

:Chapter 1: Probability and Random Variables

1) : The concept of probability and random events

1-1) : Random experimentExperiment of chance

It is any process for which we know all possible outcomes beforehand, but we determine the outcome that will appear in a particular attempt. In other words, it is an experiment where, among all possible outcomes, one result appears by chance

The set of possible outcomes for a random experiment is called the sample space Sample Space It is denoted by S . The sample space consists of n elements (possible outcomes), and these elements are usually denoted by the symbols $a, b, c, d, \dots$

like

Tossing a coin, possible outcomes: rear p front , f :Sample space $S = \{p, f\}$

:Throwing a die, possible outcomes or sample space $S = \{1, 2, 3, 4, 5, 6\}$

1-2) : EventEvent

An event or events is each subset of the sample space, and is usually denoted by the symbols $A, B, C, \dots$

like

The appearance of f's face :When flipping a coin $A = \{f\}$

:even number appeared when the die was thrown $A = \{2, 4, 6\}$

If the subset contains only one element, it is called a simple (: Simple event primary) event

to the If the subset contains elements that do not belong : Impossible event sample space (an impossible result), it is called an impossible event

The complementary event to event : The complementary event A denoted by $\bar{A}$ is the event that includes the set of elements in the sample space that do not belong to event A .

Two events : Compatible events A and B , are incompatible if the occurrence of one negates or prevents the occurrence of the other

Independent events: Two events, A and B , are independent if the occurrence of one does not affect the occurrence of the other

The union of two events : Union A, B , and is symbolized as $A \cup B$. It is an event consisting of elements that belong to A or belong to B or both, and it means the occurrence of at least one of the two events

The intersection of two events : Intersection A, B and is denoted by $A \cap B$ is , an event consisting of elements that belong to A and belong to B i.e., common elements, and it means the occurrence of the two events together

:Example 1

Throwing a die, even number event $A = \{2, 4, 6\}$:

:Event number greater than or equal to 4 $B = \{4, 5, 6\}$,

:The incidents can be identified $\bar{A} = \{1, 3, 5\}$

$$A \cup B = \{2, 4, 5, 6\}$$

$$A \cap B = \{4, 6\}$$

1-3) : Probability

1-3-1) : Probability of an element and probability of an event

If S is a sample space containing n elements then the probability of element a_i is $1/n$.

If event A contains k elements, then the probability of event A is k/n : i.e ,
 A can occur is equal to the number of ways in which the sample space S occurs .
of probability : the number of favorable cases This is the traditional definition
.divided by the total number of possible cases

:identification

if it was S is a finite sample space containing n elements $S = \{a_1, a_2, \dots, a_n\}$
 $p(a_i)$ can be assigned to the element a_i and called the
:probability of this element, such that it satisfies the two conditions

$$0 \leq p(a_i) \leq 1$$

$$\left(\text{for } \sum_{i=1}^n p(a_i) = 1\right)$$

The probability of any event A is defined as the sum of the probabilities of its
¹.constituent elements

Let S be the sample space for a random experiment, and let $\mathcal{U}(S)$ the set of events defined on S it accompanies each event ; $A \in \mathcal{U}(S)$ A given number $p(A) \in [0,1]$ is called the probability of event A such that it satisfies the following ,
:postulates

a) If $A \subset S$: then the probability is , $0 \leq p(A) \leq 1$, $p(A) \geq 0$

: b) Probability $p(S) = 1$

c) If it was A_i : Where $(i = 1, 2, 3, \dots, \infty)$ mutually exclusive events, then the ,
: probability of

$$p(\cup A_i) = \sum_1 p(A_i)$$

²:the following conclusions can be proven ,From the above premises

:An empty set (an impossible event) then $\emptyset$ Result 1: If $p(\emptyset) = 0$

¹ Ahmed Awda, Introduction to Statistical Theory. Deanship of Library Affairs , King Saud University , Riyadh, .p. 5 .1991

² .Same reference, same page

is the complement of event $\bar{A}$ Result 2: If A :then , $p(\bar{A}) = 1 - p(A)$

Result 3: If $A \subset B$:then , $p(A) \leq p(B)$

Result 4: If Two events, A and B :then , $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

: The rule of adding probabilities

:From the previous relationships, we can find

If A and B :are mutually exclusive $p(A \cup B) = p(A) + p(B)$

: 2 Example

:Throwing a die, event $A = \{2, 4, 6\}$:event , $B = \{3, 4, 5, 6\}$,

: We calculate the probabilities $p(A) = \frac{3}{6}$, $p(B) = \frac{4}{6}$

$$p(\bar{A}) = 1 - p(A) = 1 - \frac{3}{6} = \frac{1}{2}$$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{5}{6}$$

1-3-2) : probability Conditional Probability

The probability of event A occurring given that event B occurs is called conditional probability and is denoted by $p(A/B)$:and is defined as follows ,

If A and B are two events in a sample space, and $p(B) \neq 0$ then the probability of , event A given that B :occurs is

$$p(A / B) = \frac{p(A \cap B)}{p(B)}$$

: probabilities The rule of multiplying

:From the previous relationship, we can find $p(A \cap B) = p(B) * p(A / B)$

:Or also $p(A \cap B) = p(A) * p(B / A)$

:Example 3

A society made up of 48% men and 52% women, with 3% of men and 5% of .women being unemployed

We take one person at random, event H is a man, event F is a woman, event C is an .unemployed person

The possibility that a person is unemployed: unemployed man + unemployed : woman

$$\begin{aligned} p(C) &= p(H) * p(C/H) + p(F) * p(C/F) \\ &= 0.48 * 0.03 + 0.52 * 0.05 = 0.0404 \end{aligned}$$

:The possibility that the person is a man, even though he is unemployed

$$p(H / C) = \frac{p(H \cap C)}{p(C)} = \frac{p(H) * p(C/H)}{p(C)} = \frac{0.48 * 0.03}{0.0404} = 0.3564$$

1-3-3) : Independent incidents Independent Events

of one occurring is not affected by Two events are independent if the probability .the occurrence of the other

if it was A and B are independent events; the probability of A occurring is 0. It is not affected by the occurrence of B ,

: any $p(A/B) = p(A)$

: In this case it would be $p(A \cap B) = p(A) * p(B)$

This relationship is the mathematical definition of the independence of two³ events

: Example 4

We roll a die twice; event A is a 5 on the first attempt, and event B is an odd

number on the second attempt. The probability of each event is $p(A) =$

$$\frac{1}{6} , \quad p(B) = \frac{3}{6}$$

: The two events are independent $p(A \cap B) = p(A) * p(B) = \frac{1}{6} * \frac{3}{6} = \frac{3}{36}$

2) : Random variables

: Random variable Random Variable

A random variable is a variable that takes on values in a given range in a random manner. If it takes a certain value at a given moment, we cannot know beforehand what value it will take next. This is in contrast to a regular variable, which changes regularly (within a range containing predefined values) from one value to another. regularly If it passes through a certain value at a given moment, it will not take that value again

: identification

's consider a random experiment with a sample space S. We call the function X that is associated with each element $a \in S$ a real number that is one and only one, $x = X(a)$ is called⁴. A random variable

: The random variable space is $\mathcal{A} = \{ x ; x = X(a) , a \in S \}$

2-1) : Discrete random variables Discrete Random Variables

A discrete random variable X is a variable whose possible values are in the form of points or discrete values

2-1-1) : The law of probability

X It has be a discrete random variable n: possible values $x_1, x_2, \dots, x_n$

: specific probability Each of these events occurs with a probability $p(X = x_i) = p_i , i = 1, 2, \dots, n$

³ Khaldi khaled , Probabilities: approaches to courses and corrigées. OPU alger 1999. P 10.

⁴ Idem, p38.

The law of probability is the relationship that connects the possible values of the variable X with their associated probabilities p_i ,

:In table format

X	$\Pr(X=x_i)$
x_1	p_1
x_2	p_2
$\dots$	$\dots$
x_n	p_n
	1

: Or in the form of a function called the probability function $f(x) = pr(X = x_i)$

Probability function Probability Function :checks both conditions

a) For each X : value $0 \leq f(x) \leq 1$

: b) The sum of probabilities $\sum_1^n f(x_i) = 1$

:Example 4

We toss a coin twice; the random variable X is the number of times heads

.(appear (heads 1, backs 0

:Possible outcomes $(0, 0), (0, 1), (1, 0), (1, 1)$

:Values of the random variable $X = 0, 1, 2$

: Function values Probability

$$f(0) = pr(x = 0) = \frac{1}{4}$$

$$f(1) = pr(x = 1) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

$$f(2) = pr(x = 2) = \frac{1}{4}$$

2-1-2) : **Distribution function** Distribution Function

The distribution function or cumulative probability function A function is a function that gives us a probability. At what point or value does x take ?,

X :Values He takes be a discrete random variable $x_1, x_2, \dots, x_n$

:The distribution function is defined as follows

$$F(x_k) = pr(X \leq x_k) = \sum_1^k f(x_i), \quad k \leq n$$

up to the value Any cumulative function gives us the sum of the probabilities x_k ⁵.

If we set $x = a$:then the distribution function ,

$$F(a) = pr(X \leq a) = f(x_1) + f(x_2) + \dots + f(a)$$

$$pr(X > a) = 1 - pr(X \leq a) = 1 - F(a)$$

Let a and b be within the range of values of the variable X :such that $a \leq$

b , $a, b \in [x_1, x_n]$

⁵ Lipschutz , Probabilities . Translated by Sameh Daoud. Fifth Edition , International House, Cairo 2004. Seymour .p. 116

$$pr(a \leq X \leq b) = pr(X \leq b) - pr(X \leq a) = F(b) - F(a)$$

Properties

1 $F(x_k) \leq F(x_{k+1})$

2 $[x_k, x_{k+1}]$

3 $0 \leq F(x) \leq 1$

any $\lim_{x \rightarrow -\infty} F(x) = 0$, $\lim_{x \rightarrow +\infty} F(x) = 1$

Example 5

In the previous example, flipping a coin twice, the random variable X is the number of times heads appear $X = 0, 1, 2$

We calculate the probabilities

$$F(0) = pr(x \leq 0) = f(0) = \frac{1}{4}$$

$$F(1) = pr(x \leq 1) = f(0) + f(1) = \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

$$F(2) = pr(x \leq 2) = f(0) + f(1) + f(2) = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$$

2-2) random Continuous variables Random Variables

A continuous random variable X is a variable whose possible values are in the finite or infinite continuous ranges of values, which may be form of

2-2-1) : Probability density function Probability Density Function

X be :finite interval a continuous random variable defined on a $[x_1, x_n]$ Or :unlimited $]-\infty, +\infty[$.

probability density function of the variable X the function $f(X)$ which satisfies the , :following conditions

a) $f(x) \geq 0$

b) A continuous function on $\mathbb{R}$.except for a finite number of points ,

c) The integral of the function over the entire domain equals one $\int_{-\infty}^{+\infty} f(x) dx =$

1

The density function curve is continuous; each point on the curve represents the of the curve at a class or subdomain of the domain of the continuous height variable. This height and subdomain define the probability as a definite integral or area under the density function curve. Therefore, the probability at a subdomain is te integral of the probability density function , while the the area or defini .(probability at a single point or value is zero (there is no area

The probability over the total domain of a continuous variable is the total area .equal to one under the curve of the density function and is

:Example 6

A continuous variable x defined by a : probability density function

$$f(x) = \begin{cases} 2(1-x) & , 0 \leq x \leq 1 \\ 0 & , x < 0 , x > 1 \end{cases}$$

:This function satisfies the following conditions

:negative function-A continuous and non $2(1-x) \geq 0$, $\forall x \in [0, 1]$

:Limited integration $\int_0^1 2(1-x) . dx = [2x - x^2]_0^1 = 1 - 0 = 1$

2-2-2) :Distribution functionDistribution Function

The distribution function or cumulative probability function is the function that over any interval in which gives us the probability X changes .

X_{be} :a continuous random variable defined on a finite interval $[x_1, x_n]$ Or :unlimited $]-\infty, +\infty[$

:The distribution function is defined as follows

$$F(x) = pr(X \leq x) = \int_{-\infty}^x f(X) . dX$$

The distribution function is cumulative from the first value of the domain of the continuous variable up to a certain value x .

:This function verifies

$$F(-\infty) = 0$$

$$F(+\infty) = 1$$

If we set $x=a$:the distribution function ,

$$F(a) = pr(X \leq a) = \int_{-\infty}^a f(X) . dX = [F(X)]_{-\infty}^a = F(a) - 0$$

Let a and b be :such that $a \leq b$, $a, b \in]-\infty, +\infty[$

$$pr(a \leq X \leq b) = \int_a^b f(X) dX = [F(X)]_a^b = F(b) - F(a)$$

:And also

$$pr(a < X \leq b) = F(b) - F(a)$$

:Probability at a single point

$$pr(X = a) = pr(a \leq X \leq a) = [F(X)]_a^a = F(a) - F(a) = 0$$

:Properties

:gnisaercni si noitcnuf noitubirtsid ehT °1 $F(a) \leq F(b)$, $a \leq b$
no suounitnoc si noitcnuf noitubirtsid ehT °2 $\mathbb{R}$.

:Differentiable and derivative functions .3 $\frac{dF(x)}{dx} = F'(x) = f(x)$

:neewteb denifnoc si noitcnuf noitubirtsid ehT °4 $0 \leq F(x) \leq 1$

:anylim $\lim_{x \rightarrow -\infty} F(x) = 0$, $\lim_{x \rightarrow +\infty} F(x) = 1$

:Example 7

The previous example, the probability density function of the continuous variable x :

$$f(x) = \begin{cases} 2(1-x) , & 0 \leq x \leq 1 \\ 0 , & x < 0 , x > 1 \end{cases}$$

:Distribution function

$$F(x) = \int_{-\infty}^x f(X) dX = \int_{-\infty}^x (2 - 2X) dx = [2X - X^2]_{-\infty}^x = 2x - x^2$$

:It is written as follows

$$F(x) = \begin{cases} 0 , & x < 0 \\ (2x - x^2) , & 0 \leq x \leq 1 \\ 1 , & x > 1 \end{cases}$$

: Possibilities

$$F\left(\frac{1}{2}\right) = pr\left(X \leq \frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$pr\left(\frac{1}{2} \leq X \leq 1\right) = F(1) - F\left(\frac{1}{2}\right) = \frac{1}{4}$$

3) :(Numerical characteristics of random variables (moments

Moments are numerical properties or characteristics of random variables, used to determine averages and measure dispersion, skewness and kurtosis. They are also considered numerical indicators for describing . probability distributions

3-1) :Initial moments and central moments

3-1-1) : Initial moments

These are called initial moments or moments around zero .

:identification

The moment of order r or a random variable about zero, fX , is defined by the following relation

:Discrete variable $MO_r = E(X^r) = \sum x^r * f(x)$

:Continuous variable $MO_r = E(X^r) = \int_{-\infty}^{+\infty} x^r * f(x) dx$

:Using these relationships, an infinite number of moments can be defined

If $r=0$: then $MO_0 = E(X^0) = \sum f(x) = 1$

.That is, the initial moment of order 0 is equal to one

If $r=1$: then $MO_1 = E(X) = \sum x * f(x) = \mu$

⁶ Khaldi Khaled , probabilities. Op-cit , p. 42.

.That is, the initial moment of order 1 is the expected value or the average value
if it was=2 :initial moment of order 2 , $MO_2 = E(X^2) = \sum x^2 * f(x)$
.and these cases also apply to the continuous variable...

3-1-2) :Central moments

Central moments are about a central value (mean, median...). If the moments are about the median value, they are called moments around the mean .

:identification

The central moment of order r about the mean, for a random variable X , is defined :by the following relation

:Discrete variable $MC_r = E[(X - \mu)^r] = \sum (x - \mu)^r * f(x)$

:Continuous variable $MC_r = E[(X - \mu)^r] = \int_{-\infty}^{+\infty} (x - \mu)^r * f(x) dx$

infinite number of centripetal moments can be Using these relationships, an
:defined

If $r=1$:then , $MC_1 = E(X - \mu) = \sum (X - \mu) * f(x) = 0$

That is, the central moment of order 1 is the sum of the deviations from the mean
.and equals zero

If $r=2$:then , $MC_2 = E[(X - \mu)^2] = \sum (x - \mu)^2 * f(x) = \sigma^2$

.s, the central momentum of order 2 around the mean is the variance That i

if it was $r=3$:centripetal moment of order 3 , $MC_3 = E[(X - \mu)^3] = \sum (x - \mu)^3 * f(x)$

.and the same applies to a continuous variable...

3-1-3) :Applications of torques

- : Torsion coefficient Skewness Coefficient

In a symmetrical distribution, such as a normal distribution, the mean equals the median, and the mode equals the average. Asymmetrical distributions are either
. positively skewed or negatively skewed

Fisher by Laboratories For torsion , denoted γ_1 , equals the centripetal moment of
:order 3, divided by the standard deviation raised to the power of 3

$$\gamma_1 = MC_3 / \sigma^3$$

A symmetrical distribution $\gamma_1 = 0$

skewness distribution $\gamma_1 > 0$ negative skew distribution , $\gamma_1 < 0$

- : Kurtosis coefficient Kurtosis Coefficient

Compared to the normal normal distribution, in which the curve is symmetrical shaped, the distribution is either less kurtotic (pointed) or more kurtotic -and bell
.(flat)

kurtosis coefficient, denoted as γ^2 , is equal to the centripetal moment of order 4, divided by the standard deviation raised to the power of 4, minus 3

$$\gamma_2 = \frac{MC_4}{\sigma^4} - 3$$

Normal distribution $\gamma^2 = 0$

distribution (weak dispersion) $\gamma^2 > 0$,) kurtotic distribution_{large} (dispersion $\gamma^2 < 0$)

3-2) :Expected value

: identification

The expected value or mathematical expectation of a random variable X is the initial moment of order 1, which indicates the average value of the variable

:Discrete variable $E(X) = \mu = \sum x * f(x)$

:Continuous variable $E(X) = \mu = \int_{-\infty}^{+\infty} x * f(x) dx$

If $g(X)$ is a function of the variable X then the expected value of this function, is

X :Intermittent $E[g(X)] = \sum g(x) * f(x)$

X :is continuous $E[g(X)] = \int_{-\infty}^{+\infty} g(x) * f(x) dx$

:Properties

1) If a is :expectation is a constant, the

$$E(a) = a$$

:Proof

$$E(a) = \sum a * f(x) = a * \sum f(x) = a$$

2) Adding the constant a to the variable X :the expectation is ,

$$E(X + a) = E(X) + a$$

:Proof

$$E(X + a) = \sum (x + a) * f(x) = \sum x * f(x) + \sum a * f(x) = E(X) + a$$

3) Multiplying a constant by the variable X :the expected result is ,

$$E(aX) = a * E(X)$$

:Proof

$$E(aX) = \sum (a * x) * f(x) = a * \sum x * f(x) = a * E(X)$$

4) if it was X and Y :are two variables, the expected sum is

$$E(X + Y) = E(X) + E(Y)$$

:Proof

$$E(X + Y) = \sum (x + y) * f(x, y) = \sum x * f(x) + \sum y * f(y) = E(X) + E(Y)$$

⁷ Murray Spiegel, Statistics. Translated by Shaaban Abdel Hamid. Seventh Edition, International House, Cairo, p. 143 .2004

ay, it can be proven that the expected difference is In the same w

$$E(X - Y) = E(X) - E(Y)$$

5) If X and Y are independent variables, the expected product is

$$E(X.Y) = E(X) * E(Y)$$

:Proof

$$E(X.Y) = \sum (x.y) . f(x,y) = \sum x . f(x) * \sum y . f(y) = E(X) * E(Y)$$

y to the case of a continuous variable, and in the same These properties also appl
.way they can be proven using integration

3-3) :Contrast Variance

: identification

The variance of a random variable X is the central momentum around the mean of order 2, and it indicates the dispersion or the distance of the variable's values .from the mean

:Discrete variable $V(X) = \sigma^2 = \sum (x - \mu)^2 * f(x)$

:Variance in terms of expectation $V(X) = E[(X - \mu)^2]$

:Or $V(X) = E(X^2) - [E(X)]^2 = \sum x^2 * f(x) - \mu^2$

:Continuous variable $V(X) = \sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 * f(x) dx$

⁸:Variance in terms of expectation $V(X) = E[(X - \mu)^2]$

:Or $V(X) = E(X^2) - [E(X)]^2 = \int_{-\infty}^{+\infty} x^2 * f(x) dx - \mu^2$

:Properties

1) If a is a constant, the variance is

$$V(a) = 0$$

:Proof

$$V(a) = E(a^2) - [E(a)]^2 = a^2 - a^2 = 0$$

2) Adding the constant a to the variable X :the variance is ,

$$V(X + a) = V(X)$$

:Proof

$$V(X + a) = E \left[((X + a) - (\mu + a))^2 \right] = E[(X + a - \mu - a)^2] = E[(X - \mu)^2] = V(X)$$

3) Multiplying a constant by the variable X :the variance is ,

$$V(aX) = a^2 * V(X)$$

⁸ Walder Maintaining statistics and probabilities calculations. Edition Dalloz Paris 2001. P 323.

:Proof

$$V(aX) = E[(aX - a\mu)^2] = E[a^2 * (X - \mu)^2] = a^2 * V(X)$$

4) if it was X and Y :are independent variables; the variance of the sum is

$$V(X + Y) = V(X) + V(Y)$$

:Proof

$$\begin{aligned} V(X + Y) &= E[(X + Y)^2] - [E(X + Y)]^2 = E[X^2 + Y^2 + 2XY] - \\ &[E(X) + E(Y)]^2 = E(X^2) + E(Y^2) + 2E(XY) - [E(X)]^2 + [E(Y)]^2 + \\ &2E(X)E(Y)] = E(X^2) + E(Y^2) + 2E(X)E(Y) - [E(X)]^2 - [E(Y)]^2 - \\ &2E(X)E(Y) = E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2 + 2E(X)E(Y) - \\ &2E(X)E(Y) = V(X) + V(Y) \end{aligned}$$

:The difference is $V(X - Y) = V(X) + V(Y)$

:Proof

$$\begin{aligned} V(X - Y) &= E[(X - Y)^2] - [E(X - Y)]^2 = E[X^2 + Y^2 - 2XY] - \\ &[E(X) - E(Y)]^2 = E(X^2) + E(Y^2) - 2E(XY) - [E(X)]^2 + [E(Y)]^2 - \\ &2E(X)E(Y)] = E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2 - 2E(X)E(Y) + \\ &2E(X)E(Y) = V(X) + V(Y) \end{aligned}$$

5) if it was Given that X and Y are independent variables, the variance of the
:product is

$$V(X.Y) = E(X^2).E(Y^2) - [E(X)]^2.[E(Y)]^2$$

:Proof

$$\begin{aligned} V(X.Y) &= E[(X.Y)^2] - [E(X.Y)]^2 = E[X^2.Y^2] - [E(X).E(Y)]^2 = \\ &= E(X^2).E(Y^2) - [E(X)]^2.[E(Y)]^2 \end{aligned}$$