TD N°2

Ex1 :

$$\begin{cases} \dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2}, & 0 < x < 1 \\ T = 0\,°C, & \text{à } t = 0, \text{ et } 0 \leq x \leq 1 \\ T = 100\,°C, & \text{à } x = 0, \text{ et } 0 < t \leq k. \\ T = 100\,°C, & \text{à } x = 1, \text{ et } 0 < t \leq k. \end{cases}$$

Solution :

* Methode difference centrée schéma Explicite :

$\Delta x = 0{,}2$ et $\Delta t = 0{,}01$.

$$\frac{\partial T}{\partial t} = \frac{T(i,j+1) - T(i,j)}{\Delta t}$$

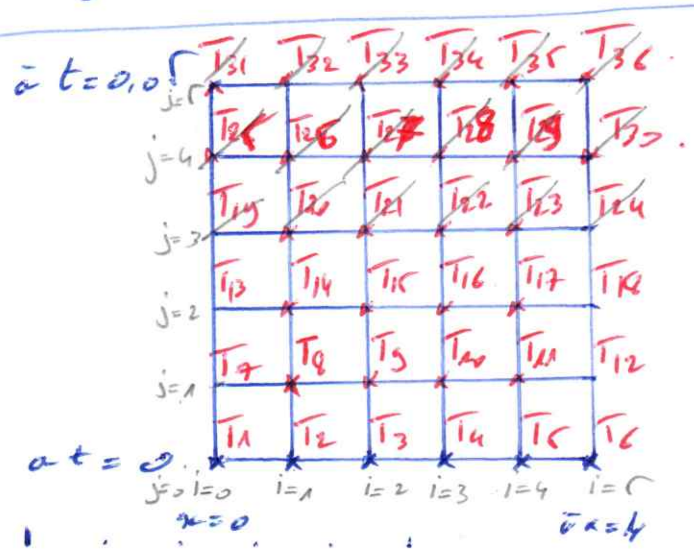
$$\frac{\partial^2 T}{\partial x^2} = \frac{T(i+1,j) - 2T(i,j) + T(i-1,j)}{\Delta x^2}$$

$$\frac{T(i,j+1) - T(i,j)}{\Delta t} = \frac{T(i+1,j) - 2T(i,j) + T(i-1,j)}{\Delta x^2}$$

$$T(i,j+1) = \lambda T(i+1,j) + (1-2\lambda) T(i,j) + \lambda T(i-1,j)$$

avec : $\lambda = \dfrac{\Delta t}{\Delta x^2} = \dfrac{0{,}01}{(0{,}2)^2} = 0{,}25 = 1/4$

$$\boxed{T(i,j+1) = 0{,}25\,T(i+1,j) + (0{,}5)\,T(i,j) + 0{,}25\,T(i-1,j)}.$$



à T = 0,01

$$\begin{cases} T_8 = \frac{1}{4}T_1 + \frac{1}{2}T_2 + \frac{1}{4}T_3 \\ T_9 = \frac{1}{4}T_2 + \frac{1}{2}T_3 + \frac{1}{4}T_4 \\ T_{10} = \frac{1}{4}T_3 + \frac{1}{2}T_4 + \frac{1}{4}T_5 \\ T_{11} = \frac{1}{4}T_4 + \frac{1}{2}T_5 + \frac{1}{4}T_6 \end{cases} \Rightarrow \begin{cases} T_8 = 0 \\ T_9 = 0 \\ T_{10} = 0 \\ T_{11} = 0 \end{cases}$$

à T = 0,02 :

$$\begin{cases} T_{14} = \frac{1}{4}T_7 + \frac{1}{2}T_8 + \frac{1}{4}T_9 \\ T_{15} = \frac{1}{4}T_8 + \frac{1}{2}T_9 + \frac{1}{4}T_{10} \\ T_{16} = \frac{1}{4}T_9 + \frac{1}{2}T_{10} + \frac{1}{4}T_{11} \\ T_{17} = \frac{1}{4}T_{10} + \frac{1}{2}T_{11} + \frac{1}{4}T_{12} \end{cases} \Rightarrow \begin{cases} T_{14} = \frac{1}{4}\times 100 + 0 + 0 = \frac{100}{4} = 25 \\ T_{15} = \frac{1}{4}\times 0 + \frac{1}{2}\times 0 + \frac{1}{4}\times 0 = 0 \\ T_{16} = \frac{1}{4}\times 0 + \frac{1}{2}\times 0 + \frac{1}{4}\times 0 = 0 \\ T_{17} = \frac{1}{4}\times 0 + \frac{1}{2}\times 0 + \frac{1}{4}\times 100 = 25 \end{cases}$$

à T = 0,03 :

$$\begin{cases} T_{20} = \frac{1}{4}T_{13} + \frac{1}{2}T_{14} + \frac{1}{4}T_{15} \\ T_{21} = \frac{1}{4}T_{14} + \frac{1}{2}T_{15} + \frac{1}{4}T_{16} \\ T_{22} = \frac{1}{4}T_{15} + \frac{1}{2}T_{16} + \frac{1}{4}T_{17} \\ T_{23} = \frac{1}{4}T_{16} + \frac{1}{2}T_{17} + \frac{1}{4}T_{18} \end{cases} \Rightarrow \begin{cases} T_{20} = \frac{100}{4} + \frac{25}{2} + 0 = 37,5 = \frac{150}{4} \\ T_{21} = \frac{25}{4} + \frac{0}{2} + \frac{0}{4} = 6,25 = \frac{25}{4} \\ T_{22} = \frac{0}{4} + \frac{0}{2} + \frac{25}{4} = 6,25 = \frac{25}{4} \\ T_{23} = \frac{0}{4} + \frac{25}{2} + \frac{100}{4} = 37,5 = \frac{150}{4} \end{cases}$$

à T = 0,04 :

$$\begin{cases} T_{26} = \frac{1}{4}T_{19} + \frac{1}{2}T_{20} + \frac{1}{4}T_{21} \\ T_{27} = \frac{1}{4}T_{20} + \frac{1}{2}T_{21} + \frac{1}{4}T_{22} \\ T_{28} = \frac{1}{4}T_{21} + \frac{1}{2}T_{22} + \frac{1}{4}T_{23} \\ T_{29} = \frac{1}{4}T_{22} + \frac{1}{2}T_{23} + \frac{1}{4}T_{24} \end{cases} \Rightarrow \begin{cases} T_{26} = \frac{100}{4} + \frac{150}{8} + \frac{25}{16} = \frac{725}{16} = 45,31 \\ T_{27} = \frac{150}{16} + \frac{25}{8} + \frac{25}{16} = \frac{225}{16} = 14,06 \\ T_{28} = \frac{25}{16} + \frac{25}{8} + \frac{150}{16} = \frac{225}{16} = 14,06 \\ T_{29} = \frac{25}{16} + \frac{150}{8} + \frac{100}{4} = \frac{725}{16} = 45,31 \end{cases}$$

* Méthode de différence centrée schéma implicite :

$$u(i,j) = -\lambda\, u(i+1,j+1) + (1+2\lambda)\, u(i,j+1) - \lambda\, u(i-1,j+1).$$

donc :

$$T(i,j) = -\frac{1}{4} T(i+1,j+1) + \frac{3}{2} T(i,j+1) - \frac{1}{4} T(i-1,j+1).$$

à $T = 0{,}01$:

$$\begin{cases} T_2 = -\frac{1}{4} T_7 + \frac{3}{2} T_8 - \frac{1}{4} T_9 \\ T_3 = -\frac{1}{4} T_8 + \frac{3}{2} T_9 - \frac{1}{4} T_{10} \\ T_4 = -\frac{1}{4} T_9 + \frac{3}{2} T_{10} - \frac{1}{4} T_{11} \\ T_5 = -\frac{1}{4} T_{10} + \frac{3}{2} T_{11} - \frac{1}{4} T_{12} \end{cases} \Rightarrow \begin{cases} \frac{3}{2} T_8 - \frac{1}{4} T_9 = T_2 + \frac{1}{4} T_7 \\ -\frac{1}{4} T_8 + \frac{3}{2} T_9 - \frac{1}{4} T_{10} = T_3 \\ -\frac{1}{4} T_9 + \frac{3}{2} T_{10} - \frac{1}{4} T_{11} = T_4 \\ -\frac{1}{4} T_{10} + \frac{3}{2} T_{11} \cancel{-\frac{1}{4} T_{12}} = T_5 + \frac{1}{4} T_{12} \end{cases} \quad \text{— (I)}$$

à $T = 0{,}02$:

$$\begin{cases} T_8 = -\frac{1}{4} T_{13} + \frac{3}{2} T_{14} - \frac{1}{4} T_{15} \\ T_9 = -\frac{1}{4} T_{14} + \frac{3}{2} T_{15} - \frac{1}{4} T_{16} \\ T_{10} = -\frac{1}{4} T_{15} + \frac{3}{2} T_{16} - \frac{1}{4} T_{17} \\ T_{11} = -\frac{1}{4} T_{16} + \frac{3}{2} T_{17} - \frac{1}{4} T_{18} \end{cases} \Rightarrow \begin{cases} \frac{3}{2} T_{14} - \frac{1}{4} T_{15} = T_8 + \frac{1}{4} T_{13} \\ -\frac{1}{4} T_{14} + \frac{3}{2} T_{15} - \frac{1}{4} T_{16} = T_9 \\ -\frac{1}{4} T_{15} + \frac{3}{2} T_{16} - \frac{1}{4} T_{17} = T_{10} \\ -\frac{1}{4} T_{16} + \frac{3}{2} T_{17} \cancel{-\frac{1}{4} T_{18}} = T_{11} + \frac{1}{4} T_{18} \end{cases} \quad \text{— (II)}$$

à $T = 0{,}03$:

$$\begin{cases} T_{14} = -\frac{1}{4} T_{19} + \frac{3}{2} T_{20} - \frac{1}{4} T_{21} \\ T_{15} = -\frac{1}{4} T_{20} + \frac{3}{2} T_{21} - \frac{1}{4} T_{22} \\ T_{16} = -\frac{1}{4} T_{21} + \frac{3}{2} T_{22} - \frac{1}{4} T_{23} \\ T_{17} = -\frac{1}{4} T_{22} + \frac{3}{2} T_{23} - \frac{1}{4} T_{24} \end{cases} \Rightarrow \begin{cases} \frac{3}{2} T_{20} - \frac{1}{4} T_{21} = T_{14} + \frac{1}{4} T_{19} \\ -\frac{1}{4} T_{20} + \frac{3}{2} T_{21} - \frac{1}{4} T_{22} = T_{15} \\ -\frac{1}{4} T_{21} + \frac{3}{2} T_{22} - \frac{1}{4} T_{23} = T_{16} \\ -\frac{1}{4} T_{22} + \frac{3}{2} T_{23} \cancel{-\frac{1}{4} T_{24}} = T_{17} + \frac{1}{4} T_{24} \end{cases} \quad \text{— (III)}$$

à $T = 0{,}04$

$$\begin{cases} T_{20} = -\frac{1}{4} T_{25} + \frac{3}{2} T_{26} - \frac{1}{4} T_{27} \\ T_{21} = -\frac{1}{4} T_{26} + \frac{3}{2} T_{27} - \frac{1}{4} T_{28} \\ T_{22} = -\frac{1}{4} T_{27} + \frac{3}{2} T_{28} - \frac{1}{4} T_{29} \\ T_{23} = -\frac{1}{4} T_{28} + \frac{3}{2} T_{29} - \frac{1}{4} T_{30} \end{cases} \Rightarrow \begin{cases} \cancel{-\frac{1}{4} T_{25}} + \frac{3}{2} T_{26} - \frac{1}{4} T_{27} = T_{20} + \frac{1}{4} T_{25} \\ -\frac{1}{4} T_{26} + \frac{3}{2} T_{27} - \frac{1}{4} T_{28} = T_{21} \\ -\frac{1}{4} T_{27} + \frac{3}{2} T_{28} - \frac{1}{4} T_{29} = T_{22} \\ -\frac{1}{4} T_{28} + \frac{3}{2} T_{29} = T_{23} + \frac{1}{4} T_{30} \end{cases} \quad \text{— (IV)}$$

I $\Leftrightarrow$ $\begin{cases} \frac{3}{2}T_8 - \frac{1}{4}T_9 = 25 \\ -\frac{1}{4}T_8 + \frac{3}{2}T_9 - \frac{1}{4}T_{10} = 0 \\ -\frac{1}{4}T_9 + \frac{3}{2}T_{10} - \frac{1}{4}T_{11} = 0 \\ -\frac{1}{4}T_{10} + \frac{3}{2}T_{11} = +25 \end{cases} \Rightarrow \left[\begin{array}{ccccc|c} \frac{3}{2} & -\frac{1}{4} & 0 & 0 & 0 & 25 \\ -\frac{1}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{3}{2} & -\frac{1}{4} & 25 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & \beta_1^{(1)} & 0 & 0 & \gamma_1^{(1)} \\ 0_1 & 1 & \beta_2^{(1)} & 0 & \gamma_2^{(1)} \\ 0 & 0_2 & 1 & \beta_3^{(1)} & \gamma_3^{(1)} \\ 0 & 0 & 0 & 1 & \gamma_4^{(1)} \end{array}\right] \Rightarrow \begin{array}{l} T_8 = 17,24 \\ T_9 = 3,448 \\ T_{10} = 3,448 \\ T_{11} = 17,24 \end{array}$$

II $\Leftrightarrow$ $T_{14} = 29,60$, $T_{15} = 8,6775$, $T_{16} = 8,6775$, $T_{17} = 29,60$

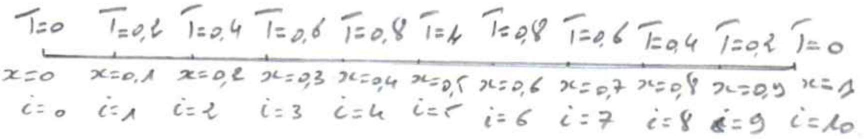
III $\Leftrightarrow$ $T_{20} = 38,8521$, $T_{21} = 14,7124$, $T_{22} = 14,7124$, $T_{23} = 38,8521$

IV $\Leftrightarrow$ $T_{26} = 46,0652$, $T_{27} = 20,9830$, $T_{28} = 20,9830$, $T_{29} = 46,0652$

V $\Leftrightarrow$ $T_{32} = 51,90$, $T_{33} = 27,1641$, $T_{34} = 27,1641$, $T_{35} = 51,90$

Exe:

a/ Pour $\Delta x = 0.1$ et $\Delta t = 0{,}01$:



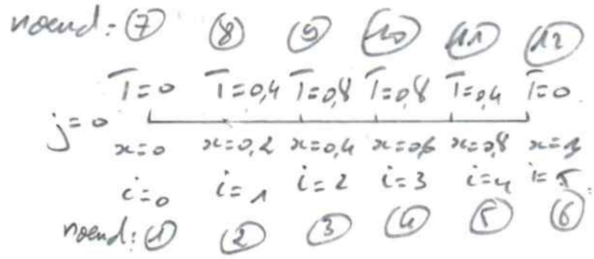
* Schéma Explicite:

$$T(i,j+1) = \lambda T(i+1,j) + (1-2\lambda) T(i,j) + \lambda T(i-1,j)$$

$$\lambda = \frac{\Delta t}{\Delta x^2} = \frac{0.01}{(0.1)^2} = 1$$

donc: $1-2\lambda > 0 \Rightarrow \lambda < 1/2$, le schéma n'est pas stable

b/ Pour $\Delta x = 0{,}2$ et $\Delta t = 0.01$:

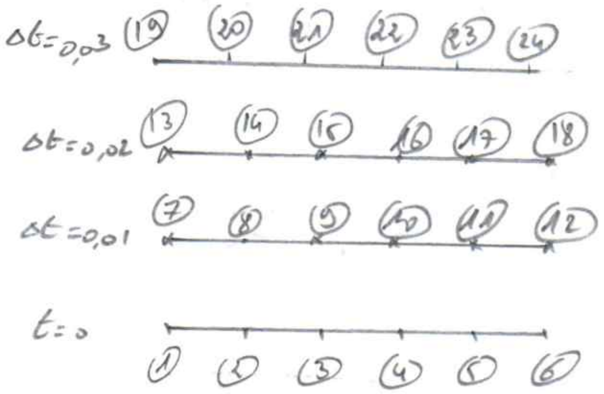


$$\lambda = \frac{\Delta t}{\Delta x^2} = \frac{0.01}{(0.2)^2} = 0{,}25$$

Alors:

$$T(i,j+1) = 0{,}25\, T(i+1,j) + 0{,}5\, T(i,j) + 0{,}25\, T(i-1,j)$$

-> Pour les noeuds internes:



à : t=0.01:

$T_8 = 0{,}25\, T_3 + 0{,}5\, T_2 + 0{,}25\, T_1 = 0{,}4$.

$T_9 = 0{,}25\, T_4 + 0{,}5\, T_3 + 0{,}25\, T_2 = 0.7$.

$T_{10} = 0{,}25\, T_5 + 0{,}5\, T_4 + 0{,}25\, T_3 = 0{,}7$.

$T_{11} = 0{,}25\, T_6 + 0{,}5\, T_5 + 0{,}25\, T_4 = 0{,}4$.

à t=0,02:

$T_{14} = 0{,}25\, T_9 + 0{,}5\, T_8 + 0{,}25\, T_7 = 0{,}375$.

$T_{15} = 0{,}25\, T_{10} + 0{,}5\, T_9 + 0{,}25\, T_8 = 0{,}625$.

$T_{16} = 0{,}25\, T_{11} + 0{,}5\, T_{10} + 0{,}25\, T_9 = 0{,}625$

$T_{17} = 0{,}25\, T_{12} + 0{,}5\, T_{11} + 0{,}25\, T_{10} = 0{,}375$.

à t=0,03:

$T_{20} = 0{,}25\, T_{15} + 0{,}5\, T_{14} + 0{,}25\, T_{13} = 0{,}344$

$T_{21} = 0{,}25\, T_{16} + 0{,}5\, T_{15} + 0{,}25\, T_{14} = 0{,}562$

$T_{22} = 0{,}25\, T_{17} + 0{,}5\, T_{16} + 0{,}25\, T_{15} = 0{,}562$

$T_{23} = 0{,}25\, T_{18} + 0{,}5\, T_{17} + 0{,}25\, T_{16} = 0{,}344$

* Schéma Implicite:

$$T(i,j) = -\lambda T(i+1,j+1) + (1+2\lambda)T(i,j+1) - \lambda T(i-1,j+1)$$

pour $\Delta x = 0.1$:

$\lambda = \dfrac{\Delta t}{\Delta x^2} = \dfrac{0.01}{(0.1)^2} = 1$ le schéma est stable, pour $\Delta x = 0,2 \Rightarrow \lambda = \dfrac{0,01}{(0,2)^2} = 0,25$

$\Delta t = 0,01$.

$$\left.\begin{aligned} T_2 &= -0,25T_9 + 1,5T_8 - 0,25T_7 \\ T_3 &= -0,25T_{10} + 1,5T_9 - 0,25T_8 \\ T_4 &= -0,25T_{11} + 1,5T_{10} - 0,25T_9 \\ T_5 &= -0,25T_{12} + 1,5T_{10} - 0,25T_{10} \end{aligned}\right\} \Leftrightarrow \begin{array}{cccc} T_8 & T_9 & T_{10} & T_{11} \end{array}$$

$$\begin{bmatrix} 1,5 & -0,25 & 0 & 0 & ; 0,4 \\ -0,25 & 1,5 & -0,25 & 0 & 0,8 \\ 0 & -0,25 & 1,5 & -0,25 & ; 0,8 \\ 0 & 0 & -0,25 & 1,5 & ; 0,4 \end{bmatrix}$$

Résolution de la matrice par la DTMA:

• calcul de $\beta_1^{(1)}$ et $\gamma_1^{(1)}$:

$$\beta_1^{(1)} = \frac{-0,25}{1,5} = -0,166, \quad \gamma_1^{(1)} = \frac{0,4}{1,5} = 0,266$$

• calcul de $\beta_2^{(1)}, \beta_3^{(1)}$:

$$\beta_2^{(1)} = \frac{\beta_2^{(0)}}{A_2 - [\beta_1^{(1)} \times C_1^{(0)}]} = \frac{-0,25}{1,5 - [+0,166 \times 0,25]} = \frac{-0,25}{1,4585} = -0,171$$

$$\beta_3^{(1)} = \frac{\beta_3^{(0)}}{A_3 - [\beta_2^{(1)} \times C_2^{(0)}]} = \frac{-0,25}{1,5 - [+0,171 \times 0,25]} = \frac{-0,25}{1,457} = -0,172$$

• calcul de $\gamma_2^{(1)}, \gamma_3^{(1)}$, et $\gamma_4^{(1)}$:

$$\gamma_2^{(1)} = \frac{\gamma_2^{(0)} - [\gamma_1^{(1)} \times C_1^{(0)}]}{A_2 - [\beta_1^{(1)} \times C_1^{(0)}]} = \frac{0,8 + [0,266 \times 0,25]}{1,4585} = 0,594$$

$$\gamma_3^{(1)} = \frac{\gamma_3^{(0)} - [\gamma_2^{(1)} \times C_2^{(0)}]}{A_3 - [\beta_2^{(1)} \times C_2^{(0)}]} = \frac{0,8 + [0,59 \times 0,25]}{1,457} = 0,651$$

$$\gamma_4^{(1)} = \frac{\gamma_4^{(0)} - [\gamma_3^{(1)} \times C_3^{(0)}]}{A_4 - [\beta_3^{(1)} \times C_3^{(0)}]} = \frac{0,4 + [0,651 \times 0,25]}{1,5 - [+0,172 \times 0,25]} = \frac{0,563}{1,457} = 0,386$$

$T_{11} = 0,39$

$T_{10} = g_3^{(1)} - \beta_3^{(1)} \times T_{11} = 0,604 + 0,172 \times 0,39 \simeq 0,671$

$T_9 = g_2^{(1)} - \beta_2^{(1)} \times T_{10} = 0,594 + 0,171 \times 0,671 = 0,71$

$T_8 = g_1^{(1)} + \beta_1^{(1)} \times T_9 = 0,266 + 0,166 \times 0,71 = 0,38$

de même manière on fait les calculs à $t = 0,02$ et à $t = 0,03$:

* Schéma de Crank Nicholson :

$-\lambda T(i-1, j+1) + 2(1+\lambda)T(i, j+1) - \lambda T(i+1, j+1) = \lambda T(i-1, j) + 2(1-\lambda)T(i,j) + \lambda T(i+1, j)$

Pour $\Delta x = 0.1$ et $\Delta t = 0.01$:

$\lambda = 1 \Rightarrow$ Schéma n'est pas stable :

Pour $\Delta x = 0,2$ et $\Delta t = 0.01$:

$\lambda = 0,25$:

Alors :

$$+2,5 T_8 - 0,25 T_9 = 1,5 T_2 + 0,25 T_3 + 0,25 T_7 = 0,8$$

$$-0,25 T_8 + 2,5 T_9 - 0,25 T_{10} = 0,25 T_2 + 1,5 T_3 + 0,25 T_4 = 1,5$$

$$-0,25 T_9 + 2,5 T_{10} - 0,25 T_{11} = 0,25 T_3 + 1,5 T_4 + 0,25 T_5 = 1,5$$

$$-0,25 T_{10} + 2,5 T_{11} = 0,25 T_4 + 1,5 T_5 + 0,25 T_6 + 0,25 T_{12} = 0,8$$

$$\begin{array}{ccccc} T_8 & T_9 & T_{10} & T_{11} & \\ \end{array}$$
$$\left[\begin{array}{cccc:c} 2,5 & -0,25 & 0 & 0 & 0,8 \\ -0,25 & 2,5 & -0,25 & 0 & 1,5 \\ 0 & -0,25 & 2,5 & -0,25 & 1,5 \\ 0 & 0 & -0,25 & 2,5 & 0,8 \end{array}\right]$$

Résolution de la matrice par la DTMA:

• Calcul de $\beta_1^{(1)}$ et $g_1^{(1)}$:

$$\beta_1^{(1)} = \frac{-0{,}25}{2{,}5} = -0{,}1 \ , \quad g_1^{(1)} = \frac{0{,}8}{2{,}5} = 0{,}32.$$

• Calcul de $\beta_2^{(1)}$ et $\beta_3^{(1)}$

$$\beta_2^{(1)} = \frac{-0{,}25}{2{,}5-\{+0{,}1\times0{,}25\}} = \frac{-0{,}25}{+2{,}475} = -0{,}101$$

$$\beta_3^{(1)} = \frac{-0{,}25}{2{,}5-\{+0{,}101\times0{,}25\}} = \frac{-0{,}25}{2{,}475} = -0{,}101$$

• Calcul de $g_2^{(1)}$, $g_3^{(1)}$ et $g_4^{(1)}$:

$$g_2^{(1)} = \frac{1{,}5+\{0{,}32\times0{,}25\}}{2{,}475} = 0{,}638$$

$$g_3^{(1)} = \frac{1{,}5+\{0{,}638\times0{,}25\}}{2{,}475} = 0{,}670$$

$$g_4^{(1)} = \frac{0{,}8+\{0{,}670\times0{,}25\}}{2{,}5-\{+0{,}101\times0{,}25\}} = \frac{0{,}968}{2{,}475} = 0{,}391$$

• Calcul des T:

$T_{11} \simeq 0{,}4$

$T_{10} = 0{,}670 + 0{,}101\times0{,}4 = 0{,}71$

$T_9 = 0{,}638 + 0{,}101\times0{,}71 = 0{,}71$

$T_8 = 0{,}32 + 0{,}1\times0{,}71 = 0{,}39$

de même manière on fait les calculs à $t=0{,}02$ et à $t=0{,}03$:

Ex 3 :

* Schéma Explicite :

$$U(i,j+1) = \lambda U(i+1,j) + (1-2\lambda) U(i,j) + \lambda U(i-1,j)$$

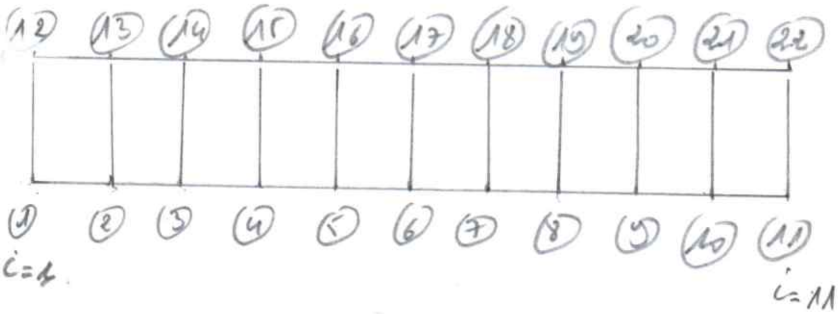
$$\lambda = \frac{\Delta t}{\Delta x^2} = \frac{0{,}0025}{(0{,}1)^2} = 0{,}25.$$

Alors :

$$U(i,j+1) = 0{,}25\, U(i+1,j) + 0{,}5\, U(i,j) + 0{,}25\, U(i-1,j)$$

* Cas (1) : Approche centrée

à $t = 0{,}0025$



Pour le noeud (12) :

$$U(1,2) = 0{,}25\, U(2,1) + 0{,}5\, U(1,1) + 0{,}25\, U(0,1) \quad i=1 \quad (*)$$

• Calcul de $U(0,1)$:

On a : $\left.\frac{\partial U}{\partial x}\right|_{x=0} = U$. Appliquant l'approche centrée : $\frac{U(i+1,j) - U(i-1,j)}{2\Delta x} = U(i,j)$

$$\Leftrightarrow \frac{U(2,1) - U(0,1)}{2\Delta x} = U(1,1) \Rightarrow U(0,1) = U(2,1) - 2\Delta x\, U(1,1) \Leftrightarrow \boxed{U(0,1) = U_2 - 0{,}2\, U_1}$$

On remplace dans l'équation :

$$U_{12} = 0{,}25\, U_2 + 0{,}5\, U_1 + 0{,}25\, U_2 - 0{,}05\, U_1$$

$$\Leftrightarrow \boxed{U_{12} = 0{,}5\, U_2 + 0{,}45\, U_1} = 0{,}5 + 0{,}45 = 0{,}95.$$

Pour les noeuds internes de (13) à (21) :

$$U_{13} = 0{,}25\, U_3 + 0{,}5\, U_2 + 0{,}25\, U_1 = 1$$

$$U_{14} = 0{,}25\, U_4 + 0{,}5\, U_3 + 0{,}25\, U_2 = 1$$

$$U_{15} = 0{,}25\, U_5 + 0{,}5\, U_4 + 0{,}25\, U_3 = 1$$

$$U_{16} = 0{,}25\, U_6 + 0{,}5\, U_5 + 0{,}25\, U_4 = 1$$

$$U_{17} = 0{,}25\, U_7 + 0{,}5\, U_6 + 0{,}25\, U_5 = 1$$

$$U_{18} = 0{,}25\, U_8 + 0{,}5\, U_7 + 0{,}25\, U_6 = 1$$

$$U_{19} = 0{,}25\, U_9 + 0{,}5\, U_8 + 0{,}25\, U_7 = 1$$

$$U_{20} = 0{,}25\, U_{10} + 0{,}5\, U_9 + 0{,}25\, U_8 = 1$$

$U_{2,1} = 0{,}25\, U_{11} + 0{,}5\, U_{10} + 0{,}25\, U_9 = 1$

- Pour le noeud U_{22}:

$U(11,2) = 0{,}25\, U(12,1) + 0{,}5\, U(11,1) + 0{,}25\, U(10,1)$.

Calcul de $U(12,1)$:

On a : $\left.\frac{\partial U}{\partial x}\right|_{x=L} = -U \Leftrightarrow \frac{U(i+1,j) - U(i-1,j)}{2\Delta x} = -U(i,j)$ Approche centrée.

$\Leftrightarrow \frac{U(12,1) - U(10,1)}{2\Delta x} = -U(11,1) \Rightarrow U(12,1) = U(10,1) - 2\Delta x\, U(11,1)$

$\Leftrightarrow \boxed{U(12,1) = U_{10} - 0{,}2\, U_{11}}$

On remplace dans l'equation:

$U_{22} = 0{,}25\, U_{10} - 0{,}05\, U_{11} + 0{,}5\, U_{11} + 0{,}25\, U_{10}$.

$\boxed{U_{22} = 0{,}5\, U_{10} + 0{,}45\, U_{11}} = 0{,}95$

b/ Cas(2): Approche à droite et à gauche.

Pour le noeud 12:

$U(1,2) = 0{,}25\, U(2,1) + 0{,}5\, U(1,1) + 0{,}25\, U(0,1)$

- Calcul de $U(0,1)$.

On a : $\left.\frac{\partial U}{\partial x}\right|_{x=0} = U \Rightarrow \frac{U(i,j) - U(i-1,j)}{\Delta x} = U(i,j) \Leftrightarrow \frac{U(1,1) - U(0,1)}{\Delta x} = U(1,1)$.

$\Leftrightarrow U(0,1) = (1-\Delta x)\, U(1,1) \Leftrightarrow \boxed{U(0,1) = 0{,}9\, U_1}$

On remplace dans l'equation:

$U_{12} = 0{,}25\, U_2 + 0{,}725\, U_1 = \boxed{0{,}975}$

Pour le noeud 22:

$U(11,2) = 0{,}25\, U(12,1) + 0{,}5\, U(11,1) + 0{,}25\, U(10,1)$

- Calcul de $U(12,1)$:

On a : $\left.\frac{\partial U}{\partial x}\right|_{x=L} = -U \Leftrightarrow \frac{U(i+1,j) - U(i,j)}{\Delta x} = -U(i,j) \Leftrightarrow \frac{U(12,1) - U(11,1)}{\Delta x} = -U(11,1)$.

$\Leftrightarrow U(12,1) = (1-0,1)\,U(11,1) \Leftrightarrow \boxed{U(12,1) = 0,9\,U(11,1)}$ où $\boxed{U(12,1) = 0,9\,U_{11}}$

<u>on remplace dans l'équation :</u>

$$U_{22} = 0,225\,U_{11} + 0,5\,U_{11} + 0,25\,U_{10} =$$

$$U_{22} = 0,725\,U_{11} + 0,25\,U_{10} = \boxed{0,975}$$

les noeuds internes de 13 à 21 = 1 : (déjà calculés)