$. Furthermore, u and v satisfy the Cauchy-Riemann equations (1.3), *then v is a conjugate of u.*

Example 1.3.10 *Find a holomorphic function f whose real part is given by*

$$u(x,y) = e^{-x}\sin y,\ \forall (x,y) \in \mathbb{R}^2.$$

We have

$$\frac{\partial u}{\partial x}(x,y) = -e^{-x}\sin y,\ \frac{\partial u}{\partial y}(x,y) = e^{-x}\cos y,\ \forall (x,y) \in \mathbb{R}^2,$$

hence

$$\frac{\partial^2 u}{\partial x^2}(x,y) = e^{-x}\sin y,\ \frac{\partial^2 u}{\partial y^2}(x,y) = -e^{-x}\sin y,\ \forall (x,y) \in \mathbb{R}^2$$

and

$$\triangle u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0,\ \forall (x,y) \in \mathbb{R}^2,$$

then u is harmonic on $\mathbb{R}^2$.

Let $v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a conjugate of u, then u and v satisfy the Cauchy-Riemann equations (1.3), *that is*

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = -e^{-x}\sin y,\ \forall (x,y) \in \mathbb{R}^2 \tag{1.4}$$

and

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = -e^{-x}\cos y,\ \forall (x,y) \in \mathbb{R}^2. \tag{1.5}$$

By integrating (1.4) *with respect to y, we get*

$$v(x,y) = e^{-x}\cos y + \varphi(x). \tag{1.6}$$

Substituting (1.6) *in* (1.5)*, we obtain*

$$-e^{-x}\cos y + \varphi'(x) = -e^{-x}\cos y,$$

then $\varphi'(x) = 0$, $\forall x \in \mathbb{R}$, hence $\varphi(x) = k \in \mathbb{R}$. Therefore,

$$f(z) = e^{-x}\sin y + ie^{-x}\cos y + ik = ie^{-z} + ik,\ k \in \mathbb{R}.$$

1.4 Exercises

Exercise 1.4.1 *Express in the form $x + iy$ the following complex numbers*

$$z_1 = 1 + i + i(2-3i).\ \ z_2 = (3+2i)(2+i).\ \ z_3 = \frac{1}{1-2i}.$$

$$z_4 = \frac{1+3i}{2-i}. \quad z_5 = \frac{2i}{1+i} + \frac{1-i}{2i}.$$

Exercise 1.4.2 *Express in trigonometric form*

$$z_1 = -8 + \frac{4}{i} + \frac{25}{3-4i}. \quad z_2 = \frac{(1-i)(\sqrt{3}+i)}{(1+i)(\sqrt{3}-i)}.$$

Exercise 1.4.3 *Solve the following equations*

$$z^2 = 2i. \quad z^4 = -16. \quad z^n = 1.$$

Exercise 1.4.4 *Let z, z_1, z_2 be complex numbers and let $\overline{z}$ be the conjugate of z. Prove that*

1) $|z| \geq 0$ *and* $|z| = 0$ *if and only if* $z = 0$.

2) $z = \overline{z} \Longleftrightarrow z \in \mathbb{R}$.

3) $|z| = \sqrt{z.\overline{z}}$.

4) $|z| = |\overline{z}|$.

5) $|z_1.z_2| = |z_1||z_2|$ *and* $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$.

6) $|z_1 + z_2| \leq |z_1| + |z_2|$, *(the triangle inequality).*

7) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

8) $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$, $\forall z \in \mathbb{C}^\star$.

Exercise 1.4.5 *By using the triangle inequality, prove that*

1) $\left|z^3 + 3z - 1\right| \leq 15$, $\forall z \in \mathbb{C}$, $|z| = 2$.

2) $\left|z^2 + iz + 2\right| \geq 6$, $\forall z \in \mathbb{C}$, $|z| = 4$.

3) $0 \leq \left|\frac{z-1}{z^2+2}\right| \leq 2$, $\forall z \in \mathbb{C}$, $|z| = 1$.

Exercise 1.4.6 *Shade the following region and determine whether they are domain.*

1) $D_1 = \left\{z \in \mathbb{C},\ 1 < |z-1| < \sqrt{3}\right\}$.

2) $D_2 = \left\{z \in \mathbb{C},\ \frac{1}{2} < \left|z - \frac{1}{2}\right| < 1\right\} \cup \left\{z \in \mathbb{C},\ \frac{1}{2} < \left|z + \frac{1}{2}\right| < 1\right\}.$

3) $D_3 = \{z \in \mathbb{C},\ |z| < |z-1|\}.$

4) $D_4 = \{z \in \mathbb{C},\ |z-1| < 1\} \cup \{z \in \mathbb{C},\ |z-3| < 1\}.$

Exercise 1.4.7 *Describe the domain of definition for each of the following functions*

$$1)\ f(z) = z^2 - iz + 5 + 4i. \quad 2)\ f(z) = \frac{z}{z-i}. \quad 3)\ f(z) = \frac{z}{z+\bar{z}}. \quad 4)\ f(z) = \frac{1}{4-|z|^2}.$$

Exercise 1.4.8 **1)** *Find the real part and the imaginary part of the function defined by* $f(z) = z^3 + 2z + 1.$

2) *Express in term of z the function defined by*

$$f(z) = x^2 - y^2 + i(2x - 2xy),\ \forall z = x + iy \in \mathbb{C}.$$

Exercise 1.4.9 *Prove that if a function has a limit, then this limit is unique.*

Exercise 1.4.10 *Find each of the following limits*

$$1) \lim_{z\to 2i} \frac{z^3+5}{iz}, \quad 2) \lim_{z\to i} \frac{z^2+1}{z-i}, \quad 3) \lim_{z\to\infty} \frac{4z^2+1}{z^2+z+3+i}, \quad 4) \lim_{z\to\infty} \frac{z^4-i}{z^3+2z+4}.$$

Exercise 1.4.11 *Discuss the continuity of the function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by*

$$f(z) = \begin{cases} \frac{z^3-1}{z-1} & \text{if } |z| \neq 1, \\ 3 & \text{if } |z| = 1 \end{cases}$$

at $z_0 = 1, z_1 = -1, z_2 = -i$ *and* $z_3 = i.$

Exercise 1.4.12 *For each of the following functions, determine* $f'(z)$.

1) $f(z) = \frac{z+2}{3z-1},\ \forall z \in \mathbb{C} \setminus \left\{\frac{1}{3}\right\}.$

2) $f(z) = \left(1 + z^2\right)^3,\ \forall z \in \mathbb{C}.$

Exercise 1.4.13 *Determine the coefficients a, b and c so that the Cauchy-Riemann equations are satisfied for the function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by* $f(z) = ay^3 + ix^3 + xy(bx + icy),\ \forall z = x + iy \in \mathbb{C}.$

Exercise 1.4.14 *Prove that the Cauchy-Riemann equations can be written, in polar coordinates as*

$$\frac{\partial u}{\partial r}(r,\theta) = \frac{1}{r}\frac{\partial v}{\partial \theta}(r,\theta), \quad \frac{1}{r}\frac{\partial u}{\partial \theta}(r,\theta) = -\frac{\partial v}{\partial r}(r,\theta).$$

Where $x = r\cos\theta$ *and* $y = r\sin\theta$.

Exercise 1.4.15 *Let* $u : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a harmonic function in* D. *Prove that*

1) *If* $v : D \longrightarrow \mathbb{R}$ *is a conjugate of* u, *then* $-u$ *is also a conjugate of* v.

2) *If* $v_1 : D \longrightarrow \mathbb{R}$ *and* $v_2 : D \longrightarrow \mathbb{R}$ *are conjugates of* u, *then* v_1 *and* v_2 *differ by a real constant.*

3) *If* $v : D \longrightarrow \mathbb{R}$ *is a conjugate of* u, *then* v *is also a conjugate of* $u + c$, *where* c *is a real constant.*

Chapter 2

Elementary complex-valued functions

This chapter is devoted to some elementary complex-valued functions: Complex exponential function, complex trigonometric functions, complex hyperbolic functions, complex logarithmic function, complex power function, inverse trigonometric and hyperbolic functions. We end the chapter by giving some exercises.

2.1 Complex exponential function

Definition 2.1.1 *The exponential function is the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by*

$$w = f(z) = e^z = e^x \cos y + ie^x \sin y, \ \forall z = x + iy \in \mathbb{C}, \ x, y \in \mathbb{R}.$$

By applying the Theorem 1.3.3 we can easily obtain the following result.

Proposition 2.1.1 *Let $w = f(z) = e^z$, $\forall z \in \mathbb{C}$, then f is entire and for all $z \in \mathbb{C}$, we have $f'(z) = e^z$.*

Proposition 2.1.2 *Let z, z_1 and z_2 be three complex numbers. Then*

1) $e^z = 1 \Leftrightarrow z = 2k\pi i$, $k \in \mathbb{Z}$.

2) $e^{z_1} = e^{z_2} \Leftrightarrow z_1 = z_2 + 2k\pi i$, $k \in \mathbb{Z}$.

3) $\overline{e^z} = e^{\bar{z}}$.

Proof.

1) $\Leftarrow$? Let $z = 2k\pi i$, $k \in \mathbb{Z}$, then

$$e^z = e^{2k\pi i} = \cos(2k\pi) + i\sin(2k\pi) = 1.$$

$\Rightarrow$? Let $z = x + iy$, $x, y \in \mathbb{R}$, then

$$\begin{aligned} e^z = 1 &\Rightarrow e^x e^{iy} = 1 \\ &\Rightarrow e^x |e^{iy}| = 1 \\ &\Rightarrow e^x = 1 \\ &\Rightarrow x = 0, \end{aligned}$$

hence

$$e^z = e^{iy} = \cos y + i\sin y = 1,$$

then

$$\cos y = 1 \text{ and } \sin y = 0.$$

Therefore, $y = 2k\pi$, $k \in \mathbb{Z}$, that is $z = 2k\pi i$, $k \in \mathbb{Z}$.

2) $\Leftarrow$? Let $z_1 = z_2 + 2k\pi i$, $k \in \mathbb{Z}$, then

$$e^{z_1} = e^{z_2 + 2k\pi i} = e^{z_2} e^{2k\pi i} = e^{z_2}.$$

$\Rightarrow$? Suppose that $e^{z_1} = e^{z_2}$, then

$$e^{z_1 - z_2} = 1.$$

By using 1), we obtain

$$z_1 - z_2 = 2k\pi i, \; k \in \mathbb{Z},$$

that is

$$z_1 = z_2 + 2k\pi i, \; k \in \mathbb{Z}.$$

3) Let $z = x + iy$, $x, y \in \mathbb{R}$, then

$$e^z = e^{x+iy} = e^x \cos y + ie^x \sin y,$$

hence

$$\overline{e^z} = e^x \cos y - ie^x \sin y = e^{x-iy} = e^{\bar{z}}.$$

■

Corollary 2.1.1 *The function $z \longmapsto e^z$ is periodic with period $T = 2\pi i$.*

2.2 Complex trigonometric functions

Definition 2.2.1 *We define the complex sine and the complex cosine on $\mathbb{C}$, respectively, by*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \; \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

From the Proposition 2.1.1 and the Definition 2.2.1, we have the following result.

Proposition 2.2.1 *The complex functions $z \longmapsto \cos z$ and $z \longmapsto \sin z$ are entire and for all $z \in \mathbb{C}$, we have*

$$\frac{d}{dz}\sin z = \cos z \text{ and } \frac{d}{dz}\cos z = -\sin z.$$

Proposition 2.2.2 *For all $z \in \mathbb{C}$, we have*

$$\overline{\sin z} = \sin \overline{z} \text{ and } \overline{\cos z} = \cos \overline{z}.$$

Proof. By using the Definition 2.2.1, the Proposition 2.1.2 and that $\overline{iz} = -i\overline{z}$, $\forall z \in \mathbb{C}$, we have

$$\begin{aligned} \overline{\sin z} &= \frac{\overline{e^{iz}} - \overline{e^{-iz}}}{\overline{2i}} \\ &= \frac{e^{\overline{iz}} - e^{\overline{-iz}}}{-2i} \\ &= \frac{e^{i\overline{z}} - e^{-i\overline{z}}}{2i} \\ &= \sin \overline{z}. \end{aligned}$$

Similarly, we can prove that $\overline{\cos z} = \cos \overline{z}$, $\forall z \in \mathbb{C}$. ■

Proposition 2.2.3 *Let z, z_1, z_2 be three complex numbers, then*

1) $\sin(z + 2\pi) = \sin z, \;\; \cos(z + 2\pi) = \cos z.$

2) $\sin(z + \pi) = -\sin z, \;\; \cos(z + \pi) = -\cos z, \;\; \sin(\frac{\pi}{2} - z) = \cos z.$

3) $\sin(-z) = -\sin z, \;\; \cos(-z) = \cos z, \;\; \sin^2 z + \cos^2 z = 1.$

4) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2.$

5) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2.$

6) $\sin(2z) = 2 \sin z \cos z, \;\; \cos(2z) = \cos^2 z - \sin^2 z.$

7) $\sin(z_1 + z_2) \sin(z_1 - z_2) = \cos^2 z_2 - \cos^2 z_1.$

8) $\cos(z_1 + z_2) \sin(z_1 - z_2) = \frac{1}{2} \left(\sin(2z_1) - \sin(2z_2)\right).$

9) $\sin z = 0 \Leftrightarrow z = k\pi, \;\; k \in \mathbb{Z}.$

10) $\cos z = 0 \Leftrightarrow z = \frac{\pi}{2} + k\pi, \;\; k \in \mathbb{Z}.$

Proof. Exercise. ■

Corollary 2.2.1 *The complex functions $z \longmapsto \cos z$ and $z \longmapsto \sin z$ are periodic with period $T = 2\pi$.*

Remark 2.2.1 *The inequalities $|\sin x| \leq 1$ and $|\cos x| \leq 1$ are satisfied for all real numbers x, but they, in general, are not satisfied for complex numbers. Indeed $|\cos i| \simeq 1.5431 > 1$ and $|\sin(2+i)| \simeq 1.4859 > 1$.*

Definition 2.2.2 1. *We define the complex tangent function by*

$$\tan z = \frac{\sin z}{\cos z}, \;\; \forall z \in \mathbb{C} \setminus \left\{\frac{\pi}{2} + k\pi, \, k \in \mathbb{Z}\right\}.$$

2. *We define the complex cotangent function by*

$$\cot z = \frac{\cos z}{\sin z}, \;\; \forall z \in \mathbb{C} \setminus \{k\pi, \, k \in \mathbb{Z}\}.$$

From the Definition 2.2.2, the Proposition 2.2.1 and the Proposition 2.2.3, we get the following result.

Proposition 2.2.4 **1)** *The complex function $z \longmapsto \tan z$ is holomorphic on $\mathbb{C} \setminus \left\{\frac{\pi}{2} + k\pi, \, k \in \mathbb{Z}\right\}$.*

2) *The complex function* $z \longmapsto \cot z$ *is holomorphic on* $\mathbb{C} \setminus \{k\pi,\ k \in \mathbb{Z}\}$.

Furthermore,

$$\frac{d}{dz}\tan z = \frac{1}{\cos^2 z},\ \ \forall \mathbb{C} \setminus \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$$

and

$$\frac{d}{dz}\cot z = -\frac{1}{\sin^2 z},\ \ \forall \mathbb{C} \setminus \{k\pi,\ k \in \mathbb{Z}\}.$$

Remark 2.2.2 *The complex tangent and cotangent functions are periodic with period* $T = \pi$.

2.3 Complex hyperbolic functions

Definition 2.3.1 *We define the complex hyperbolic functions* $z \longmapsto \cosh z$, $z \longmapsto \sinh z$, $z \longmapsto \tanh z$ *and* $z \longmapsto \coth z$ *by*

$$\sinh z = \frac{e^z - e^{-z}}{2},\ \cosh z = \frac{e^z + e^{-z}}{2},\ \tanh z = \frac{\sinh z}{\cosh z},\ \coth z = \frac{\cosh z}{\sinh z},\ \forall z \in \mathbb{C}.$$

From the Proposition 2.1.1 and the Definition 2.3.1, we have the following result.

Proposition 2.3.1 *The complex functions* $z \longmapsto \cosh z$ *and* $z \longmapsto \sinh z$ *are entire and for all* $z \in \mathbb{C}$, *we have*

$$\frac{d}{dz}\sinh z = \cosh z \text{ and } \frac{d}{dz}\cosh z = \sinh z.$$

Proposition 2.3.2 *Let* $z, z_1, z_2 \in \mathbb{C}$. *Then*

1) $\cosh(iz) = \cos z,\ \ \cos(iz) = \cosh z.$

2) $\sinh(iz) = i\sin z,\ \ \sin(iz) = i\sinh z.$

3) $\sinh(-z) = -\sinh z,\ \ \cosh(-z) = \cosh z.$

4) $\cosh^2 z - \sinh^2 z = 1.$

5) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2.$

6) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2.$

7) $\sinh(2z) = 2\sinh z \cosh z,\ \ \cosh(2z) = \cosh^2 z + \sinh^2 z.$

8) $\sinh z = 0 \Leftrightarrow z = k\pi i,\ k \in \mathbb{Z}$ *and* $\cosh z = 0 \Leftrightarrow z = \left(k + \frac{1}{2}\right)\pi i,\ k \in \mathbb{Z}$

Proof. Exercise. ■

Corollary 2.3.1 **1)** *The complex functions* $z \longmapsto \cosh z$ *and* $z \longmapsto \sinh z$ *are periodic with period* $T = 2\pi i$.

2) *The complex function* $z \longmapsto \coth z$ *is holomorphic on* $\mathbb{C} \setminus \{k\pi i,\ k \in \mathbb{Z}\}$ *and*

$$\frac{d}{dz}\coth z = -\frac{1}{\sinh^2 z},\ \forall z \in \mathbb{C} \setminus \{k\pi i,\ k \in \mathbb{Z}\}.$$

3) *The complex function* $z \longmapsto \tanh z$ *is holomorphic on* $\mathbb{C} \setminus \left\{\left(k + \frac{1}{2}\right)\pi i,\ k \in \mathbb{Z}\right\}$ *and*

$$\frac{d}{dz}\tanh z = \frac{1}{\cosh^2 z},\ \forall z \in \mathbb{C} \setminus \left\{\left(k + \frac{1}{2}\right)\pi i,\ k \in \mathbb{Z}\right\}.$$

2.4 Complex logarithmic function

Definition 2.4.1 *We define the complex logarithm function as an inverse to the complex exponential function. That is a logarithm of nonzero complex number z is any complex number w such that* $e^w = z$. *We write* $w = \log z = \ln|z| + i\arg(z)$.

Remark 2.4.1 $z \longmapsto \log z$ *is a multi-valued function.*

Example 2.4.1 **1)** $\log 3 = \ln|3| + i\arg(3) = \ln 3 + 2k\pi i,\ k \in \mathbb{Z}$.

2) $\log(-1) = \ln|-1| + i\arg(-1) = (2k+1)\pi i,\ k \in \mathbb{Z}$.

3) $\log(1+i) = \ln|1+i| + i\arg(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right),\ k \in \mathbb{Z}$.

Proposition 2.4.1 *Let* $z, z_1, z_2 \in \mathbb{C}^\star$, *then*

1) *If* $z = re^{i\theta}$, *then* $\log z = \ln r + i\theta$.

2) $\log(e^z) = z + 2k\pi i,\ k \in \mathbb{Z}$.

3) $\log(z_1 z_2) = \log z_1 + \log z_2$, *(as an equality set).*

4) $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$, *(as an equality set).*

Proof. Exercise. ■

Definition 2.4.2 *We define the principal branch of the complex logarithm function* $z \longmapsto Log\, z$ *by*

$$Log\, z = \ln|z| + iArg(z).$$

Theorem 2.4.1 *The function* $z \longmapsto Log\, z$ *is holomorphic on* $E = \mathbb{C} \setminus]-\infty, 0[$ *and*

$$\forall z \in E, \ \frac{d}{dz} Log\, z = \frac{1}{z}.$$

Proof. Let $w = Log\, z$, $z_0 \in E$ and $w_0 = Log\, z_0$, then

$$\begin{aligned} \lim_{z \to z_0} \frac{Log\, z - Log\, z_0}{z - z_0} &= \lim_{w \to w_0} \frac{w - w_0}{e^w - e^{w_0}} \\ &= \lim_{w \to w_0} \frac{1}{\frac{e^w - e^{w_0}}{w - w_0}} \\ &= \frac{1}{e^{w_0}} \\ &= \frac{1}{e^{Log\, z_0}} \\ &= \frac{1}{z_0}. \end{aligned}$$

■

2.5 Complex power function

Definition 2.5.1 *Let* $z \in \mathbb{C}^\star$ *and* $\alpha \in \mathbb{C}$. *We define the* α*th power* z^α *of* z *by*

$$z^\alpha = e^{\alpha \log z} = e^{\alpha\left(\ln|z| + i \arg(z)\right)}.$$

Remark 2.5.1 **1)** *The complex function* $z \longmapsto z^\alpha$ *is a multi-valued function.*

2) *Let* $z \in \mathbb{C}^\star$ *and* $\alpha \in \mathbb{C}$. *Then*

$$z^\alpha = e^{\alpha\left(\ln|z| + iArg(z)\right)} e^{2\alpha k\pi i}, \ k \in \mathbb{Z}.$$

Example 2.5.1 • $1^i = e^{i \log 1} = e^{-2k\pi}$, $k \in \mathbb{Z}$.

• $(-2)^i = e^{i \log(-2)} = e^{i \ln 2} e^{-(\pi + 2k\pi)}$, $k \in \mathbb{Z}$.

Definition 2.5.2 *Let $z \in \mathbb{C}^\star$ and $\alpha \in \mathbb{C}$ be a constant. The principal branch of the function $z \longmapsto z^\alpha$ is the single-valued function defined by $f(z) = z^\alpha = e^{\alpha Log\, z}$.*

Remark 2.5.2 *Since $z \longmapsto e^z$ is entire and $z \longmapsto Log\, z$ is holomorphic on $E = \mathbb{C}\backslash]-\infty, 0[$, then the principal branch of $z \longmapsto z^\alpha$ is holomorphic on E. Furthermore,*

$$\forall z \in E,\ \frac{d}{dz}(z^\alpha) = \frac{d}{dz}\left(e^{\alpha Log\, z}\right) = e^{\alpha Log\, z}\frac{d}{dz}(\alpha Log\, z) = \frac{\alpha}{z}e^{\alpha Log\, z}.$$

2.6 Inverse trigonometric and hyperbolic functions

Definition 2.6.1 **1)** *The arcsine of a complex number z, denoted as* $\arcsin z$, *is the solution to the equation*

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

2) *The arccosine of a complex number z, denoted as* $\arccos z$, *is the solution to the equation*

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}.$$

Proposition 2.6.1 **1)**

$$\arcsin z = \frac{1}{i}\log\left(iz + \left(1 - z^2\right)^{\frac{1}{2}}\right).$$

2)

$$\arccos z = \frac{1}{i}\log\left(z + i\left(1 - z^2\right)^{\frac{1}{2}}\right).$$

Proof. Exercise. ■

Definition 2.6.2 • *The principal branch of the complex arcsine function is defined by*

$$Arcsin\, z = \frac{1}{i}Log\left(iz + \left|1 - z^2\right|^{\frac{1}{2}} e^{\frac{i}{2}Arg\left(1-z^2\right)}\right).$$

• *The principal branch of the complex arccosine function is defined by*

$$Arccosin\, z = \frac{1}{i}Log\left(z + i\left|1 - z^2\right|^{\frac{1}{2}} e^{\frac{i}{2}Arg\left(1-z^2\right)}\right).$$

Proposition 2.6.2 • $\arcsin z + \arccos z = \frac{1}{i}\log i$.

Proof.

- Let $z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$, then

$$2iz = e^{iw} - e^{-iw},$$

hence

$$e^{2iw} - 2ize^{iw} - 1 = 0.$$

By letting $v = e^{iw}$, we obtain

$$v^2 - 2izv - 1 = 0,$$

hence

$$v = iz + \sqrt{1 - z^2},$$

then

$$\frac{i}{v} = \frac{i}{iz + \sqrt{1 - z^2}} = \frac{i\left(-iz + \sqrt{1 - z^2}\right)}{\left(iz + \sqrt{1 - z^2}\right)\left(-iz + \sqrt{1 - z^2}\right)} = z + i\sqrt{1 - z^2}.$$

Therefore,

$$\arcsin z + \arccos z = \frac{1}{i}\left(\log v + \log\left(\frac{i}{v}\right)\right) = \frac{1}{i}\left(\log\left(v \times \frac{i}{v}\right)\right) = \frac{1}{i}\log i.$$

■

Definition 2.6.3 **1)** *The arctangent of a complex number z, denoted as* arctan *z, is the solution to the equation*

$$z = \tan w = \frac{\sin w}{\cos w}.$$

2) *The arccotangent of a complex number z, denoted as arccot z, is the solution to the equation*

$$z = \cot w = \frac{\cos w}{\sin w}.$$

Proposition 2.6.3 **1)**

$$\arctan z = \frac{1}{2i}\log\left(\frac{i - z}{i + z}\right).$$

2)

$$\mathit{arccot}\, z = \frac{1}{2i}\log\left(\frac{z + i}{z - i}\right).$$

Proof. Exercise. ∎

Definition 2.6.4 • *The principal branch of the complex arctangent function is defined by*

$$Arctan\, z = \frac{1}{2i} Log\left(\frac{i-z}{i+z}\right).$$

• *The principal branch of the complex arccotangent function is defined by*

$$Arccot\, z = \frac{1}{2i} Log\left(\frac{z+i}{z-i}\right).$$

Proposition 2.6.4 **1)** $\arctan z + arccot\, z = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}.$

2) $Arctan\, z + Arccot\, z = \begin{cases} \frac{\pi}{2}, & \Re(z) \geq 0 \\ -\frac{\pi}{2}, & \Re(z) < 0. \end{cases}$

Proof. Exercise. ∎

Definition 2.6.5 **1)** *The complex inverse hyperbolic sine function is the solution to the equation*

$$z = \sinh w = \frac{e^{w} - e^{-w}}{2}.$$

2) *The complex inverse hyperbolic cosine function is the solution to the equation*

$$z = \cosh w = \frac{e^{w} + e^{-w}}{2}.$$

Proposition 2.6.5 **1)**

$$arcsinh\, z = \log\left(z + \left(1 + z^2\right)^{\frac{1}{2}}\right).$$

2)

$$arccosh\, z = \log\left(z + \left(z^2 - 1\right)^{\frac{1}{2}}\right).$$

Proof. Exercise. ∎

Definition 2.6.6 • *The principal branch of the complex inverse hyperbolic sine function is defined by*

$$Arcsinhz = Log\left(z + \left|1 + z^2\right|^{\frac{1}{2}} e^{\frac{i}{2} Arg\left(1+z^2\right)}\right).$$

- *The principal branch of the complex inverse hyperbolic cosine function is defined by*

$$Arccoshz = Log\left(z + \left|z^2 - 1\right|^{\frac{1}{2}} e^{\frac{i}{2}Arg\left(z^2-1\right)}\right).$$

Definition 2.6.7 **1)** *The complex inverse hyperbolic tangent function is the solution to the equation*

$$z = \tanh w = \frac{\sinh w}{\cosh w}.$$

2) *The complex inverse hyperbolic cotangent function is the solution to the equation*

$$z = \coth w = \frac{\cosh w}{\sinh w}.$$

Proposition 2.6.6 **1)**

$$arctanh\, z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

2)

$$arccoth\, z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right).$$

Proof. Exercise. ■

Definition 2.6.8 • *The principal branch of the complex inverse hyperbolic tangent function is defined by*

$$Arctanh\, z = \frac{1}{2}Log\left(\frac{1+z}{1-z}\right).$$

- *The principal branch of the complex inverse hyperbolic cosine function is defined by*

$$Arccoth\, z = \frac{1}{2}Log\left(\frac{z+1}{z-1}\right).$$

2.7 Exercises

Exercise 2.7.1 *Express in the form $x + iy$, $x, y \in \mathbb{R}$ the value of the given function*

$$\sin(2+i), \quad \cos i, \quad \tan(\pi - 2i).$$

Exercise 2.7.2 *Let z, z_1, z_2 be three complex numbers. Prove that*

1) $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$.

2) $\sin(z+\pi) = -\sin z, \;\; \cos(z+\pi) = -\cos z, \;\; \sin(\frac{\pi}{2} - z) = \cos z.$

3) $\sin(-z) = -\sin z, \;\; \cos(-z) = \cos z, \;\; \sin^2 z + \cos^2 z = 1.$

4) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2.$

5) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2.$

6) $\sin(2z) = 2\sin z \cos z, \;\; \cos(2z) = \cos^2 z - \sin^2 z.$

7) $\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos^2 z_2 - \cos^2 z_1.$

8) $\cos(z_1 + z_2)\sin(z_1 - z_2) = \frac{1}{2}\sin(2z_1) - \frac{1}{2}\sin(2z_2).$

9) $\sin z = 0 \Leftrightarrow z = k\pi, \;\; k \in \mathbb{Z}.$

10) $\cos z = 0 \Leftrightarrow z = \frac{\pi}{2} + k\pi, \;\; k \in \mathbb{Z}.$

Exercise 2.7.3 • *Prove that the complex tangent and cotangent functions are periodic with period* $T = \pi$.

Exercise 2.7.4 *Prove that for all* $z, z_1, z_2 \in \mathbb{C}$, *we have*

1) $\cosh(iz) = \cos z, \;\; \cos(iz) = \cosh z.$

2) $\sinh(iz) = i\sin z, \;\; \sin(iz) = i\sinh z.$

3) $\sinh(-z) = -\sinh z, \;\; \cosh(-z) = \cosh z.$

4) $\cosh^2 z - \sinh^2 z = 1.$

5) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2.$

6) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2.$

7) $\sinh(2z) = 2\sinh z \cosh z, \;\; \cosh(2z) = \cosh^2 z + \sinh^2 z.$

8) $\sinh z = 0 \Leftrightarrow z = k\pi i, \, k \in \mathbb{Z}$ *and* $\cosh z = 0 \Leftrightarrow z = \left(k + \frac{1}{2}\right)\pi i, \, k \in \mathbb{Z}$

Exercise 2.7.5 **1)** *Express* $\sin z$, *for all* $z = x + iy \in \mathbb{C}$, *in the form* $u(x, y) + iv(x, y)$, $x, y \in \mathbb{R}$.

2) *Express* $\cos z$*, for all* $z = x + iy \in \mathbb{C}$*, in the form* $u(x, y) + iv(x, y),\ x, y \in \mathbb{R}$.

3) *Express* $\tan z$*, for all* $z = x+iy \in \mathbb{C}\setminus\left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$*, in the form* $u(x, y)+iv(x, y),\ x, y \in \mathbb{R}$.

Exercise 2.7.6 *Prove that for all* $z = x + iy \in \mathbb{C}$*, we have*

1) $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

2) $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

3) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.

4) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$.

Exercise 2.7.7 *Let* $z, z_1, z_2 \in \mathbb{C}^\star$*. Prove that*

1) *If* $z = re^{i\theta}$*, then* $\log z = \ln r + i\theta$.

2) $\log(e^z) = z + 2k\pi i,\ k \in \mathbb{Z}$.

3) $\log(z_1 z_2) = \log z_1 + \log z_2$.

4) $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$.

Exercise 2.7.8 *Prove that*

1) $\arcsin z = \frac{1}{i}\log\left(iz + (1 - z^2)^{\frac{1}{2}}\right)$.

2) $\arccos z = \frac{1}{i}\log\left(z + (z^2 - 1)^{\frac{1}{2}}\right)$.

3) $\arctan z = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right)$.

4) $\mathit{arccot}\ z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$.

5) $\arctan z + \mathit{arccot} z = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

6) $\mathit{Arctan}\, z + \mathit{Arccot}\, z = \begin{cases} \frac{\pi}{2}, & \Re(z) \geq 0 \\ -\frac{\pi}{2}, & \Re(z) < 0. \end{cases}$

Exercise 2.7.9 **1)** *Find all values of* $(i)^{-2i}$.

2) *Determine the principal value of* $(-i)^i$.

3) *Determine* $\arcsin(-i)$.

Exercise 2.7.10 *Prove that*

1) $arcsinh\, z = \log\left(z + (1+z^2)^{\frac{1}{2}}\right)$.

2) $arccosh\, z = \log\left(z + (z^2-1)^{\frac{1}{2}}\right)$.

Chapter 3

Fundamental theorems about complex functions

In this chapter, we start by introducing the notion of complex integration. After that, we present the Cauchy integral formula, Taylor and Laurent series and zeros and singularities of complex-valued functions. We end the chapter by giving some exercises.

3.1 Complex integration

Definition 3.1.1 *Let* $z : [a,b] \longrightarrow \mathbb{C}$ *be a continuous function defined by* $z(t) = x(t) + iy(t)$, $\forall t \in [a,b]$ *where* $x, y : [a,b] \longrightarrow \mathbb{R}$ *are continuous real functions.*

- *The range* γ *of* z *is called curve in* $\mathbb{C}$, *that is* γ *is the set* $\{z(t) = (x(t), y(t)),\ t \in [a,b]\}$.
- $z(a) = x(a) + iy(a)$ *is the initial point of* γ.
- $z(b) = x(b) + iy(b)$ *is the terminal point of* γ.

Definition 3.1.2 *Let* γ *be a curve in* $\mathbb{C}$. *We say that* γ *is simple if*

$$\forall t_1, t_2 \in [a,b],\ t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2),$$

that is γ *does not cross itself. See Figure 3.1*

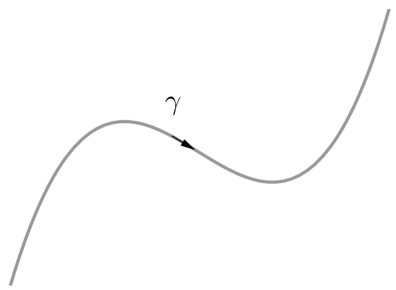


Figure 3.1: Simple curve

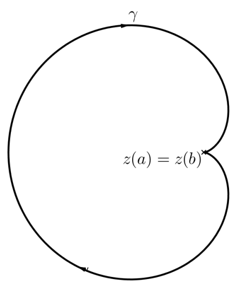


Figure 3.2: Simple closed curve

Definition 3.1.3 *Let γ be a curve in $\mathbb{C}$. We say that*

- *γ is closed if $z(a) = z(b)$.*
- *γ is simple closed if it is closed and for all $t_1, t_2 \in]a, b[$, $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$, that is γ does not cross itself except at the end points. See Figure 3.2*

Example 3.1.1 **1)** *Let $z_1, z_2 \in \mathbb{C}$. The line segment $[z_1, z_2]$ is a simple curve.*

2) *The unit circle γ given by $z(t) = (\cos t, \sin t)$ or by $z(t) = e^{it} = \cos t + i \sin t$, $t \in [0, 2\pi]$ is a simple closed curve.*

Remark 3.1.1 *Different functions may represent the same curve. For example $t \longmapsto z_1(t) = e^{it}$, $t \in [0, 2\pi]$ and $t \longmapsto z_2(t) = e^{2\pi i t}$, $t \in [0, 1]$ represent the unit circle.*

Definition 3.1.4 *Let γ be the curve given by the range of $z : [a, b] \longrightarrow \mathbb{C}$. We say that γ is piecewise continuous if*

(i) *$z(t)$ exists and is continuous for all but finitely many points in $]a, b[$.*

(ii) *At any point $t_0 \in]a, b[$ where z fails to be continuous, the limits $\lim\limits_{t \overset{<}{\to} t_0} z(t)$ and $\lim\limits_{t \overset{>}{\to} t_0} z(t)$ exist and are finite.*

(iii) *The limits $\lim\limits_{t \overset{<}{\to} b} z(t)$ and $\lim\limits_{t \overset{>}{\to} a} z(t)$ exist and are finite.*

Definition 3.1.5 *Let γ be the curve given by the range of $z : [a, b] \longrightarrow \mathbb{C}$. We say that γ is smooth if*

(i) *z' exists and is continuous on $[a, b]$.*

(ii) *$z'(t) \neq 0$, $\forall t \in]a, b[$.*

Example 3.1.2 *The line segment $[z_1, z_2]$ and the unit circle are smooth.*

Example 3.1.3 *Let γ be the curve represented by the function z defined on $[0, 2]$ by*

$$z(t) = \begin{cases} t + 2ti, & t \in [0, 1] \\ t + 2i, & t \in [1, 2] \end{cases}.$$

Then z' exists but it is not continuous at $t_0 = 1$, hence γ is not smooth. See Figure 3.3

Definition 3.1.6 *Let γ be the curve given by the range of $z : [a, b] \longrightarrow \mathbb{C}$. We say that γ is piecewise smooth if z and z' are piecewise continuous.*

Definition 3.1.7 *We say that γ is a contour if it is a sequence of smooth curves $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n$ such that the terminal point of γ_j is the initial point of γ_{j+1} for all $1 \leq j \leq n - 1$. We write, in this case, $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_n$.*

Definition 3.1.8 *Let γ be a simple closed contour. We say that γ is positively oriented or anticlockwise if the interior domain lies to the left of an observer tracing out the points in order, otherwise it is negatively oriented or clockwise.*

Remark 3.1.2 • *A contour is a continuous piecewise smooth curve.*

• *The opposite contour of γ is given by $-\gamma = (-\gamma_n) + (-\gamma_{n-1}) + \cdots + (-\gamma_1)$.*

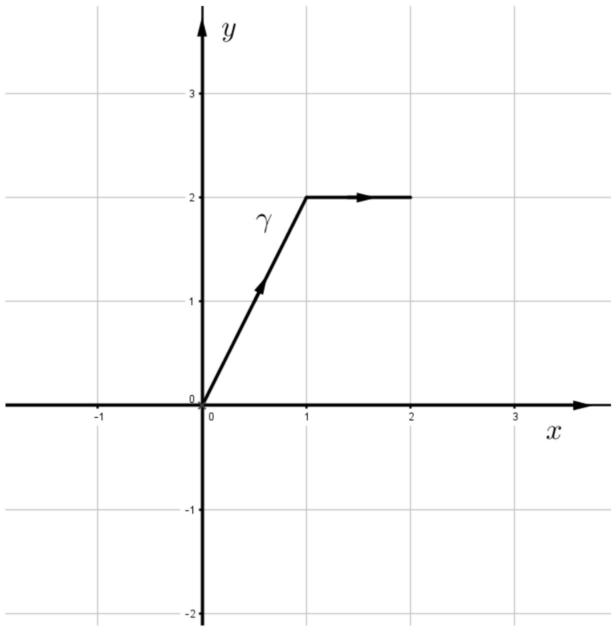


Figure 3.3: γ is not smooth.

Definition 3.1.9 *Let γ be a smooth curve given by the range of $z : [a,b] \longrightarrow \mathbb{C}$. We define the length of γ by*

$$L(\gamma) = \int_a^b |z'(t)|\, dt.$$

Example 3.1.4 **1)** *Let γ be the line segment $[z_1, z_2]$, that is the curve given by the range of $t \longmapsto z(t) = (1-t)z_1 + tz_2$, $t \in [0,1]$. Then the length of γ is*

$$L(\gamma) = \int_0^1 |z'(t)|\, dt = \int_0^1 |z_2 - z_1|\, dt = |z_2 - z_1|\,.$$

2) *Let γ be the curve given by the range of $t \longmapsto z(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then the length of γ is*

$$L(\gamma) = \int_0^{2\pi} |z'(t)|\, dt = \int_0^{2\pi} \left|ire^{it}\right| dt = 2\pi r.$$

Remark 3.1.3 *Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_n$ be a contour, then the length of γ is*

$$L(\gamma) = \sum_{j=1}^{n} L(\gamma_j).$$

Definition 3.1.10 *Let γ be a curve. We say that γ is rectifiable if the lengths L_Γ of all inscribed polygons Γ are bounded.*

Remark 3.1.4 *If γ is a piecewise smooth curve, then γ is rectifiable.*

Theorem 3.1.1 (Jordan curve theorem) *The points on any simple closed curve or simple closed contour γ are boundary points of two distinct domains, one of which is the interior of γ and is bounded. The other, which is the exterior of γ, is unbounded.*

Definition 3.1.11 *Let $E \subset \mathbb{C}$ be a domain. We say that E is a simply connected domain if the domain interior to any simple closed contour lying in E lies wholly in E.*
A domain that is not simply connected is called multiply connected.

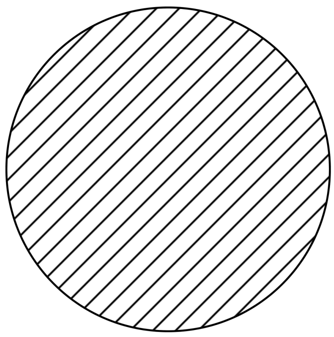


Figure 3.4: Simply connected

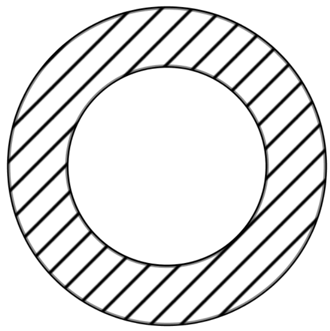


Figure 3.5: Not simply connected

Remark 3.1.5 *The interior of a simple closed curve is simply connected.*

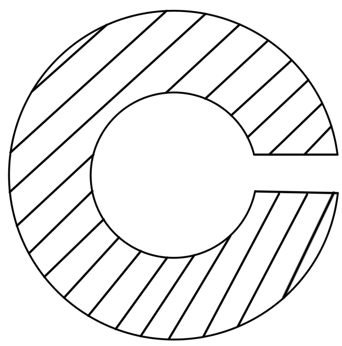


Figure 3.6: Simply connected

Definition 3.1.12 *Let $E \subset \mathbb{C}$ be an open set, $f : E \longrightarrow \mathbb{C}$ be a continuous complex-valued function and let γ be a smooth curve in E given by the range of $z : [a, b] \longrightarrow \mathbb{C}$. The integral of f along γ is defined by*

$$\int_\gamma f(z)dz = \int_a^b f(z(t))\, z'(t)dt.$$

Proposition 3.1.1 *Let $E \subset \mathbb{C}$ be an open set, $f, g : E \longrightarrow \mathbb{C}$ be continuous complex-valued functions and let γ be a smooth curve in E. Then*

1. $\int_\gamma (f(z) \pm g(z))\, dz = \int_\gamma f(z)dz \pm \int_\gamma g(z)dz.$
2. $\int_\gamma (cf(z))dz = c \int_\gamma f(z)dz,\ \forall c \in \mathbb{C}.$
3. $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$

Remark 3.1.6 *Let $E \subset \mathbb{C}$ be an open set, $f : E \longrightarrow \mathbb{C}$ be a continuous complex-valued function and let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_n$ be a contour in E, then*

$$\int_\gamma f(z)dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)dz.$$

Theorem 3.1.2 *Let $E \subset \mathbb{C}$ be an open set, γ be a contour in E and let $f : E \longrightarrow \mathbb{C}$ be a continuous complex-valued function such that $\left|f(z)\right| \leq M$, $\forall z \in \gamma$, where $M \in \mathbb{R}_+^\star$, then*

$$\left|\int_\gamma f(z)dz\right| \leq M\, L(\gamma).$$

3.2 Antiderivative of a complex-valued function

Definition 3.2.1 *Let $E \subset \mathbb{C}$ be a domain and let $f : E \longrightarrow \mathbb{C}$ be a continuous complex-valued function. An antiderivative of f on E is a function F such that $F'(z) = f(z)$, $\forall z \in E$.*

Remark 3.2.1 • *An antiderivative of a continuous complex-valued function is holomorphic.*

• *Any two antiderivatives of a continuous complex-valued function differ by a constant.*

Example 3.2.1 **1)** *The function $z \longmapsto \sin z$ is an antiderivative of the function $z \longmapsto \cos z$ on $\mathbb{C}$.*

2) *The function $z \longmapsto \frac{z^{n+1}}{n+1}$ is an antiderivative of the function $z \longmapsto z^n$ on $\mathbb{C}$.*

Theorem 3.2.1 *Let $E \subset \mathbb{C}$ be a domain, $z_1, z_2 \in E$, γ be a contour in E joining z_1 and z_2. Let $f : E \longrightarrow \mathbb{C}$ be a continuous complex-valued function. Suppose that f has an antiderivative F in E, then*

$$\int_\gamma f(z)dz = F(z_2) - F(z_1).$$

Corollary 3.2.1 • *The integral only depends on the end points and not on the choice of γ.*

• *If γ is a closed contour in E, then $\int_\gamma f(z)dz = 0$.*

Example 3.2.2 *Let $n \in \mathbb{N}$, γ_r be the circle centered at $z_0 \in \mathbb{C}$ and with radius $r > 0$ traversed once in the counterclockwise direction. Let $E = \mathbb{C} \setminus \{z_0\}$, then E is a domain and the function $z \longmapsto (z - z_0)^n$ is continuous in E and has an antiderivative $z \longmapsto \frac{(z-z_0)^{n+1}}{n+1}$. Since γ_r is a closed contour that lies in E, then $\int_{\gamma_r} (z - z_0)^n dz = 0$.*

Example 3.2.3 *Let γ_1, γ_2 and γ_3 be the curves given, respectively, by $z_1(t) = t(1+i)$, $z_2(t) = ti$ and $z_3(t) = t + i$, $t \in [0, 1]$. We define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ by*

$$f(z) = y - x + 3ix^2, \ \forall z = x + iy \in \mathbb{C}.$$

Then,

$$\int_{\gamma_1} f(z)dz = \int_0^1 f(z_1(t))\, z_1'(t)dt = \int_0^1 (3it^2 - 3t^2)dt = -1 + i$$

and

$$\int_{\gamma_2+\gamma_3} f(z)dz = \int_0^1 f(z_2(t))\, z_2'(t)dt + \int_0^1 f(z_3(t))\, z_3'(t)dt$$
$$= \int_0^1 itdt + \int_0^1 \left(1 - t + 3t^2 i\right) dt = \frac{1}{2} + \frac{3}{2}i.$$

It is clear that γ_1 and $\gamma = \gamma_2 + \gamma_3$ have the same end points but $\int_{\gamma_1} f(z)dz \neq \int_\gamma f(z)dz$, hence f cannot have an antiderivative.

3.3 Cauchy's integral theorem

Theorem 3.3.1 (Cauchy-Goursat theorem) *Let $E \subset \mathbb{C}$ be a simply connected domain, $f : E \longrightarrow \mathbb{C}$ be a holomorphic complex-valued function and let γ be a simple closed rectifiable contour in E, then $\int_\gamma f(z)dz = 0$.*

Theorem 3.3.2 *Let γ be a simple closed contour and f be a complex-valued function such that f is holomorphic at each point on and inside γ, then $\int_\gamma f(z)dz = 0$.*

Example 3.3.1 *Let us compute the integral $\int_\gamma \frac{dz}{z-2i}$, where γ is the circle centered at $z_0 = 0$ and with radius $r = \frac{3}{2}$ traversed once counterclockwise.*
The function $z \longmapsto \frac{1}{z-2i}$ is holomorphic in $\mathbb{C} \setminus \{2i\}$ but the point $z = 2i$ lies exterior to the contour, see Figure 3.7, then by Cauchy-Goursat theorem, $\int_\gamma \frac{dz}{z-2i} = 0$.

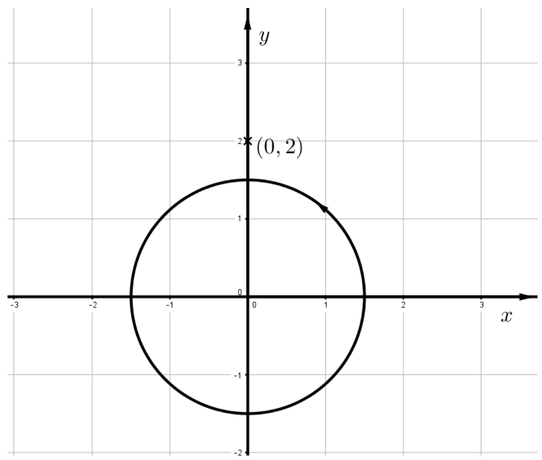


Figure 3.7:

3.4 Cauchy's integral formula

Theorem 3.4.1 *Let $E \subset \mathbb{C}$ be a simply connected domain, $f : E \longrightarrow \mathbb{C}$ be a holomorphic complex-valued function and let γ be a simple closed positively oriented contour. If E contains γ and z_0 is a point inside γ, then*

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}dz.$$

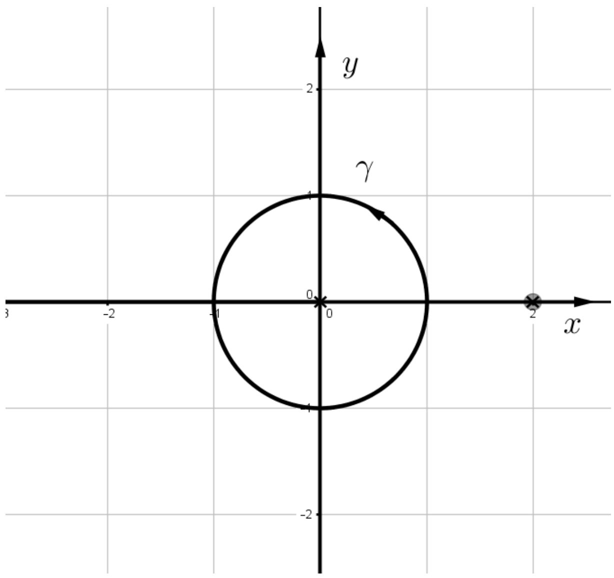
Example 3.4.1 **1)** *Evaluate the integral $I = \int_\gamma \frac{e^z}{z(z-2)}dz$ where γ is the unit circle.*

Let $g(z) = \frac{e^z}{z(z-2)}$, then g is holomorphic in $\mathbb{C} \setminus \{0, 2\}$ but only $z_0 = 0$ lies inside γ, then

$$g(z) = \frac{f(z)}{z} \text{ where } f(z) = \frac{e^z}{z-2}.$$

By applying Cauchy's integral formula, we obtain

$$I = \int_\gamma \frac{f(z)}{z}dz = 2\pi i f(0) = -\pi i.$$



2) *Evaluate the integral $I = \int_\gamma \frac{e^z}{z(z-2)}dz$ where γ is the contour given in the Figure 3.8*

It is clear that $\gamma = \gamma_1 + \gamma_2$ where γ_1 and γ_2 are the positively oriented left lobe and the negatively oriented right lobe, respectively. Then

$$I = \int_\gamma \frac{e^z}{z(z-2)}dz = \int_{\gamma_1} \frac{e^z}{z(z-2)}dz + \int_{\gamma_2} \frac{e^z}{z(z-2)}dz.$$

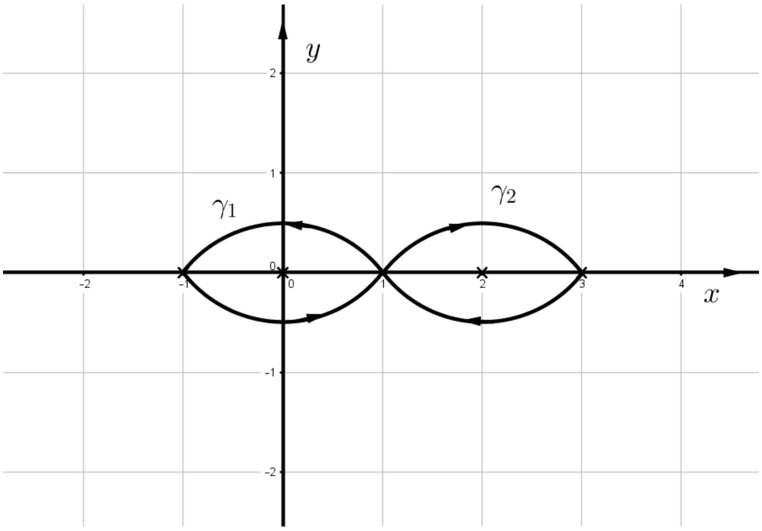


Figure 3.8:

But

$$\int_{\gamma_1} \frac{e^z}{z(z-2)}dz = \int_{\gamma_1} \frac{f(z)}{z}dz,\ f(z) = \frac{e^z}{z-2}$$

and

$$\int_{\gamma_2} \frac{e^z}{z(z-2)}dz = \int_{\gamma_2} \frac{g(z)}{z-2}dz,\ g(z) = \frac{e^z}{z}.$$

By applying Cauchy's integral formula, we obtain

$$\int_{\gamma_1} \frac{e^z}{z(z-2)}dz = 2\pi i f(0) = -\pi i \text{ and } \int_{\gamma_2} \frac{e^z}{z(z-2)}dz = -2\pi i g(2) = -e^2\pi i,$$

then

$$I = -\pi i - e^2\pi i = -(1+e^2)\pi i.$$

3) *Evaluate the integral* $I = \int_\gamma \frac{e^z}{z(z-2)}dz$ *where* γ *is the contour given in the Figure 3.9.*

We have

$$\frac{e^z}{z(z-2)} = \frac{e^z}{2(z-2)} - \frac{e^z}{2z},$$

then

$$\int_\gamma \frac{e^z}{z(z-2)}dz = \int_\gamma \frac{e^z}{2(z-2)}dz - \int_\gamma \frac{e^z}{2z}dz.$$

By applying Cauchy's integral formula, we obtain

$$\int_\gamma \frac{e^z}{2(z-2)}dz = \frac{1}{2}\int_\gamma \frac{e^z}{z-2}dz = \pi i e^2$$

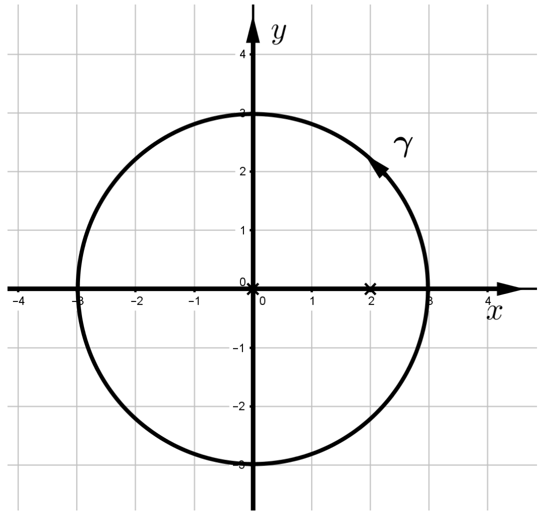


Figure 3.9:

and

$$\int_\gamma \frac{e^z}{2z}dz = \frac{1}{2}\int_\gamma \frac{e^z}{z}dz = \pi i e^0 = \pi i.$$

Hence,

$$I = \pi i e^2 - \pi i = (e^2 - 1)\pi i.$$

Theorem 3.4.2 *Let $E \subset \mathbb{C}$ be a domain and $f : E \longrightarrow \mathbb{C}$ be a holomorphic complex-valued function. Then the derivatives $f', f'', \ldots, f^{(n)}, \ldots$ exist and they are holomorphic in E.*

Theorem 3.4.3 (Cauchy's integral formula for derivatives) *Let γ be a simple closed positively oriented contour and z_0 be a point inside γ. If f is a complex-valued function such that it is holomorphic inside and on γ, then for all $n \in \mathbb{N}^\star$ we have*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Example 3.4.2 *Evaluate the integral $\int_\gamma \frac{2z+1}{z^2}dz$ along the unit circle traversed once in the counterclockwise direction. Let $f(z) = 2z + 1$ and $z_0 = 0$, then by Cauchy's integral formula for $n = 1$, we get*

$$\int_\gamma \frac{2z+1}{z^2}dz = \frac{2\pi i}{1!}f'(0) = 4\pi i.$$

3.5 Taylor series

Definition 3.5.1 *A sequence* $\{z_n\}_{n=0}^{+\infty}$ *of complex numbers is a function* $z : \mathbb{N} \longrightarrow \mathbb{C}$ *defined by* $z(n) = z_n,\ \forall n \in \mathbb{N}$.

Definition 3.5.2 *We say that* $\{z_n\}_{n=0}^{+\infty}$ *has a limit* $z \in \mathbb{C}$ *as* $n \to +\infty$ *if*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ |z_n - z| < \varepsilon.$$

In this case, we say that $\{z_n\}_{n=0}^{+\infty}$ *converges to* z *with* $\lim\limits_{n\to+\infty} z_n = z$.
We say that $\{z_n\}_{n=0}^{+\infty}$ *diverges if it has no limit or* $\lim\limits_{n\to+\infty} z_n = \infty$.

Definition 3.5.3 **1)** *We say that* $\{z_n\}_{n=0}^{+\infty}$ *is bounded if*

$$\exists M \in \mathbb{R}_+^\star,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ |z_n| \leq M.$$

2) *We say that* $\{z_n\}_{n=0}^{+\infty}$ *is a Cauchy sequence if*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n, m \in \mathbb{N},\ n > m \geq n_0,\ |z_n - z_m| < \varepsilon.$$

Proposition 3.5.1 $\{z_n\}_{n=0}^{+\infty}$ *is convergent if and only if it is a Cauchy sequence.*

Proposition 3.5.2 *Let* $z_0 \in \mathbb{C}$ *and let* f *be a complex-valued function defined in a neighborhood of* z_0*, then* f *is continuous at* z_0 *if and only if for any sequence* $\{z_n\}_{n=0}^{+\infty}$ *converging to* z_0 *we have* $\lim\limits_{n\to+\infty} f(z_n) = f(z_0)$.

Remark 3.5.1 • *A convergent sequence has a unique limit.*

• *A convergent sequence is bounded.*

Proposition 3.5.3 *Let* $\{w_n\}_{n=0}^{+\infty}$ *be a complex sequence and* $z_0, w_0, \alpha, \beta \in \mathbb{C}$ *such that* $\lim\limits_{n\to+\infty} z_n = z_0$ *and* $\lim\limits_{n\to+\infty} w_n = w_0$. *Then*

1) $\lim\limits_{n\to+\infty} (\alpha z_n + \beta w_n) = \alpha z_0 + \beta w_0$.

2) $\lim\limits_{n\to+\infty} (z_n w_n) = z_0 w_0$.

3) $\lim\limits_{n\to+\infty} \left(\frac{z_n}{w_n}\right) = \frac{z_0}{w_0},\ w_0 \neq 0$.

4) $\lim_{n\to+\infty} \overline{z_n} = \overline{z_0}$ *and* $\lim_{n\to+\infty} |z_n| = |z_0|$.

Proposition 3.5.4 *If* $z_n = x_n + iy_n$, $\forall n \in \mathbb{N}$, $z_0 = x_0 + iy_0$, $x_0, y_0 \in \mathbb{R}$, *and* $\{x_n\}_{n=0}^{+\infty}$ *and* $\{y_n\}_{n=0}^{+\infty}$ *are real sequences, then*

$$\lim_{n\to+\infty} z_n = z_0 \Longleftrightarrow \begin{cases} \lim_{n\to+\infty} x_n = x_0 \\ \lim_{n\to+\infty} y_n = y_0. \end{cases}$$

Proposition 3.5.5 *Let* $\{w_n\}_{n=0}^{+\infty}$ *be a complex sequence.*

1) *If* $|w_n| \le |z_n|$, $\forall n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} z_n = 0$, *then* $\lim_{n\to+\infty} w_n = 0$.

2) *If* $\lim_{n\to+\infty} z_n = 0$ *and* $\{w_n\}_{n=0}^{+\infty}$ *is bounded, then* $\lim_{n\to+\infty} w_n z_n = 0$.

Definition 3.5.4 *Let* $z_0 \in \mathbb{C}$. *We say that* z_0 *is an accumulation point of* $\{z_n\}_{n=0}^{+\infty}$ *if for every neighborhood* $V(z_0)$ *of* z_0, *there exists a subsequence* $\{z_{nk}\}$ *for which all terms belong to* $V(z_0)$.

Example 3.5.1 ♣ *Let* $z_n = \frac{3n+2i}{n}$, $n \in \mathbb{N}^\star$, *then* $\lim_{n\to+\infty} z_n = \lim_{n\to+\infty} \frac{3n}{n} = 3$ *and* $\{z_n\}_{n=1}^{+\infty}$ *converges to* 3.

♣ *Let* $z_n = \frac{3n+7ni}{2n+5i}$, $n \in \mathbb{N}$, *then* $\lim_{n\to+\infty} z_n = \lim_{n\to+\infty} \frac{3n+7ni}{2n} = \frac{3+7i}{2}$ *and* $\{z_n\}_{n=0}^{+\infty}$ *converges to* $\frac{3+7i}{2}$.

♣ *Let* $z_n = \frac{n^2-i}{2n+5i}$, $n \in \mathbb{N}$, *then* $\lim_{n\to+\infty} z_n = \lim_{n\to+\infty} \frac{n^2}{2n} = +\infty$ *and* $\{z_n\}_{n=0}^{+\infty}$ *diverges.*

Definition 3.5.5 **1)** *A series of complex numbers is a formal expression* $\sum_{n=0}^{+\infty} z_n$ *where* $z_n \in \mathbb{C}$, $\forall n \in \mathbb{N}$.

2) *We denote by* S_n *the nth partial sum of the series* $\sum_{n=0}^{+\infty} z_n$ *and it is defined by* $S_n = \sum_{k=0}^{n} z_k$.

3) *We say that the series* $\sum_{n=0}^{+\infty} z_n$ *converges if the sequence* $\{S_n\}_{n=0}^{+\infty}$ *converges to* $S \in \mathbb{C}$. *In this case, S is the sum of the series* $\sum_{n=0}^{+\infty} z_n$ *and we can write* $S = \sum_{n=0}^{+\infty} z_n$.

4) *If the series* $\sum_{n=0}^{+\infty} z_n$ *does not converge, then we say that it diverges.*

5) *We say that the series* $\sum_{n=0}^{+\infty} z_n$ *is absolutely convergent if the series* $\sum_{n=0}^{+\infty} |z_n|$ *converges.*

6) *We say that the series* $\sum_{n=0}^{+\infty} z_n$ *is conditionally convergent if it is convergent but not absolutely convergent.*

Example 3.5.2 *Let us discuss the convergence of the geometric series* $\sum_{n=0}^{+\infty} pq^n$.
We have

$$S_n = \sum_{k=0}^{n} pq^k = p + pq + pq^2 + \cdots + pq^n$$

and

$$\frac{p}{1-q} - S_n = \frac{pq^{n+1}}{1-q}.$$

Then if $|q| < 1$, *we have* $\frac{|p||q|^{n+1}}{1-q}$ *converges to* 0 *as* $n \to +\infty$, *hence* $\{S_n\}_{n=0}^{+\infty}$ *converges to* $\frac{p}{1-q}$. *Therefore the geometric series* $\sum_{n=0}^{+\infty} pq^n$ *converges if* $|q| < 1$ *and its sum is* $S = \sum_{n=0}^{+\infty} pq^n = \frac{p}{1-q}$.

Proposition 3.5.6 **1)** *If* $\sum_{n=0}^{+\infty} z_n$ *converges, then* $\lim_{n\to+\infty} z_n = 0$.

2) *If* $\sum_{n=0}^{+\infty} z_n$ *converges, then* $\lim_{n\to+\infty} \sum_{k=n+1}^{+\infty} z_k = 0$.

Proposition 3.5.7 *Let* $\sum_{n=0}^{+\infty} w_n$ *be a series of complex numbers.*

1) *If* $|z_n| \leq w_n$, $\forall n \in \mathbb{N}$ *and* $\sum_{n=0}^{+\infty} w_n$ *converges, then* $\sum_{n=0}^{+\infty} z_n$ *converges (Comparison test).*

2) *If* $\sum_{n=0}^{+\infty} z_n$ *and* $\sum_{n=0}^{+\infty} w_n$ *converge, then for all* $\alpha, \beta \in \mathbb{C}$, $\sum_{n=0}^{+\infty} (\alpha z_n + \beta w_n)$ *converges and*

$$\sum_{n=0}^{+\infty} (\alpha z_n + \beta w_n) = \alpha \sum_{n=0}^{+\infty} z_n + \beta \sum_{n=0}^{+\infty} w_n.$$

Proposition 3.5.8 **(Cauchy's root test)** *Let* $\lim_{n\to+\infty} \sqrt[n]{|z_n|} = l < \infty$, *then*

1) *The series* $\sum_{n=0}^{+\infty} z_n$ *converges if* $l < 1$.

2) *The series* $\sum_{n=0}^{+\infty} z_n$ *diverges if* $l > 1$.

3) *The test is inconclusive if* $l = 1$.

Proposition 3.5.9 **(D'Alembert's rational test)** *Let* $\lim_{n\to+\infty} \left|\frac{z_{n+1}}{z_n}\right| = l < \infty$, *then*

1) *The series* $\sum_{n=0}^{+\infty} z_n$ *converges if* $l < 1$.

2) *The series* $\sum_{n=0}^{+\infty} z_n$ *diverges if* $l > 1$.

3) *The test is inconclusive if* $l = 1$.

Definition 3.5.6 *Let* $E \subset \mathbb{C}$ *be a domain and let* $\{f_n\}_{n=0}^{+\infty}$ *be a sequence of complex-valued functions defined on* E.

1) *We say that* $\{f_n\}_{n=0}^{+\infty}$ *converges to* f *pointwise on* E *if for any* $z \in E$, *the complex sequence* $\{f_n(z)\}_{n=0}^{+\infty}$ *converges and* $\lim_{n\to+\infty} f_n(z) = f(z)$, $\forall z \in E$.

2) *We say that* $\{f_n\}_{n=0}^{+\infty}$ *converges to* f *uniformly on* E *if*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall z \in E,\ \forall n \geq n_0,\ \left|f_n(z) - f(z)\right| < \varepsilon.$$

Example 3.5.3 *Let* $f_n(z) = z^n$, $\forall z \in \mathbb{C}$, $n \in \mathbb{N}$. *Then the sequence* $\{f_n\}_{n=0}^{+\infty}$ *converges pointwise to* 0 *on the open unit disk* $B(0,1)$ *and it is convergent uniformly to* 0 *on the closed disk* $\overline{B}(0,r)$ *for all* $0 < r < 1$.

Definition 3.5.7 **1)** *A series of complex-valued functions is a formal expression* $\sum_{n=0}^{+\infty} f_n$ *where* f_n *are complex-valued functions defined on* $E \subset \mathbb{C}$.

2) *We denote by* $\{S_n\}_{n=0}^{+\infty}$ *the sequence of partial sums of the series* $\sum_{n=0}^{+\infty} f_n$ *and it is defined on* E *by* $S_n(z) = \sum_{k=0}^{n} f_k(z)$, $\forall z \in E$.

3) *We say that the series* $\sum_{n=0}^{+\infty} f_n$ *converges pointwise on* E *if its sequence of partial sums* $\{S_n\}_{n=0}^{+\infty}$ *converges pointwise to the complex-valued function* S *and for all* $z \in E$, *we have* $S(z) = \sum_{n=0}^{+\infty} f_n(z)$.

4) *If the series* $\sum_{n=0}^{+\infty} f_n$ *does not converge, then we say that it diverges.*

5) *We say that the series* $\sum_{n=0}^{+\infty} f_n$ *converges absolutely on* E *if its sequence of partial sums* $\{S_n\}_{n=0}^{+\infty}$ *converges absolutely to the complex-valued function* S.

6) *We say that the series* $\sum_{n=0}^{+\infty} f_n$ *converges uniformly on* E *if its sequence of partial sums* $\{S_n\}_{n=0}^{+\infty}$ *converges uniformly to the complex-valued function* S.

Proposition 3.5.10 *Let* $E \subset \mathbb{C}$ *be a domain and* f_n *be complex-valued functions defined on* E. *If each* f_n *is continuous on* E *and the series* $\sum_{n=0}^{+\infty} f_n$ *converges uniformly on* E *to the function* S, *then* S *is continuous on* E.

Proposition 3.5.11 *Let* $E \subset \mathbb{C}$ *be a domain and* γ *be a contour in* E. *If the series* $\sum_{n=0}^{+\infty} f_n$ *converges uniformly on* E *to the function* S, *then*

$$\int_{\gamma} S(z)dz = \int_{\gamma} \sum_{n=0}^{+\infty} f_n(z)dz = \sum_{n=0}^{+\infty} \int_{\gamma} f_n(z)dz.$$

Proposition 3.5.12 *Let* $E \subset \mathbb{C}$ *be a domain. If each* f_n *is holomorphic on* E *and the series* $\sum_{n=0}^{+\infty} f_n$ *converges uniformly on every compact subset of* E *to the function* S, *then*

$$S'(z) = \frac{d}{dz}\left(\sum_{n=0}^{+\infty} f_n(z)\right) = \sum_{n=0}^{+\infty} f_n'(z),\ \forall z \in E.$$

Definition 3.5.8 *A power series around* $z_0 \in \mathbb{C}$ *is a series of functions of the form*

$$\sum_{n=0}^{+\infty} a_n (z - z_0)^n \tag{3.1}$$

where $a_n \in \mathbb{C}$, $\forall n \in \mathbb{N}$.
$\{a_n\}_{n\geq 0}$ *are the coefficients of the series* (3.1).

Definition 3.5.9 *The radius of convergence of the series* (3.1) *is a real number* $0 \leq R \leq +\infty$ *such that* (3.1) *converges for all* z *such that* $|z - z_0| < R$ *and it diverges for all* z *such that* $|z - z_0| > R$ *and the set* $\{z \in \mathbb{C},\ |z - z_0| < R\}$ *is called the disk of convergence of* (3.1).

Remark 3.5.2 • *The radius of convergence is unique.*

- *If* (3.1) *converges nowhere except at* z_0, *then* $R = 0$.
- *If* (3.1) *converges for all* $z \in \mathbb{C}$, *then* $R = +\infty$.

Theorem 3.5.1 *Suppose that* (3.1) *converges at* $z = z_1 \neq z_0$, *then it converges absolutely and uniformly in the closed disk* $\overline{B}(z_0, r)$ *where* $r < |z_1 - z_0|$.

Corollary 3.5.1 ♣ *If* (3.1) *diverges at some point* $z = z_1$*, then it diverges at all points* z *that satisfy* $|z - z_0| > |z_1 - z_0|$.

♣ *If* R *is the radius of convergence of* (3.1)*, then the series* (3.1) *converges uniformly on the closed disk* $\overline{B}(z_0, R_1)$ *where* $0 < R_1 < R$.

Remark 3.5.3 *Let* R *be the radius of convergence of* (3.1)*. Then*

- *If* $\lim\limits_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$, $l \in]0, +\infty[$, *then* $R = \frac{1}{l}$. *(D'Alembert's test).*
- *If* $\lim\limits_{n\to+\infty} \sqrt[n]{|a_n|} = l$, $l \in]0, +\infty[$, *then* $R = \frac{1}{l}$, *(Cauchy's root test).*
- $R = \frac{1}{l}$ *where* $l = \limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}$, $l \in]0, +\infty[$ *(Cauchy-Hadamard formula).*
- *If* $l = 0$, *then* $R = +\infty$ *and if* $l = +\infty$, *then* $R = 0$.

Example 3.5.4 **1)** $\sum\limits_{n=0}^{+\infty} z^n$, $a_n = 1$, $\forall n \in \mathbb{N}$. *We have* $\lim\limits_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, *then* $R = \frac{1}{1} = 1$.

2) $\sum\limits_{n=0}^{+\infty} (n+1)^n (z+i)^n$, $a_n = (n+1)^n$, $\forall n \in \mathbb{N}$, $z_0 = -i$. *We have* $\lim\limits_{n\to+\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to+\infty} (n+1) = +\infty$, *then* $R = 0$.

Theorem 3.5.2 *Let* R *be the radius of convergence of* (3.1) *and* $B(z_0, R)$ *be its disk of convergence. Then*

1) *The sum* S *defined by* $S(z) = \sum\limits_{n=0}^{+\infty} a_n (z - z_0)^n$ *is holomorphic on* $B(z_0, R)$.

2) *If* γ *is a contour in* $B(z_0, R)$ *then*

$$\int_\gamma \sum_{n=0}^{+\infty} a_n (z - z_0)^n dz = \sum_{n=0}^{+\infty} a_n \int_\gamma (z - z_0)^n dz.$$

3) $S'(z) = \sum\limits_{n=1}^{+\infty} n a_n (z - z_0)^{n-1}$, $\forall z \in B(z_0, R)$. *Furthermore, the radius of convergence of the series* $\sum\limits_{n=1}^{+\infty} n a_n (z - z_0)^{n-1}$ *is also* R.

Corollary 3.5.2 *Let* R *be the radius of convergence of* (3.1) *and* $B(z_0, R)$ *be its disk of convergence. Then its sum* S *is infinitely differentiable for all* $z \in B(z_0, R)$ *and for all* $k \in \mathbb{N}^\star$*, we have*

$$S^{(k)}(z) = \sum_{n=k}^{+\infty} n(n-1)\ldots(n-k+1) a_n (z - z_0)^{n-k}, \ \forall z \in B(z_0, R).$$

Furthermore,

$$a_k = \frac{S^{(k)}(z_0)}{k!}, \ \forall k \in \mathbb{N}^\star.$$

Example 3.5.5 *We have*

$$\sum_{n=0}^{+\infty} z^n = \frac{1}{1-z}, \ z \in B(0,1)$$

and

$$\sum_{n=1}^{+\infty} nz^n = \sum_{n=0}^{+\infty} (n+1)z^n = \frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)', \ z \in B(0,1).$$

Theorem 3.5.3 *Let $E \subset \mathbb{C}$ be a domain, $z_0 \in E$ and let $f : E \to$ be a holomorphic complex-valued function in E. Suppose that $B(z_0, R)$ is the largest open disk in E where $R > 0$. Then f can be written as follows*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \ \forall z \in B(z_0, R). \quad (3.2)$$

Remark 3.5.4
- *The series* (3.2) *is called the Taylor series of f at z_0 and it converges uniformly on the closed disk $\overline{B}(z_0, r)$ where $0 \le r < R$.*
- *If we put $z_0 = 0$, then Taylor's series* (3.2) *is called the Maclaurin series of f.*

By using the definition of the Maclaurin series of a function, we can obtain

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{+\infty} \frac{z^n}{n!}, \ R = +\infty.$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \ R = +\infty.$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \ R = +\infty.$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{+\infty} \frac{z^{2n}}{(2n)!}, \ R = +\infty.$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{+\infty} \frac{z^{2n+1}}{(2n+1)!}, \ R = +\infty.$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{+\infty} z^n, \ R = 1.$$

Definition 3.5.10 *An analytic function at a point $z_0 \in \mathbb{C}$ is a function that can be expanded in a convergent power series centered at z_0.*
We say that f is analytic on $E \subset \mathbb{C}$ if it is analytic at any point $z \in E$.

From Theorem 3.5.2 and Theorem 3.5.3, we have the following corollary

Corollary 3.5.3 *Let $E \subset \mathbb{C}$ be an open set and $f : E \longrightarrow \mathbb{C}$ be a complex-valued function. Then f is holomorphic on E if and only if f is analytic on E.*

3.6 Laurent's series

Theorem 3.6.1 *Let $z_0 \in \mathbb{C}$, $\rho_1, \rho_2 \in \mathbb{R}_+$ such that $\rho_1 < \rho_2$. Let f be a holomorphic function in the annulus domain $C_{z_0}(\rho_1, \rho_2) = \{z \in \mathbb{C},\ \rho_1 < |z - z_0| < \rho_2\}$, then f can be represented by the following series*

$$f(z) = \sum_{n\geq 0} a_n (z - z_0)^n + \sum_{n\geq 1} \frac{b_n}{(z - z_0)^n},\ \ \forall z \in C_{z_0}(\rho_1, \rho_2) \tag{3.3}$$

where

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,\ n \in \mathbb{N}$$

and

$$b_n = \frac{1}{2\pi i} \int_\gamma f(z)(z - z_0)^{n-1} dz,\ n \in \mathbb{N}^\star$$

and γ is a positively oriented simple closed contour around z_0 lying in $C_{z_0}(\rho_1, \rho_2)$.

Remark 3.6.1 **1)** *The series in* (3.3) *is called Laurent's series of f at z_0.*

2) $\sum\limits_{n\geq 1} \frac{b_n}{(z-z_0)^n}$ *is called the principal part of the Laurent sereis* (3.3).

Example 3.6.1 *Let us expand the function f defined by $f(z) = \sin z$ in the Laurent series around the point $z_0 = 0$. We have*

$$\sin w = \sum_{n\geq 0} (-1)^n \frac{w^{2n+1}}{(2n + 1)!},\ \forall w \in \mathbb{C}.$$

Let $w = \frac{1}{z}$, $z \neq 0$, then

$$\sin\left(\frac{1}{z}\right) = \sum_{n\geq 0} (-1)^n \frac{1}{(2n + 1)! z^{2n+1}},\ \forall z \in \{z \in \mathbb{C},\ 0 < |z| < \infty\}.$$

Example 3.6.2 *Let us expand the function f defined by $f(z) = \frac{1}{1-z}$ in the Laurent series around the point $z_0 = 0$ in the domain $\{z \in \mathbb{C},\ 1 < |z| < \infty\}$. We have*

$$\frac{1}{1-w} = \sum_{n\geq 0} w^n,\ \forall w \in \{w \in \mathbb{C},\ |w| < 1\}.$$

Since

$$\frac{1}{1-z} = \frac{1}{z\left(\frac{1}{z}-1\right)} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}},\ \left|\frac{1}{z}\right| < 1,$$

then

$$\frac{1}{1-z} = -\frac{1}{z}\sum_{n\geq 0}\left(\frac{1}{z}\right)^n = -\sum_{n\geq 0}\left(\frac{1}{z}\right)^{n+1} = -\sum_{n\geq 0}\frac{1}{z^{n+1}},\ \forall z \in \{z \in \mathbb{C},\ 1 < |z| < \infty\}.$$

Example 3.6.3 *Let us find the Laurent series expansion for the function f defined on $\mathbb{C} \setminus \{1, 2\}$ by*

$$f(z) = \frac{1}{(1-z)(z-2)},$$

1) *in the domain $\{z \in \mathbb{C},\ |z| < 1\}$.*

2) *In the domain $\{z \in \mathbb{C},\ 1 < |z| < 2\}$.*

3) *In the domain $\{z \in \mathbb{C},\ 2 < |z| < \infty\}$.*

We have

$$f(z) = \frac{1}{(1-z)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1},\ \forall z \in \mathbb{C} \setminus \{1, 2\}.$$

Then

1) *If z is a complex number such that $|z| < 1$, then*

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n\geq 0} z^n,\ |z| < 1.$$

$$\frac{1}{z-2} = \frac{1}{2\left(\frac{z}{2}-1\right)} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}}.$$

Since $\left|\frac{z}{2}\right| < 1$, then

$$\frac{1}{1-\frac{z}{2}} = \sum_{n\geq 0}\left(\frac{z}{2}\right)^n \text{ and } \frac{1}{z-2} = -\sum_{n\geq 0}\frac{z^n}{2^{n+1}}.$$

Hence,

$$f(z) = \sum_{n\geq 0} z^n - \sum_{n\geq 0}\frac{z^n}{2^{n+1}} = \sum_{n\geq 0}\left(1 - \frac{1}{2^{n+1}}\right)z^n,\ \forall z \in \{z \in \mathbb{C},\ |z| < 1\}.$$

2) *If z is a complex number such that $1 < |z| < 2$, then*

$$-\frac{1}{z-1} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n\geq 0}\left(\frac{1}{z}\right)^n, \left|\frac{1}{z}\right| < 1.$$

$$\frac{1}{z-2} = \frac{1}{2\left(\frac{z}{2}-1\right)} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\sum_{n\geq 0}\frac{z^n}{2^{n+1}}.$$

Hence,

$$f(z) = -\sum_{n\geq 0}\frac{1}{z^{n+1}} - \sum_{n\geq 0}\frac{z^n}{2^{n+1}} = -\sum_{n\geq 0}\left(\frac{z^n}{2^{n+1}} + \frac{1}{z^{n+1}}\right), \ \forall z \in \{z \in \mathbb{C}, \ 1 < |z| < 2\}.$$

3) *If z be a complex number such that $2 < |z| < \infty$, then*

$$-\frac{1}{z-1} = -\sum_{n\geq 0}\frac{1}{z^{n+1}}$$

and

$$\frac{1}{z-2} = \frac{1}{z\left(1-\frac{2}{z}\right)} = \frac{1}{z}\sum_{n\geq 0}\left(\frac{2}{z}\right)^n = \sum_{n\geq 0}\frac{2^n}{z^{n+1}}, \left|\frac{2}{z}\right| < 1.$$

Hence,

$$f(z) = \sum_{n\geq 0}(2^n - 1)\frac{1}{z^{n+1}}, \ \forall z \in \{z \in \mathbb{C}, \ 2 < |z| < \infty\}.$$

3.7 Zeros of holomorphic functions

Definition 3.7.1 *Let $z_0 \in \mathbb{C}$ and let f be a complex-valued function. Suppose that f is holomorphic in a neighbourhood of z_0. We say that z_0 is a zero of f if $f(z_0) = 0$.*
We say that z_0 is a zero of order $k \in \mathbb{N}^\star$ of f if $f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$, but $f^{(k)}(z_0) \neq 0$.

Remark 3.7.1 *If z_0 is a zero of order $k = 1$ of f, then we say that z_0 is a simple zero of f.*

Example 3.7.1 **1)** *$z_k = k\pi$, $k \in \mathbb{Z}$ are simple zeros of the function $z \longmapsto \sin z$.*

2) *$z_0 = 0$ is a zero of order $k = 3$ of the function f defined by $f(z) = z\sin(z^2)$.*

Theorem 3.7.1 *Let $E \subset \mathbb{C}$ be a domain, $z_0 \in E$ and $f : E \longrightarrow \mathbb{C}$ be a holomorphic function. Then f has a zero of order $k \in \mathbb{N}^\star$ if and only if $f(z) = (z - z_0)^k g(z)$, $\forall z \in E$, where g is holomorphic at z_0 and $g(z_0) \neq 0$.*

Corollary 3.7.1 ♣ *If f and g are two complex-valued functions such that f (respectively, g) has a zero of order k (respectively, m) at $z_0 \in \mathbb{C}$, then fg has a zero of order $k+m$ at z_0.*

♣ *If f is holomorphic at z_0 and $f(z_0) = 0$, then either f is identically zero in a neighbourhood of z_0 or there is a punctured disk about z_0 in which f has no zeros, that is the zeros of a holomorphic function are isolated.*

Theorem 3.7.2 *Let γ be a positively oriented contour and let f be a complex-valued function. Suppose that f is holomorphic inside and on γ and $f(z) \neq 0$ on γ. Then*

$$z_f = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz$$

where z_f is the number of zeros (counting with multiplicities) of f that lie inside γ.

3.8 Analytic continuation

Definition 3.8.1 *Let $E, D \subset \mathbb{C}$ be non-empty open subsets such that $E \subset D$. If f and F are analytic complex-valued functions defined on E and D respectively such that $F(z) = f(z)$ for all $z \in E$, then we say that F is an analytic continuation of f, that is f is a restriction of F to E.*

Theorem 3.8.1 *Let $E \subset \mathbb{C}$ be a connected region. Suppose that f and g are two analytic complex-valued functions on E. If $f(z_n) = g(z_n)$ on a set of points $z_n \in A$ that converges to a point $z_0 \in E$, then $f(z) = g(z)$, $\forall z \in A$.*

Remark 3.8.1 *If analytic continuation exists, then it is unique.*

Proposition 3.8.1 *Let $E, D \subset \mathbb{C}$ be two domains. Suppose that F is the analytic continuation of f to D, then the single-valued function G defined by*

$$G(z) = \begin{cases} f(z), & z \in E \\ F(z), & z \in D \end{cases}$$

is analytic in the domain $E \cup D$ and G is the analytic continuation of either f or F to $E \cup D$. f and F are called elements of G.

3.9 Singularities

Definition 3.9.1 *Let $z_0 \in \mathbb{C}$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex-valued function. We say that z_0 is a singular point of f if f is not analytic at z_0 but it is analytic at some point in the open disk $B(z_0, r)$, for all $r > 0$.*
We say that a singular point z_0 of f is isolated if there exists $r > 0$ such that f is analytic on some punctured open disk $\{z \in \mathbb{C},\ 0 < |z - z_0| < r\}$.

Definition 3.9.2 *$f : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex-valued function and $\rho > 0$. We say that f has an isolated singularity point at ∞ if f is analytic in $\{z \in \mathbb{C},\ \rho < |z| < \infty\}$.*

Example 3.9.1 *Let $f(z) = \frac{e^z}{z(z^2+1)}$, then the singular points of f are $z = 0$ and $z = \pm i$ and they are isolated.*

Example 3.9.2 *Each point in $]-\infty, 0]$ is a singular point of the function $z \longmapsto Log z$ and all these points are not isolated.*

Let z_0 be an isolated singular point of f, then f has the following Laurent series expansion around z_0.

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n,\ \forall z \in \{z \in \mathbb{C},\ 0 < |z - z_0| < r\},\ r > 0. \tag{3.4}$$

Definition 3.9.3 **1)** *We say that z_0 is a removable singularity or regular point of f if $a_n = 0,\ \forall n < 0$.*

2) *We say that z_0 is a pole of f of order $k \in \mathbb{N}^\star$ if $a_{-k} \neq 0$, but $a_n = 0,\ \forall n < -k$.*
If $k = 1$ we say that z_0 is a simple pole of f.

3) *We say that z_0 is an essential singularity of f if $a_n \neq 0$ for an infinite number of negative values of n.*

Remark 3.9.1 • *If z_0 is a removable singularity of f, then its Laurent series takes the form*

$$f(z) = a_0 + a_1(z - z_0) + \cdots,\ 0 < |z - z_0| < r \text{ and } \lim_{z \to z_0} f(z) = a_0.$$

Theorem 3.9.1 *Let $z_0 \in \mathbb{C}$ and f be a complex-valued function. Then z_0 is a removable singularity if and only if any one of the following conditions holds*

1) *f has a limit as z approaches z_0.*

2) *f can be redefined at z_0 so that the new function is analytic at z_0.*

3) $\lim\limits_{z \to z_0} (z - z_0) f(z) = 0.$

4) *f is bounded in some punctured neighbourhood of z_0.*

Example 3.9.3 *Let $f(z) = \frac{\sin z}{z}$, then $z_0 = 0$ is a removable singularity of f and we can remove the singularity by defining F at z_0 by $F(z_0) = \lim\limits_{z \to z_0} f(z) = 1$.*

Remark 3.9.2 *Let z_0 be a pole of order k of f, then the Laurent series of f takes the form*

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \frac{a_{-(k-1)}}{(z - z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots , \ a_{-k} \neq 0$$

which is valid in some punctured neighbourhood of z_0.

Example 3.9.4 **1)** *Let $f(z) = \frac{\sin z}{z^2}$, then*

$$f(z) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots ,$$

then $z_0 = 0$ is a simple pole of f.

2) *Let $f(z) = \frac{\sin z}{z^3}$, then*

$$f(z) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots ,$$

then $z_0 = 0$ is a pole of order 2 of f.

Theorem 3.9.2 *Let $z_0 \in \mathbb{C}$ and f be a complex-valued function. Then z_0 is a pole of order k of f if and only if*

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

in some punctured neighbourhood of z_0 where g is analytic at z_0 and $g(z_0) \neq 0$.

Example 3.9.5 *Let $f(z) = \frac{e^z}{(z-1)^3(z^2+1)}$, then*

♠ $z_0 = 1$ *is a pole of order* 3 *of* f, *since* $f(z) = \frac{g(z)}{(z-1)^3}$, *where* $g(z) = \frac{e^z}{z^2+1}$, $g(1) = \frac{e}{2} \neq 0$.

♠ $z_0 = i$ *is a simple pole of* f, *since* $f(z) = \frac{g(z)}{z-i}$, *where* $g(z) = \frac{e^z}{(z-1)^3(z+i)}$, $g(i) = \frac{e^i}{(i-1)^3 2i} \neq 0$.

♠ $z_0 = -i$ *is a simple pole of* f, *since* $f(z) = \frac{g(z)}{z+i}$, *where* $g(z) = \frac{e^z}{(z-1)^3(z-i)}$, $g(-i) = -\frac{e^{-i}}{(-i-1)^3 2i} \neq 0$.

Corollary 3.9.1 ♣ *If* f *and* g *have poles of orders* k *and* m *respectively at* z_0, *then* fg *has a pole of order* $k+m$ *at* z_0.

♣ *If* f *has a zero of order* k *at* z_0, *then* $\frac{1}{f}$ *has a pole of order* k *at* z_0.

Example 3.9.6 *Let* $f(z) = \frac{1}{z(z-1)^2}$. *It is clear that* $z_0 = 0$ *is a simple zero of* $\frac{1}{f}$, *then* $z_0 = 0$ *is a simple pole of* f. *We have also,* $z_1 = 1$ *is a zeros of order* 2 *of* $\frac{1}{f}$, *then* $z_1 = 1$ *is a pole of order* 2 *of* f.

Proposition 3.9.1 *Let* $z_0 \in \mathbb{C}$ *and* f, g *be two complex-valued functions. Suppose that* f *and* g *have zeros of orders* k *and* m *respectively, then the function* $\frac{f}{g}$ *has*

1) *a removable singularity at* z_0 *if* $k \geq m$.

2) *a pole of order* $m-k$ *at* z_0 *if* $k < m$.

Corollary 3.9.2 *A rational complex-valued function has only removable singularities or poles.*

Proposition 3.9.2 *Let* $z_0 \in \mathbb{C}$ *and* f *be a complex-valued functions. Then* z_0 *is a pole of* f *if and only if* $\lim_{z \to z_0} \left| f(z) \right| = \infty$.

Definition 3.9.4 *Let* f *be a complex-valued function. We say that* $z_0 = \infty$ *is*

1) *a removable singularity of* f *if* $w = 0$ *is a removable singularity of* $F(w)$ *where* $F(w) = f\left(\frac{1}{w}\right)$.

2) *A pole of order* k *if* $w = 0$ *is a pole of order* k *of* $F(w)$ *where* $F(w) = f\left(\frac{1}{w}\right)$.

3) *An essential singularity of* f *if* $w = 0$ *is an essential singularity of* $F(w)$ *where* $F(w) = f\left(\frac{1}{w}\right)$.

Remark 3.9.3 *Suppose that* f *has a removable singularity at* $z_0 = \infty$ *and* $\lim_{z \to \infty} f(z) = 0$, *then we say that* f *has a zero at* $z_0 = \infty$.

Example 3.9.7 **1)** *Let* $f(z) = \frac{z^2+1}{z^2}$. *Let us consider the function* F *defined by* $F(w) = f\left(\frac{1}{w}\right)$, *then*

$$F(w) = \frac{\left(\frac{1}{w}\right)^2 + 1}{\left(\frac{1}{w}\right)^2} = w^2 + 1.$$

Then $w = 0$ *is a removable singularity of* F, *hence* $z_0 = \infty$ *is a removable singularity of* f. *In addition,* $\lim\limits_{z \to \infty} f(z) = 1$, *then* f *dose not have a zero at* $z_0 = \infty$.

2) *Let* $g(z) = z^3$. *Let us consider the function* G *defined by*

$$G(w) = g\left(\frac{1}{w}\right) = \left(\frac{1}{w}\right)^3 = \frac{1}{w^3}.$$

Then $w = 0$ *is a pole of order* 3 *of* G, *hence* $z_0 = \infty$ *is a pole of order* 3 *of* g.

Theorem 3.9.3 *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be an entire function. Then*

1) f *has a removable singularity at* $z_0 = \infty$ *if and only if it is constant.*

2) f *has a pole of order* k *at* $z_0 = \infty$ *if and only if it is a polynomial of degree* k.

3) f *is rational if and only if it has at most poles on the extended complex plane* $\mathbb{C} \cup \{\infty\}$.

Definition 3.9.5 *Let* $E \subset \mathbb{C}$ *be a region and let* $f : E \longrightarrow \mathbb{C}$ *be a complex-valued function. We say that* f *is meromorphic if it is analytic in* E *except for poles.*

3.10 Exercises

Exercise 3.10.1 *Let* γ *be the curve given by* $z : [0, 1] \longrightarrow \mathbb{C}$, $z(t) = t + it^2$, $t \in [0, 1]$ *and* $f(z) = z - 1$. *Compute the integral* $\int_\gamma f(z)dz$.

Exercise 3.10.2 *Compute the integral* $\int_\gamma f(z)dz$ *where* $f(z) = z + \frac{1}{z}$, $\forall z \in \mathbb{C}^\star$ *and* γ *is the upper semi-circle at the origin of radius* $r = 1$.

Exercise 3.10.3 *Let* $\gamma = \gamma_1 + \gamma_2$ *where* γ_1 *is given by the rang of* $z_1(t) = t$, $t \in [0, 1]$ *and* γ_2 *is given by the rang of* $z_2(t) = 1 + i(t - 1)$, $t \in [1, 2]$.
Compute the integral $\int_\gamma f(z)dz$ *where* $f(z) = z - 1$, $\forall z \in \mathbb{C}$.

Exercise 3.10.4 *Let γ be the curve given by $z : [0, 2\pi] \longrightarrow \mathbb{C}$, $z(t) = 2e^{it}$, $t \in [0, 2\pi]$. Show that*

$$\left|\int_{\gamma} \frac{e^z}{z^2+1} dz\right| \leq \frac{4\pi e^2}{3}.$$

Exercise 3.10.5 *Compute the integral $\int_{\gamma} e^z dz$ where γ is the part of the unit circle joining* 1 *to i in the counterclockwise direction.*

Exercise 3.10.6 *Compute the integral $\int_{\gamma} \frac{e^z}{z^2-16} dz$ where γ is the unit circle traversed once in the counterclockwise direction.*

Exercise 3.10.7 *Compute the integral $\int_{\gamma} \frac{e^{2z}+\sin z}{z-\pi} dz$ where γ is the circle $\{z \in \mathbb{C}, |z-2| = 2\}$ traversed once in the counterclockwise direction.*

Exercise 3.10.8 *Compute the integral $\int_{\gamma} \frac{\sin(3z)}{z^4} dz$ where γ is the unit circle traversed once in the counterclockwise direction.*

Exercise 3.10.9 *Let γ be the contour given by $z = e^{i\theta}, \theta \in [-\pi, \pi]$ traversed in the positive direction.*

1) *Show that $\int_{\gamma} \frac{e^{\alpha z}}{z} dz = 2\pi i$, $\forall \alpha \in \mathbb{R}$.*

2) *Deduce that $\int_0^{\pi} e^{\alpha \cos\theta} \cos(\alpha \sin\theta) d\theta = \pi$.*

Exercise 3.10.10 *Compute the radius of convergence of the following series*

$$\sum_{n\geq 0} \frac{z^n}{(1+3i)^{n+1}}, \ \sum_{n\geq 0} \frac{(n!)^2}{(2n)!}(z-i)^n, \ \sum_{n\geq 0} \left(\frac{11n+9}{2n+5}\right)^n z^n, \ \sum_{n\geq 0} \left(\frac{4n^2}{2n+1} - \frac{6n^2}{3n+4}\right)^n (z-3i)^n.$$

Exercise 3.10.11 *Expand in Maclaurin series each of the following functions*

$$z \mapsto f(z) = \frac{z}{z^2+4}, \ z \mapsto g(z) = \frac{1}{z^2-1}, \ z \mapsto h(z) = \frac{1}{z^2+1}.$$

Exercise 3.10.12 *Expand in Laurent series on the indicated domain each of the following functions*

1) $f(z) = \frac{1}{z^2+1}$, *in a punctured disk centered at $z_0 = i$.*

2) $f(z) = \frac{1}{(z-1)(z-2)(z-3)}$ *in the domain $\{z \in \mathbb{C}, 1 < |z| < 2\}$.*

Exercise 3.10.13 *Determine the zeros and the order of each of them of the following functions*

$$z \mapsto f(z) = e^{2z} - e^{z}, \ \ z \mapsto g(z) = z^2 \sinh z, \ \ z \mapsto h(z) = z^4 \cos^2 z, \ \ z \mapsto H(z) = z^3 \cos z^2.$$

Exercise 3.10.14 *Determine and classify the singularities of each of the following functions*

$$z \mapsto f(z) = \frac{z^2+2}{z^3(z+i)}, \ \ z \mapsto g(z) = z^2 e^{\frac{1}{z}}, \ \ z \mapsto h(z) = \frac{z+1}{\sin z},$$

$$z \mapsto H(z) = \frac{(z^2-1)\cos(\pi z)}{(z+2)(2z-1)(z^2+1)\sin^2(\pi z)}.$$

Chapter 4

The residue theorem and its applications

In this chapter, we start by introducing the notion of residue of a complex-valued function at a singular point z_0, then we present the Cauchy residue theorem and show how we can apply this important theorem to compute some real integrals. We end the chapter by giving some exercises.

4.1 Cauchy's residue theorem

Let f be a complex-valued function.

Definition 4.1.1 *Let z_0 be an isolated singular point of f. Suppose that f can be expanded in Laurent series around z_0*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (4.1)$$

The residue of f at z_0 is the coefficient a_{-1} in (4.1) *and we write $Res(f, z_0) = a_{-1}$.*

Example 4.1.1 **1)** *Let $f(z) = e^{\frac{1}{z}}$, then $z_0 = 0$ is an isolated singular point of f and for all $z \in \mathbb{C}$ such that $0 < |z| < \infty$, we have*

$$f(z) = e^{\frac{1}{z}} = \sum_{n \geq 0} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots ,$$

then $Res(f, 0) = 1$.

2) *Let* $f(z) = \frac{\sin z}{z}$, *then* $z_0 = 0$ *is an isolated singular point of* f *and for all* $z \in \mathbb{C}$ *such that* $0 < |z| < \infty$, *we have*

$$f(z) = \frac{\sin z}{z} = \sum_{n\geq 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots,$$

then $Res(f, 0) = 0$.

Theorem 4.1.1 *Suppose that* f *has a pole of order* $k \in \mathbb{N}^\star$ *at* z_0. *Then*

$$Res(f, z_0) = \lim_{z\to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z - z_0)^k f(z)\right).$$

Remark 4.1.1 • *If* f *has a simple pole at* z_0, *then*

$$Res(f, z_0) = \lim_{z\to z_0} (z - z_0) f(z).$$

• *If* f *has a simple pole at* z_0 *and* g *is a complex-valued function such that* g *is holomorphic at* z_0, *then*

$$Res(fg, z_0) = g(z_0) Res(f, z_0).$$

Example 4.1.2 **1)** *Let* $f(z) = \frac{e^z}{z(z-2)}$. *It is clear that* f *has simple poles at* $z_0 = 0$ *and* $z_1 = 2$, *then*

$$Res(f, 0) = \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{e^z}{z-2} = -\frac{1}{2}.$$

And

$$Res(f, 2) = \lim_{z\to 2} (z - 2) f(z) = \lim_{z\to 2} \frac{e^z}{z} = \frac{e^2}{2}.$$

2) *Let* $f(z) = \frac{\cos z}{z^2(z-\pi)^3}$. *It is clear that* f *has a pole of order* $k = 2$ *at* $z_0 = 0$, *then*

$$\begin{aligned} Res(f, 0) &= \lim_{z\to 0} \frac{1}{1!} \frac{d}{dz} \left(z^2 f(z)\right) \\ &= \lim_{z\to 0} \frac{1}{1!} \frac{d}{dz} \left(\frac{\cos z}{(z-\pi)^3}\right) \\ &= \lim_{z\to 0} \frac{-(z-\pi)^3 \sin z - 3(z-\pi)^2 \cos z}{(z-\pi)^6} \\ &= -\frac{3}{\pi^4}. \end{aligned}$$

Proposition 4.1.1 *Let P and Q be two complex-valued functions. Suppose that P and Q are holomorphic at z_0, Q has a simple zero at z_0 and $P(z_0) \neq 0$. Let f be the function defined by $f(z) = \frac{P(z)}{Q(z)}$, then*

$$Res(f, z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Example 4.1.3 *Let $f(z) = \frac{\sin z}{\cos z}$, $z_0 = \frac{\pi}{2}$. It is clear that $f(z) = \frac{P(z)}{Q(z)}$ where $P(z) = \sin z$ and $Q(z) = \cos z$. P and Q are holomorphic at $z_0 = \frac{\pi}{2}$, Q has a simple zero at $z_0 = \frac{\pi}{2}$ and $P\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \neq 0$. Then*

$$Res\left(f, \frac{\pi}{2}\right) = \frac{P\left(\frac{\pi}{2}\right)}{Q'\left(\frac{\pi}{2}\right)} = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = -1.$$

Theorem 4.1.2 (Cauchy's residue theorem) *Let γ be a positively oriented simple closed contour. Suppose that f is holomorphic inside and on γ except at the points $z_1, z_2, \ldots, z_n$ inside γ, then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res\left(f, z_k\right).$$

Example 4.1.4 *Compute the integral $\int_\gamma f(z)dz$ where $f(z) = \frac{1}{z^4+1}$ and γ is the contour in Figure 4.1*

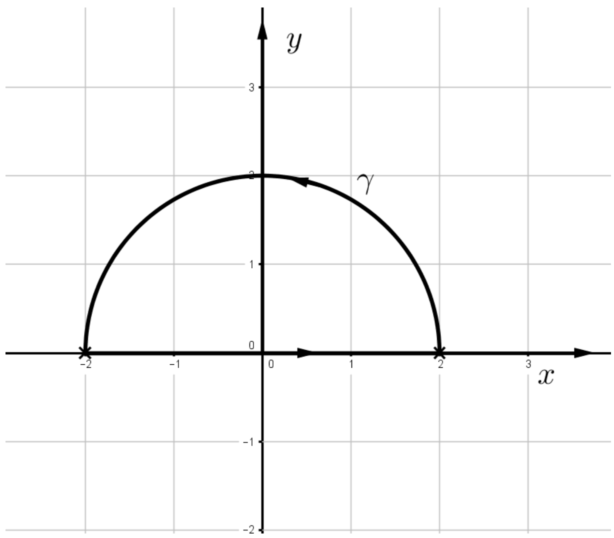


Figure 4.1:

Let $f(z) = \frac{1}{z^4+1}$*, then* f *has four singularities* $z_1 = e^{i\frac{\pi}{4}}$, $z_2 = e^{i\frac{3\pi}{4}}$, $z_3 = e^{i\frac{5\pi}{4}}$ *and* $z_4 = e^{i\frac{7\pi}{4}}$*, but only* z_1 *and* z_2 *are inside* γ*. By using Cauchy's residue theorem, we get*

$$\int_\gamma f(z)dz = 2\pi i\left(Res\left(f, e^{i\frac{\pi}{4}}\right) + Res\left(f, e^{i\frac{3\pi}{4}}\right)\right).$$

We have $f(z) = \frac{P(z)}{Q(z)}$ *where* $P(z) = 1$ *and* $Q(z) = z^4 + 1$*. We can easily see that* Q *has simple zeros at* z_1 *and* z_2 *and* $P(z_1) = P(z_2) \neq 0$*, then*

$$Res\left(f, e^{i\frac{\pi}{4}}\right) = \frac{P(z_1)}{Q'(z_1)} = \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3} \text{ and } Res\left(f, e^{i\frac{3\pi}{4}}\right) = \frac{P(z_2)}{Q'(z_2)} = \frac{1}{4\left(e^{i\frac{3\pi}{4}}\right)^3},$$

hence

$$\int_\gamma f(z)dz = 2\pi i\left(\frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3} + \frac{1}{4\left(e^{i\frac{3\pi}{4}}\right)^3}\right) = \frac{2\pi i}{4}\left(e^{-i\frac{3\pi}{4}} + e^{-i\frac{9\pi}{4}}\right) = \frac{\pi}{\sqrt{2}}.$$

4.2 Residue at infinity

Definition 4.2.1 *Let* f *be a complex-valued function. Suppose that* f *has an isolated singularity point at* ∞*, that is* f *is analytic in* $\{z \in \mathbb{C},\ \rho < |z| < \infty\}$ *where* $\rho > 0$*. The residue of* f *at* ∞ *is given by*

$$Res(f, \infty) = \frac{1}{2\pi i}\int_{\gamma_r} f(z)dz,$$

where $r > \rho$ *and* $\gamma_r = \{z \in \mathbb{C},\ |z| = r\}$ *negatively oriented.*

Definition 4.2.2 *Let* $\rho > 0$*. Suppose that for all* $z \in \mathbb{C}$ *such that* $\rho < |z| < \infty$*, the complex-valued function* f *can be expanded in a Laurent series*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n = \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots \tag{4.2}$$

The residue of f *at* $z_0 = \infty$ *is given by* $Res(f, \infty) = -a_{-1}$*.*

Theorem 4.2.1 *Let* f *be a complex-valued function. Suppose that* f *is holomorphic except for a finite number of singular points interior to a positively oriented simple closed contour* γ*. Then*

$$\int_\gamma f(z)dz = 2\pi i Res\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right). \tag{4.3}$$

Theorem 4.2.2 *Let f be a holomorphic complex-valued function in $\mathbb{C} \cup \{\infty\}$ except at the isolated singular points $z_1, z_2, \dots, z_n$, then*

$$\sum_{k=1}^{n} Res(f, z_k) + Res(f, \infty) = 0. \tag{4.4}$$

Corollary 4.2.1 *Let f be a complex-valued function. suppose that f is holomorphic except for a finite number of singularities. From Theorem 4.1.2, Theorem 4.2.1 and Theorem 4.2.2, we have*

$$Res(f, \infty) = -Res\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Example 4.2.1 *Let us compute the integral $\int_\gamma \frac{dz}{z^{15}-1}$ where γ is the circle $\{z \in \mathbb{C}, |z| = 2\}$ traversed once in the counterclockwise direction.*
Let $f(z) = \frac{1}{z^{15}-1}$, it is clear that f has 15 simple poles and all of them are inside γ, then by using the Cauchy residue theorem, we get

$$I = \int_\gamma \frac{dz}{z^{15}-1} = 2\pi i \sum_{k=1}^{15} Res(f, z_k),$$

but we know that

$$\sum_{k=1}^{15} Res(f, z_k) + Res(f, \infty) = 0,$$

then

$$\sum_{k=1}^{n} Res(f, z_k) = -Res(f, \infty),$$

hence

$$I = \int_\gamma \frac{dz}{z^{15}-1} = -2\pi i Res(f, \infty).$$

Let us evaluate $Res(f, \infty)$. We have

$$f\left(\frac{1}{z}\right) = \frac{1}{\left(\frac{1}{z}\right)^{15} - 1} = \frac{z^{15}}{1 - z^{15}} = z^{15} \sum_{n \geq 0} \left(z^{15}\right)^n = \sum_{n \geq 0} z^{15n+15},$$

then

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n \geq 0} z^{15n+13}.$$

Hence

$$Res(f, \infty) = -Res\left(\frac{1}{z^2} f(z), 0\right) = 0 \text{ and } I = 0.$$

4.3 Integrals of rational functions

Definition 4.3.1 *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous real function. The improper integral of g over $]-\infty, +\infty[$ is defined by*

$$\int_{-\infty}^{+\infty} g(x)dx = \lim_{a\to+\infty}\int_{-a}^{0} g(x)dx + \lim_{b\to+\infty}\int_{0}^{b} g(x)dx$$

provided these limits exist.

Theorem 4.3.1 *Let f be a rational complex-valued function such that $f(z) = \frac{P(z)}{Q(z)}$ where P and Q do not have a common zero, Q has no zeros on the real axis and degree $Q \geq 2+$ degree P, then*

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum_k Res\,(f, z_k)$$

where z_k are the singular points of f situated in the upper half-plane, $\Im(z_k) > 0$.

Example 4.3.1 *Let us compute the integral $I = \int_{-\infty}^{+\infty} \frac{dx}{x^4+16}$.*
First, we consider the function defined by $f(z) = \frac{1}{z^4+16}$. We have $f(z) = \frac{P(z)}{Q(z)}$ where $P(z) = 1$ and $Q(z) = z^4 + 16$. We can easily see that P and Q do not have a common zero, Q has no zeros on the real axis and degree $Q \geq 2+$ degree P. In addition, the singular points of f are $z_0 = 2e^{i\frac{\pi}{4}}$, $z_1 = 2e^{i\frac{3\pi}{4}}$, $z_2 = 2e^{i\frac{5\pi}{4}i}$ and $z_3 = 2e^{i\frac{7\pi}{4}}$ and they are simple poles. It is clear that only z_0 and z_1 are in the upper half-plane. Then by Theorem 4.3.1, we get

$$I = 2\pi i\,(Res(f, z_0) + Res(f, z_1))\,.$$

Since

$$Res(f, z_0) = \frac{P(z_0)}{Q'(z_0)} = \frac{1}{4z_0^3} = \frac{1}{32e^{i\frac{3\pi}{4}}}$$

and

$$Res(f, z_1) = \frac{P(z_1)}{Q'(z_1)} = \frac{1}{4z_1^3} = \frac{1}{32e^{i\frac{9\pi}{4}}},$$

then

$$I = \frac{2\pi i}{32}\left(e^{-i\frac{3\pi}{4}} + e^{-i\frac{9\pi}{4}}\right) = \frac{\pi i}{16}\left(-\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}i\right) = \frac{\pi}{16\sqrt{2}}.$$

Example 4.3.2 *Let us evaluate the integral $I = \int_0^{+\infty} \frac{dx}{\left(x^2+4\right)^2}$.*
It is clear that the function $x \longmapsto \frac{1}{\left(x^2+4\right)^2}$ is even, then $I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+4\right)^2}$. We consider the

function defined by $f(z) = \frac{1}{\left(z^2+4\right)^2}$. *We have* $f(z) = \frac{P(z)}{Q(z)}$ *where* $P(z) = 1$ *and* $Q(z) = (z^2+4)^2$. *We can easily see that P and Q do not have a common zero, Q has no zeros on the real axis and degree* $Q \geq 2+$ *degree P. In addition, the singular points of f are* $z_0 = 2i$ *and* $z_1 = -2i$ *and each of them is a pole of order two. We can also, see that only* z_0 *is in the upper half-plane. Then by Theorem 4.3.1, we get*

$$I = \pi i Res(f, z_0) = \pi i Res(f, 2i).$$

Since

$$Res(f, 2i) = \lim_{z\to 2i} \frac{d}{dz}\left((z-2i)^2 f(z)\right) = \lim_{z\to 2i} \frac{d}{dz}\frac{1}{(z+2i)^2} = \lim_{z\to 2i}\frac{-2}{(z+2i)^3} = \frac{1}{32i},$$

then $I = \frac{\pi}{32}$.

4.4 Trigonometric integrals

In this section, we consider the integral

$$J = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

where R is a rational function with real coefficients of $\cos\theta$ and $\sin\theta$ and whose denominator does not vanish on $[0, 2\pi]$. We put $z = e^{i\theta}$, then

$$\frac{1}{z} = e^{-i\theta},\ \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),\ \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \text{ and } \frac{dz}{d\theta} = ie^{i\theta} = iz.$$

Hence

$$J = \int_\gamma R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz}$$

where γ is the positively oriented unit circle.

Example 4.4.1 *Let us evaluate the integral* $J = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$. *Let* $z = e^{i\theta}$, *then*

$$\frac{1}{z} = e^{-i\theta},\ \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \text{ and } \frac{dz}{d\theta} = ie^{i\theta} = iz.$$

Hence

$$2 + \sin\theta = 2 + \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2 + 4iz - 1}{2iz}.$$

Therefore

$$J = \int_\gamma \frac{d\theta}{2+\sin\theta} = \int_\gamma \frac{2}{z^2+4iz-1}dz$$

where γ *is the positively oriented unit circle. It is clear that the function defined by* $g(z) = \frac{1}{z^2+4iz-1}$ *has two simple poles* $z_0 = -\left(2+\sqrt{3}\right)i$ *and* $z_1 = \left(-2+\sqrt{3}\right)i$ *but only* z_1 *is inside* γ *since* $|z_0| = 2+\sqrt{3} > 1$ *and* $|z_1| = \sqrt{3}-2 < 1$, *then by the Cauchy residue theorem, we get*

$$J = 4\pi i Res(g, z_1) = 4\pi i \lim_{z\to\to z_1}(z-z_1)g(z) = 4\pi i \lim_{z\to(-2+\sqrt{3})i}\left(\frac{1}{z+2i+\sqrt{3}i}\right) = \frac{2\pi}{\sqrt{3}}$$

4.5 Evaluation of integrals involving multi-valued functions

In this section, we will evaluate the integral $\int_0^{+\infty} x^{\alpha-1}f(x)dx$ where $0 < \alpha < 1$, and the function f satisfies the following hypothesises

$H_1:\ z \longmapsto f(z)$ is a single-valued analytic function except for a finite number of isolated singularities not on the positive real semiaxis.

$H_2:\ z \longmapsto f(z)$ has a zero of order at least one at $z = \infty$.

$H_3:\ z \longmapsto f(z)$ has a removable singularity at $z = 0$.

Let E be the domain $\{z \in \mathbb{C},\ 0 < \arg(z) < 2\pi\}$. It is clear that $z \longmapsto z^{\alpha-1}f(z)$ is single-valued in E and its singularities are the same as those of the function $z \longmapsto f(z)$. Let $R > 0$ be large enough so that all singularities z_k of the function $z \longmapsto f(z)$ lie inside the circle $\gamma_R = \{z \in \mathbb{C},\ |z| = R\}$ and let $r > 0$ be small enough so that all singularities z_k of the function $z \longmapsto f(z)$ lie outside the circle $\gamma_r = \{z \in \mathbb{C},\ |z| = r\}$. We consider the closed contour $\gamma = \gamma_r \cup [A,B] \cup \gamma_R \cup [C,D]$ in Figure 4.2

By the Cauchy residue theorem, we have

$$\int_\gamma z^{\alpha-1}f(z)dz = 2\pi i \sum_k Res\left(z^{\alpha-1}, z_k\right),$$

then

$$\int_r^R x^{\alpha-1}f(x)dx + \int_{\gamma_R} z^{\alpha-1}f(z)dz + \int_R^r z^{\alpha-1}f(z)dz + \int_{\gamma_r} z^{\alpha-1}f(z)dz = 2\pi i \sum_k Res\left(z^{\alpha-1}f, z_k\right),. \tag{4.5}$$

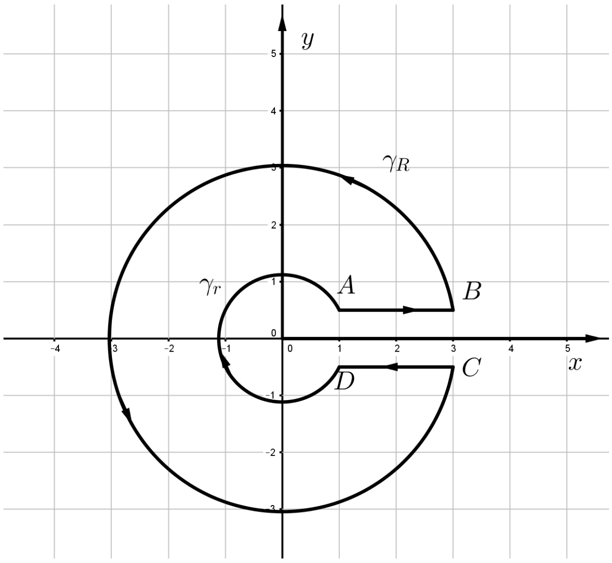


Figure 4.2:

From H_2 : we get $|f(z)| \leq \frac{M}{|z|}$ for some $M > 0$ and for all sufficiently large $|z|$, then

$$\left|\int_{\gamma_R} z^{\alpha-1} f(z)dz\right| \leq \frac{MR^{\alpha-1}}{R} 2\pi R = 2\pi M R^{\alpha-1} \underset{R\to+\infty}{\longrightarrow} 0.$$

From H_3 : we obtain

$$\left|\int_{\gamma_r} z^{\alpha-1} f(z)dz\right| \leq K r^{\alpha-1} 2\pi r \underset{r\to 0}{\longrightarrow} 0, \text{ where } K > 0.$$

On [C,D], we have $\arg(z) = 2\pi$, then $z = xe^{2\pi i}$, $x > 0$, hence

$$\int_R^r z^{\alpha-1} f(z)dz = -e^{2\pi i(\alpha-1)} \int_r^R x^{\alpha-1} f(x)dx.$$

By letting $r \to 0$ and $R \to +\infty$ in (4.6), we get

$$\int_0^{+\infty} x^{\alpha-1} f(x)dx - e^{2\pi i(\alpha-1)} \int_0^{+\infty} x^{\alpha-1} f(x)dx = 2\pi i \sum_k Res\left(z^{\alpha-1} f, z_k\right). \tag{4.6}$$

Then

$$\int_0^{+\infty} x^{\alpha-1} f(x)dx = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum_k Res\left(z^{\alpha-1} f, z_k\right). \tag{4.7}$$

Example 4.5.1 *Let us compute the integral* $I = \int_0^{+\infty} \frac{x^{\alpha-1}}{x+1} dx$ *where* $0 < \alpha < 1$. *It is clear that* $I = \int_0^{+\infty} x^{\alpha-1} f(x)dx$ *where* $f(x) = \frac{1}{x+1}$. *Then from* (4.7), *we obtain*

$$I = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum_k Res\left(z^{\alpha-1} f, z_k\right).$$

But the function $z \mapsto \frac{z^{\alpha-1}}{z+1}$ *has only one singularity at* $z_0 = -1$ *that lies inside the contour* γ *represented in Figure 4.2, (here, we choose* $r < 1$*), hence*

$$I = \frac{2\pi i}{1 - e^{2\pi i\alpha}} Res\left(\frac{z^{\alpha-1}}{z+1}, -1\right).$$

Since the point z_0 *is a simple pole of the function* $z \mapsto \frac{z^{\alpha-1}}{z+1}$*, then*

$$Res\left(\frac{z^{\alpha-1}}{z+1}, -1\right) = \lim_{z\to -1} z^{\alpha-1} = (-1)^{\alpha-1}.$$

By using the definition of the power complex-valued function, we get

$$I = \frac{2\pi i(-1)^{\alpha-1}}{1 - e^{2\pi i\alpha}} = \frac{2\pi i e^{\pi i(\alpha-1)}}{1 - e^{2\pi i\alpha}} = \frac{\pi}{\sin(\alpha\pi)}.$$

4.6 Exercises

Exercise 4.6.1 *Compute the following residues*

$$1)\, Res\left(\frac{z^3-1}{z-1}, 1\right), \quad 2)\, Res\left(\frac{e^{2z}}{(z-1)^3}, 1\right), \quad 3)\, Res\left(\frac{e^{z^3}}{z(z-1)}, 0\right), \quad 4)\, Res\left(\frac{e^{z^3}}{z(z-1)}, 1\right),$$

$$5)\, Res\left(\frac{1}{(z-4)(z+1)^3}, 4\right),\ 6)\, Res\left(\frac{1}{(z-4)(z+1)^3}, -1\right), 7)\, Res\left(\frac{z^2}{z+1}, \infty\right),\ 8)\, Res\left(\frac{2z^3+7}{z(z-1)^3}, \infty\right).$$

Exercise 4.6.2 *Use the residue theorem to compute the following integrals*

1) $I_1 = \int_\gamma \frac{z^2}{(z-1)^2(z+2)} dz$ *where* γ *is the counterclockwise circle around* $z_0 = 0$ *of radius* $r = 3$.

2) $I_2 = \int_\gamma \frac{dz}{z^4-1}$ *where* γ *is the rectangle defined by* $x = -0.5$, $x = 2$, $y = -2$ *and* $y = 2$.

3) $I_3 = \int_\gamma \frac{z}{(z^2-1)(z^2+1)} dz$ *where* γ *is the counterclockwise circle around* $z_0 = 1$ *of radius* $r = \sqrt{3}$.

4) $I_4 = \int_\gamma \tan(z) dz$ *where* γ *is the counterclockwise circle around* $z_0 = 0$ *of radius* $r = 2$.

Exercise 4.6.3 *Use the residue theorem to evaluate the following integrals*

$$I_1 = \int_{-\infty}^{+\infty} \frac{dx}{x^4+1}, \quad I_2 = \int_{-\infty}^{+\infty} \frac{dx}{x^2+x+1}, \quad I_3 = \int_0^{+\infty} \frac{dx}{(x^2+1)(x^2+4)}.$$

Exercise 4.6.4 *Use the residue theorem to evaluate the following integrals*

$$I_1 = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}, \quad I_2 = \int_0^{2\pi} \frac{\cos(3\theta)}{5-4\sin\theta} d\theta, \quad I_3 = \int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}.$$

Chapter 5

Applications

In this chapter, we present Liouville's theorem, the maximum modulus principle and Rouché's theorem. We end the chapter by giving some examples about calculus of integrals by using the residue theorem.

5.1 Liouville's theorem

Theorem 5.1.1 (Morera's theorem) *Let $E \subset \mathbb{C}$ be a domain and let $f : E \longrightarrow \mathbb{C}$ be a complex-valued function. If f is continuous on E and $\int_\gamma f(z)dz = 0$ for every simply closed contour γ in E, then f is analytic.*

Example 5.1.1 *Let us prove that the function $f : \mathbb{C}^\star \longrightarrow \mathbb{C}$ defined by $f(z) = \frac{1-e^{-z^2}}{z^2}$ is analytic. It is clear that $f(z) = \int_0^1 e^{-z^2 t}dt$. Let γ be a simple closed contour in $\mathbb{C}^\star$, then*

$$\int_\gamma f(z)dz = \int_\gamma \left(\int_0^1 e^{-z^2 t}dt\right)dz = \int_0^1 \left(\int_\gamma e^{-z^2 t}dz\right)dt = 0,$$

hence f is analytic.

Theorem 5.1.2 (Cauchy's inequality) *Let γ_R be the circle of radius $R > 0$ and centered at $z_0 \in \mathbb{C}$. Let f be a complex-valued function. Suppose that f is analytic inside and on γ_R and for all $z \in \gamma_R$ we have $|f(z)| \leq M$ for some $M > 0$, then*

$$\left|f^n(z_0)\right| \leq \frac{n!M}{R^n}, \ \forall n \in \mathbb{N}^\star.$$

Theorem 5.1.3 (Liouville's theorem) *The only bounded entire functions are the constant functions.*

5.2 Maximum modulus principle

Lemma 5.2.1 *Let f be a complex-valued function. Suppose that f is analytic on an open disk centered at z_0 and that the maximum value of $|f(z)|$ over this disk is $|f(z_0)|$, then $|f|$ is constant on the disk.*

Theorem 5.2.1 (Maximum modulus principle) *Let $E \subset \mathbb{C}$ be a domain. If the complex-valued function f is analytic on E and $|f(z)|$ achieves its maximum value at a point $z_0 \in E$, then f is constant on E.*

Theorem 5.2.2 *A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.*

Remark 5.2.1 *The maximum modulus principle fails on unbounded domains.*

Example 5.2.1 *Let us find the maximum of $|f(z)|$ where $f(z) = ze^z + z^2$ on the set $E = \{z \in \mathbb{C},\ |z| \leq 1$ and $\Im(z) \geq 0\}$. We have f is an entire function, so it is analytic on E, then by Theorem 5.2.2 the maximum of $|f(z)|$ occurs on the boundary of E. Moreover, if $|z| \leq 1$, we have*

$$|f(z)| \leq |z||e^z + z| = |e^z + z| \leq |e^z| + |z| \leq e^{|z|} + |z| = e + 1$$

and if $z = 1,\ |f(1)| = e + 1$.

5.3 Rouché's theorem

Theorem 5.3.1 *Let γ be a positively oriented contour and let f be a complex-valued function. Suppose that f is meromorphic inside and on γ and $f(z) \neq 0$ on γ. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Z_f - P_f$$

where Z_f (respectively, P_f) is the number of zeros (respectively, of poles) counting multiplicities of f that lie inside γ.

Theorem 5.3.2 (Rouché's theorem) *Let E be a domain and let f and g be two meromorphic complex-valued functions in E. Let γ be a simply closed positively oriented contour in E. Suppose that on γ we have $|f(z)| > |g(z)|$ and f and g have no zeros or poles on γ, then*

$$Z_f - P_f = Z_{f+g} - P_{f+g}.$$

Corollary 5.3.1 *Let E be a domain and γ be a simply closed positively oriented contour in E. If f and g are analytic on E and $|f(z)| > |g(z)|$ for all z on γ, then f and $f+g$ have the same number of zeros inside γ counting multiplicities.*

Example 5.3.1 *Show that the function defined by $h(z) = z^8 - 5z^5 - 2z + 1$ has five zeros inside the unit circle.*
Let $f(z) = -5z^5$ and $g(z) = z^8 - 2z + 1$, then for all z such that $|z| = 1$, we have

$$|f(z)| = 5|z|^5 = 5 \text{ and } |g(z)| \leq |z|^8 + 2|z| + 1 = 4,$$

that is

$$|f(z)| > |g(z)|, \ \forall z, \ |z| = 1.$$

By the Corollary 5.3.1, we obtain that f and $f+g = h$ have the same number of zeros inside the unit circle, hence h has five zeros inside the unit circle.

5.4 Evaluating real integrals using complex integration

Lemma 5.4.1 1. *For all $\theta \in]0, \frac{\pi}{2}[$, we have*

$$\frac{2}{\pi}\theta < \sin\theta. \tag{5.1}$$

2.

$$I = \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{5.2}$$

Proof.

1. Let f be the function defined by $f(\theta) = \sin\theta - \frac{2\theta}{\pi}$, $\forall\theta \in]0, \frac{\pi}{2}[$. Then for all $\theta \in]0, \frac{\pi}{2}[$, we have

$$f'(\theta) = \cos\theta - \frac{2}{\pi} \text{ and } f''(\theta) = -\sin\theta < 0,$$

hence

$$\min_{\theta\in]0,\frac{\pi}{2}[} f(\theta) = f(0) = f\left(\frac{\pi}{2}\right) = 0,$$

that is

$$f(\theta) = \sin\theta - \frac{2\theta}{\pi} > f(0) = 0,$$

then (5.1) is satisfied.

2. It is clear that

$$I^2 = \left(\int_0^{+\infty} e^{-x^2}dx\right)\left(\int_0^{+\infty} e^{-y^2}dy\right) = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}dxdy.$$

By using the polar coordinates

$$x = \rho\cos\theta \text{ and } y = \rho\sin\theta,\ \rho > 0,\ \theta \in \left[0, \frac{\pi}{2}\right],$$

we get

$$I^2 = \int_0^{\frac{\pi}{2}}\int_0^{+\infty} e^{-\rho^2}\rho d\rho d\theta = \frac{\pi}{2}\left[-\frac{1}{2}e^{-\rho^2}\right]_0^{+\infty} = \frac{\pi}{4}.$$

Then $I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.

■

Example 5.4.1 *Consider the Fresnel integrals*

$$J_1 = \int_0^{+\infty} \cos x^2 dx \text{ and } J_2 = \int_0^{+\infty} \sin x^2 dx.$$

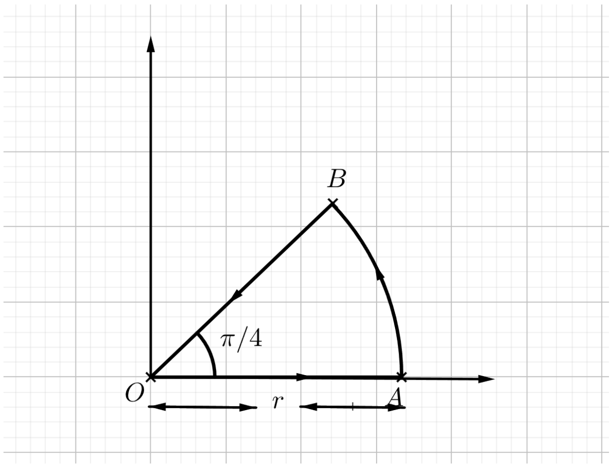
To compute the integrals J_1 and J_2, we consider the complex-valued function f defined by $f(z) = e^{iz^2}$ and the contour $\gamma = OA + AB + BO$ as in Figure below
It is clear that the function f is entire, then by using the Theorem 3.3.2, we obtain

$$\int_\gamma f(z)dz = \int_{OA} e^{iz^2}dz + \int_{AB} e^{iz^2}dz + \int_{BO} e^{iz^2}dz = 0. \tag{5.3}$$

◊ *On the line OA we have $z = x$, $0 \le x \le r$.*

◊ *On the arc AB we have $z = re^{i\theta}$, $0 \le \theta \le \frac{\pi}{4}$.*

◊ *On the line BO we have $z = \rho e^{i\frac{\pi}{4}}$, $0 \le \rho \le r$.*



Then (5.3) *becomes*

$$\int_0^r e^{ix^2}dx + \int_0^{\frac{\pi}{4}} e^{ir^2e^{2i\theta}} rie^{i\theta}d\theta + \int_r^0 e^{i\rho^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}} d\rho = 0. \tag{5.4}$$

The use of (5.1) *gives*

$$\left|\int_0^{\frac{\pi}{4}} e^{ir^2e^{2i\theta}} rie^{i\theta}d\theta\right| \leq r\int_0^{\frac{\pi}{4}} \left|e^{ir^2(\cos(2\theta)+i\sin(2\theta))}\right| d\theta = r\int_0^{\frac{\pi}{4}} e^{-r^2\sin(2\theta)}d\theta \leq r\int_0^{\frac{\pi}{4}} e^{-r^2\left(4\frac{\theta}{\pi}\right)}d\theta,$$

hence

$$\left|\int_0^{\frac{\pi}{4}} e^{ir^2e^{2i\theta}} rie^{i\theta}d\theta\right| \leq \frac{\pi}{4}\frac{1-e^{-r^2}}{r} \underset{r\to+\infty}{\longrightarrow} 0. \tag{5.5}$$

From (5.2), *we get*

$$\lim_{r\to+\infty}\int_r^0 e^{i\rho^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}} d\rho = -e^{i\frac{\pi}{4}}\int_0^{+\infty} e^{-\rho^2}d\rho = -\frac{1+i}{\sqrt{2}}\frac{\sqrt{\pi}}{2}. \tag{5.6}$$

By combining (5.4), (5.11) *and* (5.12), *we obtain*

$$\lim_{r\to+\infty}\int_0^r e^{ix^2}dx = \int_0^{+\infty}\left(\cos x^2 + i\sin x^2\right)dx = \frac{\sqrt{\pi}}{2\sqrt{2}} + i\frac{\sqrt{\pi}}{2\sqrt{2}},$$

then

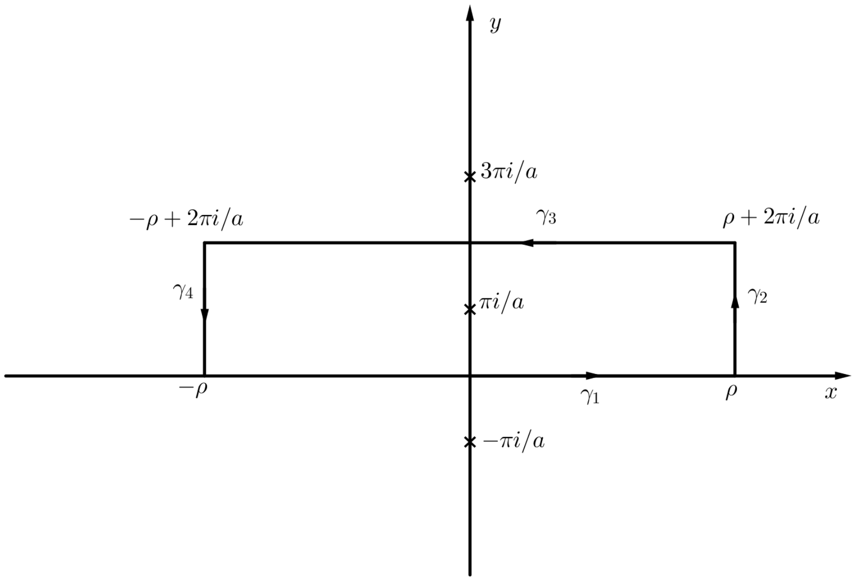
$$\int_0^{+\infty}\cos x^2 dx = \int_0^{+\infty}\sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Example 5.4.2 *Let us evaluate the integral*

$$J = \int_{-\infty}^{+\infty}\frac{e^x}{e^{ax}+1}dx,\ a>1.$$

For this, we consider the complex-valued function f defined by

$$f(z) = \frac{e^z}{e^{az}+1} \tag{5.7}$$



and the contour $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ as in Figure below

It is clear that the function f has only singularity $z_0 = \frac{i\pi}{a}$ that lies inside γ, then by using the residue theorem, we obtain

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz = 2\pi i Res\left(f, \frac{i\pi}{a}\right). \quad (5.8)$$

◇ *On γ_1 we have $z = x$, $-\rho \leq x \leq \rho$.*

◇ *On γ_2 we have $z = \rho + iy$, $0 \leq y \leq \frac{2\pi}{a}$.*

◇ *On γ_3 we have $z = x + i\frac{2\pi}{a}$, $-\rho \leq x \leq \rho$.*

◇ *On γ_4 we have $z = -\rho + iy$, $0 \leq y \leq \frac{2\pi}{a}$.*

In addition, z_0 is a simple pole of f, then

$$Res\,(f, z_0) = \left.\frac{e^z}{ae^{az}}\right|_{z=\frac{i\pi}{a}} = \frac{e^{\frac{i\pi}{a}}}{ae^{i\pi}} = -\frac{e^{i\frac{\pi}{a}}}{a}.$$

Then (5.8) *becomes*

$$\int_{-\rho}^{\rho} f(x)dx + \int_0^{\frac{2\pi}{a}} f(\rho + iy)idy + \int_\rho^{-\rho} f\left(x + i\frac{2\pi}{a}\right)dx + \int_{\frac{2\pi}{a}}^0 f(-\rho + iy)idy = -2\pi i\frac{e^{i\frac{\pi}{a}}}{a}. \quad (5.9)$$

The use of (5.7) *gives*

$$\int_{\rho}^{-\rho} f\left(x+i\frac{2\pi}{a}\right)dx = \int_{\rho}^{-\rho} \frac{e^{x+i\frac{2\pi}{a}}}{e^{ax+2\pi i}+1}dx = -e^{i\frac{2\pi}{a}}\int_{-\rho}^{\rho}\frac{e^x}{e^{ax}+1}dx. \tag{5.10}$$

$$\left|\int_0^{\frac{2\pi}{a}} f(\rho+iy)idy\right| = \left|\int_0^{\frac{2\pi}{a}} \frac{e^{\rho+iy}}{e^{a(\rho+iy)}+1}idy\right| \leq \frac{2\pi}{a}\frac{e^{\rho}}{e^{a\rho}-1} \underset{\rho\to+\infty}{\longrightarrow} 0. \tag{5.11}$$

$$\left|\int_{\frac{2\pi}{a}}^{0} f(-\rho+iy)idy\right| = \left|\int_{\frac{2\pi}{a}}^{0} \frac{e^{-\rho+iy}}{e^{a(-\rho+iy)}+1}idy\right| \leq \frac{2\pi}{a}\frac{e^{-\rho}}{1-e^{-a\rho}} \underset{\rho\to+\infty}{\longrightarrow} 0. \tag{5.12}$$

Letting $\rho \to +\infty$ *and combining* (5.9)-(5.12), *we obtain*

$$\int_{-\infty}^{+\infty}\frac{e^x}{e^{ax}+1}dx + 0 - e^{i\frac{2\pi}{a}}\int_{-\infty}^{+\infty}\frac{e^x}{e^{ax}+1}dx + 0 = -2\pi i\frac{e^{i\frac{\pi}{a}}}{a},$$

then

$$\left(1-e^{i\frac{2\pi}{a}}\right)J = -2\pi i\frac{e^{i\frac{\pi}{a}}}{a},$$

therefore,

$$J = -\frac{2\pi i}{a}\frac{e^{i\frac{\pi}{a}}}{1-e^{i\frac{2\pi}{a}}} = -\frac{2\pi i}{a}\frac{1}{e^{-i\frac{\pi}{a}}-e^{i\frac{\pi}{a}}} = -\frac{2\pi i}{a}\frac{1}{-2i\sin\left(\frac{\pi}{a}\right)} = \frac{\pi}{a\sin\left(\frac{\pi}{a}\right)}.$$

Corollary 5.4.1 *If we put* $t = e^x$, *we get*

$$\int_0^{+\infty}\frac{dt}{t^{\alpha}+1} = \int_{-\infty}^{+\infty}\frac{e^x}{e^{ax}+1}dx = \frac{\pi}{a\sin\left(\frac{\pi}{a}\right)},\ a>1.$$

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